Aspects of learning and understanding in multivariable calculus

by

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Abstract

In this thesis we study the processes by which university students solve problems in multivariable calculus. Our data consists of a series of questionnaires and interviews with students enrolled in a vector calculus class at MIT. We interpret our observations in the light of previous research into the acquisition of mathematical knowledge and understanding.

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Chapter 1

Introduction and statement of purpose

This project started as an attempt to study how students learn in mathematics. In particular I was interested in the distinction between learning algorithms and learning concepts, and the role they play together in producing a complete understanding of the material.

Kannemeyer (2005) explains it nicely:

> Expertise in mathematics is dependent on all categories of understanding and effective learning of mathematics cannot be based only on any type of understanding.
>
> These different kinds of understanding are facets of a dynamic process that sees them impacting on one another, at times reinforcing or even hindering each other.
>
> Although the boundaries between kinds of understanding are not considered to be static, the distinctions between them usefully provide criteria to make judgments about a student's abilities.

Therefore, I decided to investigate these aspects of learning focusing my attention on a course offered at MIT in multivariable calculus (18.02). I wanted to see how these abilities can and should be developed, in particular looking at the different aspects of the class: lecture, recitation, personal study, study in group, office hours.

To study these questions I gave the students two surveys and interviewed some of them, asking to solve a problem with me.

During my research I discovered that the nature of mathematics and the learning processes have been classified in several ways.

Mathematical understanding has been categorized into conceptual and procedural, but within conceptual learning some researchers distinguished two stages, calling them participatory and anticipatory (Tzur, Simon, 2004). Moreover the type of understanding one achieves has deep impact on the capability of developing operational and structural conceptions (Sfard, 1991).

Along the way I had to acknowledge the need to clarify which are the abilities/knowledges that we expect the students to have and to analyze how well we do in teaching what we say is important.

Chapter 2

Literature review

Many researchers have addressed conceptual and procedural knowledge. To be more precise, we can find in the literature many classifications and definitions of conceptual/procedural knowledge, understanding and thinking. Therefore, we need to state what we mean by these and to present some of the different theories which the educational community has agreed upon.

Some of the literature that we'll discuss is not limited to college level education. However, we find that the contributions that those authors offer apply to our purpose. Therefore, we won't go into details about the contexts of their investigations.

By conceptual knowledge we refer to the ability to identify and apply principles, knowledge of facts and definitions, and understanding of their meaning.

To have an example to work with, we could consider the fact that regions are described by inequalities. This is conceptual since one has to know that regions are characterized by being bounded by a subspace of dimension one less than the dimension of the space they are immersed in. This implies that the boundary is expressed by one equation, and then one has to select one side, which corresponds to the inequality.

But this definition is not enough when we talk about conceptual *understanding*. What we have in mind in that case is something like the ability to look at an object from different points of view. Sfard (1991) offers a nice example, following Hadamard's metaphor comparing this idea to what happens in face recognition:

"pictures . . . preserve their identity and meaning when observed from different points of view and in different contexts" (Hadamard, 1949).

An example of procedural understanding is comprehending the algorithms. However it is useful to talk about procedural fluency, which is the ability to carry out the procedures in a way that is accurate and efficient, but also appropriate to the context and hence flexible.

Referring to the example above, one has to "translate" the two components of the region (boundary and interior) into the "algebraic components" of the inequality: the equation associated with it describes the boundary, and the strict inequality identifies the interior. This is a fundamental procedural step, necessary to perform the right computations.

It is evident even in this case that the distinction between conceptual and procedural requires careful attention. At the same time it is of great interest to explore how concepts and procedures are related.

Of course, there are different types of understanding that one can achieve, and that can be reflected in the approach towards a task (and this could be referred to as conceptual/procedural thinking), as well as the capability for developing different ways to perceive a mathematical notion.

The topic of learning approaches has been addressed by Ramsden (2004). He introduces the word *conception* to talk about perceptions of notions, and explains it as the way a person makes sense of something. He says that "when we talk about a student understanding something, what we are really saying is if the student is capable of relating to a concept in the way that an expert does." It is from this perspective that he analyses the approaches to learning, pointing out two aspects: whether the student is searching for meaning (deep/surface approach), and how the student organizes the task (atomistic/holistic approach). He clarifies:

> In practice, these two aspects of approaches are fused together. It makes no sense to talk about the meaning attributed to something unless one also talks about how the meaning is constituted. On the other hand, how a student structures a task cannot be considered in isolation from what

> he or she is intending to structure. (Ramsden, 2004)

He goes further, stating that approaches to learning are not something a student has. Rather, they represent what a learning task means to the learner. And that might bear upon the level of understanding achieved. Quality of understanding and learning approaches are linked one another:

> An approach is not about learning facts versus learning concepts: it is about learning *just* the unrelated facts (or procedures) versus learning the facts *in relation to* the concepts. Surface is, at best, about quantity without quality; deep is about quality and quantity. (Ramsden, 2004)

In his research he used both interviews and surveys to investigate this topic, and he explains that while some students stressed *memorizing* and *arranging* disconnected pieces, others engaged in the process of *linking together* and abstracting personal meaning.

Many researchers agree that conceptual knowledge increases procedural proficiency, and conversely, that increasing procedural knowledge assists in expanding conceptual understanding and knowledge. This is usually described as a learning process in which procedural knowledge is used to manipulate existing concepts (Piaget).

This model has been studied in detail by Anna Sfard, in [5]. Her reaserch is based on a comparison between historical development of mathematical concepts and how individual students learn them. The latter is based upon empirical evidence, psychological studies (pioneered by Piaget) and theoretical arguments, and we are going to present some of her research here.

We have already introduced the word "conception," relating it to the way in which we make sense of something. Sfard uses this word to mean "the whole cluster of internal representations and associations evoked by the concept", and introduces it as a response to the need to deal with abstract entities such as the mathematical concepts.

In opposition to the previously presented decomposition of mathematical knowledge, she posits a unified whole in which emerges a dual nature of conceptions, where the two aspects, *structural* and *operational*, are complementary and necessary to each other.

First of all, we need to stress that what she calls structural conception is very close to our original idea of conceptual understanding. In fact it refers to the ability of seeing a mathematical entity as an object, which means "being capable of referring to it as an object,... being able to recognize the idea *at a glance* and to manipulate it as a whole, without going into details" (Sfard, 1991), like recognizing the face of a man.

Operational refers to algorithms and processes, and could be related to our idea of procedural understanding. Sfard explains (1991) that "interpreting a notion as a process implies regarding it as a potential rather than actual entity, which comes into existence upon request in a sequence of actions", and is therefore very detailed.

She argues that doing things is the only way to get in touch with abstract constructions. Therefore, it would be unreasonable to think that a student arrives at a structural conception without a previous operational understanding. On the other hand, insight and mastery of procedures "should sometimes be viewed as a basis for understanding" the underlying "concepts rather than its outcome" (Sfard, 1991). In her historical review, she points out that "the majority of ideas originated in processes rather than in objects" even though the "insight necessary for mathematical creation can hardly be achieved without the ability to 'see' abstract objects".

In other words, the ability to see a notion both as a process and as an object is indispensable for a deep understanding, but "our capability for developing operational and structural conceptions bears upon the type of understanding we achieve" (Sfard, 1991).

In her research, Sfard reports that many students don't reach the "object-level" of understanding. Tzur and Simon (2004) have studied this issue, and suggest that the "transition from process to object involves two stages of conceptual transformation" that they call *participatory* and *anticipatory*. They argue that "a critical component of a conception is an abstracted relationship (anticipation) between an activity and its effects. Thus an activity both generates and is a constituent of a conception." (By *activity* they refer to any mental process.) Their work describes the development of a mathematical conception. They identify a first stage, called participatory, in which "the learners have learned to anticipate the effects of an activity -*effect* is the outcome of their activity- and may also be able to explain why the effects derive from the activity. However, this knowledge is only available to the learners in the context of the activity through which it was developed." This "means either that the learners are engaged in the activity or are somehow . . . prompted to use or think about the activity. In contrast, at the anticipatory (second) stage . . . a learner independently calls up and uses a newly formed activity-effect relationship appropriate to the situation at hand."(Tzur, Simon, 2004)

Chapter 3

Methods

We based our study on two questionnaires and on a series of interviews, in which we asked the students to work on a problem in our presence, and to think aloud about it.

Their participation was voluntary, anonymous and did not affect the assessment for the course.

We worked with the students attending a multivariable calculus class at MIT, in the Spring of 2006. This class consisted of 35 hours in lecture, 26 in recitation, and each instructors were available for two hours a week for office hours. There were 199 students enrolled in this class.

3.1 Questionnaires

Two questionnaires were given to the students: one at the beginning of the term (Pre-Class Survey, see Appendix A) and one at the end (Post-Class Survey, see Appendix B) . They focused on students' perceptions of their learning, understanding of concepts and development of problem solving abilities. They were asked to analyze the different components of the class (recitation, lecture, personal study, office hours, interaction with other students) and to give opinions on what helped them most. After the first survey, we grouped the components of the course that could contribute to learning as follows:

PI: Passive Interaction (recitation, lecture)

AI: Active Interaction (office hours, students interactions, one-to-one interaction)

Th: Theory (readings)

C: Connections (with other subjects, pictures)

Pr: Practice (problem-sets, personal study/elaboration)

and used these categories for the Post-Class Survey. This was done in order to narrow the range of responses and attempt an analysis of their opinions.

Students' responses to the Pre-Class Survey led us to the conclusion that the students expected to rely mostly on some sort of passive teaching, as the students often regard lecture and recitation. In other words, any kind of input that didn't require apparent effort (except attention) from the students fell into the category named Passive Interaction.

Within Active Interaction, we included all those activities in which students participation was more marked, or in which each student was somehow helped (or forced) to actively discuss or ask questions.

We needed a category to collect the set of written sources that the students could access, and we named it Theory. Internet resources also belong to this category.

Some students also mentioned the usefulness of connections with other subjects or within the course material, and of visual aids such as pictures and graphs, and we grouped them together as Connections, interpreting this as the need for different points of view.

Finally, all those activities regarding personal elaboration, various inputs (like problem sets) and even just repetition and time to absorb what learned were referred to as Practice.

Description of Surveys

The main purpose of these surveys was to determine the students' approach to the course in terms of the content and the methodologies offered and required.

The Post-Class Survey was meant to check how their approach changed during the semester, based on their experience. For this reason the questions asked were the altered versions of the first three in the Pre-Class Survey. Only the open ended question was changed in its structure, in the sense that we asked them to choose among items as we grouped the answers they gave in the Pre-Class Survey (this is explained in Section 3.1).

3.2 Interviews

Together with the questionnaires, we collected data from a series of interviews with 16 students. Their selection was based on their grades on their first test, since we wanted to have sample representation of the whole class. We invited about 80 students distributed over different levels of performance. Sexteen agreed to participate and each of them has been interviewed once.

Among the students who participated, five got A as final grade, six got B, three got C and two got D. This is consistent with the distribution of grades over the whole class.

Seven of these 16 students were males and 9 females. We will refer to each of them using "he."

Choice of Problems

Methods of investigation of students' understanding have been studied by Kannemeyer (2005) in a very systematic way. The aim of his research was to "develop and assess a system of instruments to investigate students' understanding in mathematics based upon the way they answer questions on topics in a first year university course."

We found Kannemeyer's framework particularly appropriate to our purpose, so we decided to follow his recommendation for our interviews. His suggestion is to test students' understanding in terms of linkages - that is, how students link together the knowledge they acquire - by using non-routine questions. In particular, he proposes choosing problems in contexts that are quite unfamiliar to the students.

Here we describe the implementation of Kannemeyer's recommendation. These are the problems that we used:

Problem 1

Given $f : A \to \mathbb{R}$, determine $f(A)$, where

$$A: \quad 1 \leq x^2 + y^2 \leq 4$$

and

$$f(x, y) = x^2 - y^2.$$

Problem 2

Describe the region

$$\begin{cases} x^2 + y^2 + z^2 \leq 9 \\ x^2 + y^2 \leq 3z^2 \end{cases}$$

and compute its volume.

Problem 3

Compute the average distance from the origin on the region described by:

$$\begin{cases} x^2 + y^2 \leq 4 \\ 1 \leq x + y \leq 2. \end{cases}$$

3.3 Analysis of Problems

Following Kannemayer's classification, we decided to focus on three aspects of understanding:

Instrumental: refers to the ability to reason and compute (knowing how).

Relational: refers to the ability to justify the reasoning used (knowing why).

Formal : refers to the use of formal symbolism and notation, which might indicate either misconceptions or clarity.

This framework includes procedural knowledge (instrumental understanding), which is recognized as an important part of understanding, and will be further detailed in order to analyze the data collected from the interviews.

Relational understanding can be further elaborated into three categories:

1. Identification of Goal (**IG**): interpretation of task and/or plan of action.
2. Identification of Keypoints (**IK**): specification of markers signaling progress towards goal.
3. Choice of Procedures (**PC**): linking of procedural knowledge to conceptual knowledge.

This classification articulates the process of making the knowledge available in order to solve the problem. It is worth noticing that the choice of procedures falls into relational understanding since it includes selection of procedures and how legitimate those selections are.

Instrumental understanding concerns mostly mastery of algorithms, and is referred to as Application of Procedures (**PA**).

We are going now to give a detailed analysis of each problem based on the classification just explained.

3.3.1 Problem 1

For this problem the Identification of Goal amounts to understanding what the image of a function is and that it is related to the critical values.[1]

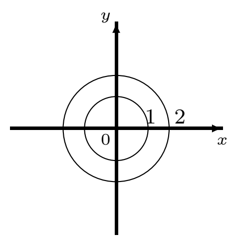
The determination of the image $f(A)$ requires the following steps (IK):

1. Realize that the image in this case is an interval, so the goal becomes to find the minimum and the maximum of the function f over the region A;

[1] The critical values are the values attained by the function at the critical points, which are the points that either are on the boundary or which make the partial derivatives vanish.

This step is actually quite complex since the fact that $f(A)$ is an interval is based on a deeper understanding of the region and some basic knowledge of topology. In fact one has to recognize that A is connected,[2] and to observe that f is continuous. These conditions force the image to be connected as well, and since it is a subset of the real numbers, one can conclude that the image must be an interval.

2. Understand the region, identifying interior and boundary;



the annulus A: $1 \leq x^2 + y^2 \leq 4$
interior : $1 < x^2 + y^2 < 4$
boundary : $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

Figure 3-1: Problem 1

3. Find the critical points in the interior and classify them (if any; in this case there are not);

4. Find the critical points on the boundary;

5. Compare the values that the function takes in these points.

An example of Choice of Procedure (PC) for the search of critical points in the interior of the region could be the computation of the gradient, while, on the boundary one could choose between the method of Lagrange's multipliers or the parametrization of each portion.

[2] A region is called connected if it is not decomposable into two disjoint parts

3.3.2 Problem 2

Here the problem asks explicitly for two goals: the description of the region and the computation of its volume. The keypoints (IK) are:

1. Understand that in order to describe the portion of the space given by

$$\begin{cases} x^2 + y^2 + z^2 \leq 9 \\ x^2 + y^2 \leq 3z^2 \end{cases}$$

   one has to study the region described by each relation, then take the intersection;

2. Split the study of each expression considering first the associated equality then the inequality. Here, for the first one, the equation

$$x^2 + y^2 + z^2 = 9$$

   describes a sphere; then

$$x^2 + y^2 + z^2 \leq 9$$

   selects the inside. For the second expression,

$$x^2 + y^2 = 3z^2$$

   gives a cone, and its interior is found by considering

$$x^2 + y^2 \leq 3z^2;$$

3. Understand their intersection;

4. Compute the volume.

The Choice of Procedure (PC) for the last step concerns the appropriate coordinate system and the integral computing the volume. In fact one could approach this task by a one-dimensional, two-dimensional or three-dimensional integral. In the first case, one sees the region as a solid of rotation (we need to point out though that this method was not expected to be considered since it involves topics included in previous classes), thus the volume is computed by

$$2\int_0^{\frac{3\sqrt{3}}{2}} 2\pi x(\sqrt{9-x^2} - \frac{\sqrt{3}}{3}x)dx$$

In the second case the region can be seen as bounded by graphs of functions, namely $f_1(x,y) = \sqrt{9-x^2-y^2}$ and $f_2(x,y) = \sqrt{\frac{x^2+y^2}{3}}$, thus computing the volume as

$$2\iint_{x^2+y^2\leq 9} (f_1 - f_2)dx\,dy.$$

Finally, in the third case one can compute the volume of a solid

$$\iiint_R dV = 2\int_0^{2\pi}\int_0^{\pi/3}\int_0^3 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

where R is the region inside the solid.

3.3.3 Problem 3

For this problem one has to remember that the average distance is computed by

$$\frac{1}{\text{Area}}\iint_R f\,dx\,dy.$$

where f is the distance, i.e. $f(x,y) = \sqrt{x^2+y^2}$, and the Area (of R) can be computed either by geometric argument or by the integral $\iint_R dx\,dy$.

So the main difficulty is to understand the region R (this would be both IK and PC), given by

$$\begin{cases} x^2+y^2 \leq 4 \\ 1 \leq x+y \leq 2, \end{cases}$$

and that amounts to the following steps:

1. Recognize the necessity to study the two relations one at a time;

2. Understand the region described by the first expression, by considering first the associated equality and then the inequality;

3. Understand the region described by the second expression in the same way;

4. Understand their intersection.

Finally, one of the challenges of this problem is to be able to describe this region (PA), since the numerical choices of the relations lead to non standard values for the angles involved, so the application of procedures needs some careful work.

3.4 Comments

We discussed the choice of the problems with the course professor, in order to establish a reference with the material as it has been taught and to the learning objectives of the class. We present here his analysis of the problems with the difficulties that he expected the students would have.

3.4.1 Problem 1

The choice of notation was not familiar to the students. In particular, the statement of the problem relies too much on set theory, which the students haven't been exposed to at this point of their education. In particular, he referred to the notations

$$f : A \to \mathbb{R} \quad \text{and} \quad f(A).$$

Moreover the step necessary to relate the idea of range to the task of finding max/min is too distracting. A less intimidating version would be:

Find the max/min of the function

$$f(x, y) = x^2 - y^2$$

defined on the annulus

$$1 \leq x^2 + y^2 \leq 4.$$

Even this version would be a nontrivial problem since the understanding of subsets of the plane and their description are still challenging for many students. For instance, they would find it easier to have the region presented as bounded by the two circles.

For these reasons he would expect 15% of the students to be able to describe the annulus at the beginning of the semester and maybe 60% by the end of it. Those percentages would increase to 30% − 80% for the same problem *in isolation* (this means when the problem is considered in itself, out of the context that could be a source of distraction).

3.4.2 Problem 2

Here the main points are:

1. Understand the region;

2. Compute the volume.

The professor stated that the second part is quite standard once the student is able to describe the region, so the focus should be on the difficulties present in the first task.

There are two expressions describing two portions of the plane, so the first step is to understand each of them. In order to do that, one has to split the process by first considering the associated equality and then examining the inequality. Then one has to take the intersection of these regions, as prescribed with the notation

$$\begin{cases} x^2 + y^2 \leq 4 \\ 1 \leq x + y \leq 2. \end{cases} \tag{3.1}$$

This part relies again on set theory and might look unfamiliar.

For the analysis of each expression in (3.1), the professor would expect most students being able to split the process, but less than 40% of them to be able to understand the region even when solving this problem in isolation (that is when trying to determine a region expressed by an inequality), even assuming they were given this task at the end of the semester.

3.4.3 Problem 3

This problem has difficulties very similar to those in Problem 2 regarding the understanding of the region, which is indeed expressed as the intersection of two subsets of the plane. An additional distracting factor could be the choice of numerical values, but the goal of the problem is to compute

$$\frac{1}{A}\iint (\text{dist})\, dx\, dy.$$

and most students should be able to set it up.

Chapter 4

Findings

4.1 Surveys

In order to analyze the data from the questionnaires, we need to specify the amount of information we received and the way we sorted it. The number of the students who responded to the Pre-Class Survey and the Post-Class Survey is respectively 146 and 88, out of the 199 students enrolled in the class. For the closed ended questions, we report the percentage of the occurrence of each answer. For the open ended questions, where we asked to provide (or select, in the case of the Post-Class Survey) up to two choices, the percentage refers to the occurrence of that choice among what the students listed: some of them gave two answers, some only one.

Based on the classification explained above (Section 3.1), we could determine what the students thought was most helpful in understanding concepts and in solving problems. For instance, for the understanding of concepts, 44% of what the students listed in the the Pre-Class Survey reported that methods in the category PI (Passive Interaction) would be the most helpful, while the percentage in the Post-Class Survey is 37% for the same category (see Table 4.1).

We were also able to order the different components of the class based on the importance that the students gave to each of them for their learning (Table 4.3): in the Pre-Class Survey Lectures got the highest score, followed by Recitations. Then, in order, Homework, Individual Study, Interaction with Other Students, TA's Office

	Pre-Class	Post-Class
PI	44%	37%
AI	14%	19%
Th	14%	14%
C	6%	2%
Pr	22%	28%

Table 4.1: What students find helpful in understanding concepts

	Pre-Class	Post-Class
PI	45%	31%
AI	19%	22%
Th	10%	17%
C	1%	1%
Pr	25%	29%

Table 4.2: What students find helpful in understanding how to solve problems

	Pre-Class	Post-Class
Lectures	1	3
Recitation	2	2
Homework	3	4
Individual Study	4	1
Interaction with Other Students	5	5
TA's Office Hours	6	6
Professor's Office Hours	7	8
Tutoring Room	8	7

Table 4.3: Importance of the components of the class.

Hours, Professor's Office Hours and Tutoring Room.

The last four components stay in their positions at the end of the semester, but there is a significant change in the order among the first four: students list as the most important source for learning Individual Study, followed by Recitations, Lectures and Homework.

Other data obtained from the surveys were the number of hours expected to be spent (and then actually spent) per week on each component of the class (Table 4.4) and the percentage of attendance (Table 4.5), again as anticipated (Pre-class Survey) and as actually done (Post-class Survey).

	Pre-class	Post-class
Homework	6.5	5.6
Individual Study	2.6	2.6
Interactions with Other Students	3	2.2
Professor's Office Hours	0.3	0.2
TA's Office Hours	0.6	0.6
Tutoring Room	0.8	0.7

Table 4.4: Hours per week

	Pre-class	Post-class
Lectures	96%	92%
Recitation	93%	87%
Professor's Office Hours	12%	4%
TA's Office Hours	21%	19%
Tutoring Room	15%	14%

Table 4.5: Percentage of attendance

4.2 Interviews

4.2.1 Problem 1

Ten students tried to solve this problem, and all of them were confused by the statement because, as explained above, it used unfamiliar terminology. As the professor confirmed (cf. 3.4.1), they were not used to writing $\mathbb{R}$ for the set of real numbers or $f : A \to \mathbb{R}$ for a function f defined on a domain A and taking real values, or considering the image of such a function and denoting it by $f(A)$.

So I went through all these notions and gave some examples of images of functions.

All but two could draw the picture without problems. The main difficulties they encountered at this point were the meaning of the equation of the circle, namely that it is a constraint on the distance, and that it is a property which is true or false for each point of the plane.

Four students followed the analysis of the domain by an algebraic study of the function, trying to get information about the image. One used a geometric approach. Two students ignored the search of critical points in the interior of the region, assuming they were to be found on the boundary. Seven decided to express the boundary in polar coordinates. One of them extended this procedure to the interior and used it to find the interval and an explanation of the fact that it was an interval.

In more detail, this student used polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

to rewrite the function $f(x, y) = x^2 - y^2$ as

$$f(r, \theta) = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta,$$

and since the region A is given by

$$\begin{cases} 1 \le r \le 2 \\ 0 \le \theta < 2\pi \end{cases}$$

he deduced

$$-4 \le f(r, \theta) \le 4.$$

He then argued that the functions r^2 and $\cos 2\theta$ are continuous and the region is connected, hence the image has to be connected too.

Another student (the one who used the geometric argument) was able to prove that the image must be an interval. After showing that -4 and 4 bound the image (algebraically), he provided a path through the region on which the function takes all values ranging from -4 to 4. In detail, he observed that on the segment from $(0, 2)$ to $(0, 1)$ the function is given by

$$f(x, y) = f(0, y) = -y^2, \quad y \in [1, 2]$$

hence its image is $[-4, -1]$; on the segment from $(0, 1)$ to $(1, 1)$ the function becomes

$$f(x, 1) = x^2 - 1, \quad x \in [0, 1]$$

and takes all values in $[-1, 0]$; on the segment from $(1, 1)$ to $(1, 0)$ it is

$$f(1, y) = 1 - y^2, \quad y \in [0, 1]$$

so its image is $[0, 1]$ and finally on the segment from $(1, 0)$ to $(2, 0)$

$$f(x, y) = f(x, 0) = x^2, \quad x \in [1, 2]$$

ranges from 1 to 4. What this student did was reduce the function to more familiar ones, so he could easily see their behaviour and work with a new concept like the image.

4.2.2 Problem 2

Four students attempted the solution of this problem, three of them tried to understand the region described by each expression in

$$\begin{cases} x^2 + y^2 + z^2 \leq 9 \\ x^2 + y^2 \leq 3z^2 \end{cases} \tag{4.1}$$

and had no problem with the sphere. The cone, however, gave them more to think about. Two of them weren't able to assign surfaces or regions to equations or inequalities, so they needed help figuring out the solid they were asked to compute the volume of.

The fourth student didn't have the same approach and tried to determine the region described by (4.1) by manipulating algebraically the two inequalities, plugging one into the other, but he couldn't make use of the information he got. Then I suggested he look at the equalities first, but still he was unable to figure out by himself the subset defined by what he found.

All four students attempted the second part, namely the computation of the volume. Three could set up the integral: one of them decided to look at the region as bounded by graphs of functions; the other two preferred the three-dimensional method and wrote $\iiint dV$ as iterated integrals by switching to spherical coordinates. However, one of them couldn't determine bounds for the variable ϕ (the angle with the positive z-axis).

4.2.3 Problem 3

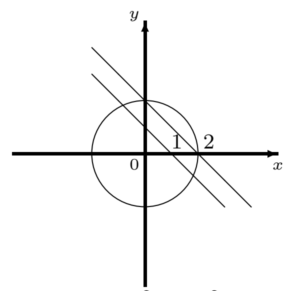
This problem was attempted by only two students. Neither remembered how to compute the average value of a function, so I provided them with the formula

$$\bar{f} = \frac{1}{\text{Area}} \iint_R f \, dA$$

asking them what was the function f inside the integral. They both said it was the distance, but couldn't write an expression for it, so I had to help them remember how to compute it.

The computation of the area was discussed with only one of them; this student noticed some symmetry in the domain, but didn't really know how to make use of it. In particular, he was confused about the portion below the x-axis, thinking that the integral $\iint dA$ over that part would give a negative value.

Both of the students struggled with the description of the region. I suggested that they look at the relations one at a time and to consider first the equalities associated with them. One student needed help recognizing the equation of a circle, but then both were able to draw the two lines and the circle involved in the picture (see Fig.4-1).



Circle : $x^2 + y^2 = 4$
lines : $x + y = 1$, $x + y = 2$

Figure 4-1: Problem 3

The student who struggled with the circle guessed at first that it was a parabola, intercepting the axes at the points $(0, 2)$, $(2, 0)$ and $(0, -2)$. It is of interest to note that he was able to point out the relevant numerical values of the equation, and this allowed him to draw the right circle once assured that it was a circle. They were both able to guess the portion of the plane cut off by these three curves, but weren't able to explain precisely why.

Finally, only one student tried to set up bounds for the region, but, even though the idea behind the computation was correct (fix a value of x in the range present in the region, then let y vary inside the interval cut by the region), he wasn't able to carry on the procedure (offering an example of good PC but weak PA). In particular, he was confused by having to change the choice of the curves bounding the region (namely sometimes a line and a portion of the circle, sometimes the two lines) and the complexity of the computations required (indeed not standard).

Chapter 5

Discussion

5.1 Questionnaires

The data collected from the surveys are particularly interesting when we compare the responses that the students gave at the beginning of the semester with what they said at the end of it.

If we look at the ranking of the components of the class in terms of the importance the students gave each for their learning, we can note that Lectures, Recitations, Homework and Individual Study appear in the first four spots in both surveys. In the Pre-Class Survey they were listed in this order, but the Post-Class Survey showed that the students indicate Individual Study as the most useful thing, much more than what they expected. Moreover, they found Recitations were more helpful than Lectures.

Another result worth noting comes from the open ended questions, where the students were asked to list two things that help them most in understanding concepts and the two things that help them most in problem solving. In this case, it is not the comparison between Pre-Class and Post-Class Survey that attracts our attention. Rather, it is impressive that almost half of the students indicated, in both cases, an activity which we have classified as Passive Interaction. This was not the word they used - they often listed recitation and lecture - but they referred to these things as to activities in which they were given information and sets of rules and instructions.

Practice come next, but the numbers are halved.

5.2 Interviews

We believe that visual thinking may help in deciding the appropriateness of a process, in the choice of procedure and in the identification of keypoints. Visual thinking is not just the ability to recognize the graph of a function from the equation defining it. Rather it refers to the approach of solving a problem and involves skills of multiple representation. As Habre (1999) explains: "Visualization is the process of using geometry to illustrate mathematical concepts." He continues, "proficiency in visual thinking requires from the students an ability to understand that algebra and geometry are alternatives for expressing mathematical ideas, and that a graph may contain information needed for a better overall picture of a problem." (Habre, 1999)

Let's look at some examples.

In Problem 2, an option was to determine the surface described by

$$x^2 + y^2 = 3x^2$$

by taking slices of it. This process usually consists of the following steps:

a) intersection with each of the coordinate planes:

- with the xy-plane:

$$\begin{cases} x^2 + y^2 = 3z^2 \\ z = 0 \end{cases}$$

- with the xz-plane:

$$\begin{cases} x^2 + y^2 = 3z^2 \\ y = 0 \end{cases}$$

- with the yz-plane:

$$\begin{cases} x^2 + y^2 = 3z^2 \\ x = 0 \end{cases}$$

and/or

b) intersection with a family of parallel planes:

- parallel to the xy-plane:

$$\begin{cases} x^2 + y^2 = 3z^2 \\ z = c \quad c \text{ constant} \end{cases}$$

- parallel to the xz-plane:

$$\begin{cases} x^2 + y^2 = 3z^2 \\ y = c \quad c \text{ constant} \end{cases}$$

- parallel to the yz-plane:

$$\begin{cases} x^2 + y^2 = 3z^2 \\ x = c \quad c \text{ constant} \end{cases}$$

This method relies heavily on a good knowledge of analytic geometry. In fact one has to be fluent with both the concept of intersection (what it means to intersect the surface with a plane) and what that implies algebraically (namely that you can substitute, say $z = 0$, in the equation of the surface) and geometrically (realize that you get the horizontal section through the origin).

In Problem 1, a visual approach could be based on the knowledge that

$$z = x^2 - y^2$$

is a hyperbolic paraboloid, or at least that the origin is a saddle point for the function

$$f(x, y) = x^2 - y^2.$$

This would give an insight about the graph of the function over the annulus $1 \leq x^2 + y^2 \leq 4$, so one could guess where the minima and the maxima were, or maybe just visualize that they were on the boundary.

But visual thinking could help in a much deeper way, even if one doesn't remember the graph of that function. In fact, the step from the conception of the image of a function as a set of points and its visualization as a subset of the real numbers is what allows the connection with the search of critical points.

An analysis of the interviews conducted in this study reveals that only a small number of students demonstrated a confidence in visual thinking, showing that this is not an approach they are familiar with. It seems more common for them to rely solely on algebraic manipulation, usually revealing a lack of knowledge of the corresponding geometric meaning of their computations.

Over all, we could observe some difficulty with multiple representation in general, with the striking exception of a remarkable insistence on searching for symmetries. Indeed, the students were always looking for them, and this was certainly due to a specific curricular choice made by the professor.

However, they were often unable to carry on with the procedure or to explain what they were doing and why. It was a common mistake, for instance, to forget the multiplicative factor after reducing to a portion of the region.

This weakness in procedural knowledge was not limited to this context. Almost everybody made computational mistakes, but some of them might have felt pressure due to the unfamiliarity of the problem or even to the uncommon circumstance of explaining how they were solving a problem to someone else.

Some habits of thought were common to most of the students. First of all, they seemed very reluctant to use the material just learned, usually trying to use knowledge from previous classes, even high school, rather than looking for tools introduced in the course they were attending. Only after being prompted to go through the list of topics presented in class, could they point out (and sometimes explain and/or use) the one they needed in order to solve the problem. It is remarkable though that occasionally some of the methods they came out with were not standard and quite efficient. It seems that they don't appreciate the power of the tools they're learning.

Most of the students were reluctant to make deductions. Their approach mostly relied on memory and often consisted of *arranging* disconnected pieces.

Finally, we could observe that the students usually don't have a strategic approach: with very few exceptions, they set up no frame - that is they don't have a plan of action. This was particularly evident in the difficulty they had in identifying the goal of their problem and the key points that would lead them to the solution (IG and IK in our classification, see Section 3.1).

Chapter 6

Conclusions, limitations and suggestions for further research

Our research has shown many weaknesses of the students. However, we have to acknowledge that the choice of the problems proposed was particularly challenging. In part this was on purpose, as an attempt to realize the conditions described by Kennemayer, as introduced in Section 3.2. It is also true though, that the unfamiliarity of the type of question and of the notation produced a situation in which students tended to rely too much on terminology. This was not necessary, and it resulted in an inefficient research tool.

Moreover, the purpose of our study changed a bit along the way, mostly in order to follow the lead of the data. Initially, we were trying to determine in which parts of the class different abilities are developed. We ended up analyzing whether some abilities were developed.

We think, though, that the results are of great interest.

From our discussions, it appears that students look for general lists of steps to do to solve the problem, without really trying to understand why. The result is that their procedures and approaches are not very flexible.

Let's examine the difference between the expert and the novice, in any field of knowledge:

> Experts are more likely than novices to recognize meaningful patterns of information ([8]).

and

> Experts' abilities to reason and solve problems depend on well-organized knowledge that affects what they notice and how they represent problems. Experts are not simply *general problem solvers* who have learned a set of strategies ([8]).

These strategies are what the students aimed for, instead of the transfer of expertise. As Ramsden says, "What students learn is associated with how they go about learning it. Approaches are related to how much satisfaction students experience in their learning."Students who did not get the point [of the text] were not looking for it".(Ramsden, 2004)

Of course, if this is the attitude towards learning that the students have, it might be also because the instructors don't put enough attention to the aspect of transfer.

It would be of great interest to investigate what the instructors see as their educational purpose, and whether the students get what they aim to transfer. Also, it would be useful to determine where this kind of approach that the students have has originated, and how can it be corrected.

Appendix A

Pre-Class Survey

The following two pages show the questionnaire given to the students at the beginning of the semester.

QUESTIONNAIRE
18.02, Multivariable Calculus
February 2006

Statement of Your Rights

Thank you for your help on this project. Please note that:

- Participation is voluntary.
- You may decline to answer any question, and can withdraw from the study at any time. If this happens, this questionnaire will be destroyed.
- Your responses will be anonymous and will not affect your grade in any way.
- No attempt will be made to link any responses to individuals.

1. How important do you expect that each of the following will be to your learning the material covered in 18.02?	Not At All Important				Extremely Important
a. Lectures	1	2	3	4	5
b. Recitations	1	2	3	4	5
c. Homework	1	2	3	4	5
d. Individual study	1	2	3	4	5
e. Interactions with other students	1	2	3	4	5
f. Professor's office hours	1	2	3	4	5
g. TA's office hours	1	2	3	4	5
h. Tutoring room	1	2	3	4	5
i. Other (please specify): ______	1	2	3	4	5

Please list the two things that help you most understand concepts:

______

______

Please list the two things that help you most understand how to solve problems:

______

______

Figure A-1: Pre-Class Survey, page 1

2. On average, how many hours per week do you expect to spend on each of the following for 18.02?

	Hours per Week
a. Homework	____
b. Individual study	____
c. Interactions with other students	____
d. Professor's office hours	____
e. TA's office hours	____
f. Tutoring room	____
g. Other (please specify): ______________________________	____

3. What percentage of each of the following do you expect to attend in 18.02?

	Percent
a. Lectures	____
b. Recitations	____
c. Professor's office hours	____
d. TA's office hours	____
e. Tutoring room	____

4. ***Please complete this question if you took 18.01 last semester …***

In 18.01 you were asked to complete reading assignments in preparation for lectures.

- ☐ I *always* did the reading
- ☐ I did the reading *except when time constraints prevent it*
- ☐ I did the reading *on subjects that I found challenging*
- ☐ I did the reading *when I was lost*
- ☐ I did the reading *on subjects that I found interesting*
- ☐ I *seldom* did the reading
- ☐ I *never* did the reading
- ☐ Other(s), please specify: ______________________________

5. How useful did you find the pre-lecture readings in 18.01?

Not at all Useful				Extremely Useful
1	2	3	4	5

Figure A-2: Pre-Class Survey, page 2

Appendix B

Post-Class Survey

The following two pages show the questionnaire given to the students at the end of the semester.

QUESTIONNAIRE
18.02, Multivariable Calculus
May 2006

Statement of Your Rights

Thank you for your help on this project. Please note that:

- Participation is voluntary.
- You may decline to answer any question, and can withdraw from the study at any time. If this happens, this questionnaire will be destroyed.
- Your responses will be anonymous and will not affect your grade in any way.
- No attempt will be made to link any responses to individuals.

1. How important each of the following was to your learning the material covered in 18.02?	Not At All Important				Extremely Important
a. Lectures	1	2	3	4	5
b. Recitations	1	2	3	4	5
c. Homework	1	2	3	4	5
d. Individual study	1	2	3	4	5
e. Interactions with other students	1	2	3	4	5
f. Professor's office hours	1	2	3	4	5
g. TA's office hours	1	2	3	4	5
h. Tutoring room	1	2	3	4	5
i. Other (please specify): ________________	1	2	3	4	5

Please check the two things that help you most understand concepts:

- ☐ recitation, lecture
- ☐ office hours, interaction with students, 1-1 tutor
- ☐ readings, course notes, book
- ☐ connections with other topics/courses, pictures
- ☐ homework, personal study
- ☐ Other(s), please specify:

Please check the two things that help you most understand how to solve problems:

- ☐ recitation, lecture
- ☐ office hours, interaction with students, 1-1 tutor
- ☐ readings, course notes, book
- ☐ connections with other topics/courses, pictures
- ☐ homework, personal study
- ☐ Other(s), please specify:

Figure B-1: Post-Class Survey, page 1

2. On average, how many hours per week did you spend on each of the following for 18.02?

	Hours per Week
a. Homework	____
b. Individual study	____
c. Interactions with other students	____
d. Professor's office hours	____
e. TA's office hours	____
f. Tutoring room	____
g. Other (please specify): ____________________	____

3. What percentage of each of the following did you attend in 18.02?

	Percent
a. Lectures	____
b. Recitations	____
c. Professor's office hours	____
d. TA's office hours	____
e. Tutoring room	____

Figure B-2: Post-Class Survey, page 2

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