Noncommutative ring spectra

by

Vigleik Angeltveit

Cand. Mag., University of Oslo (1999)
Cand. Scient., University of Oslo (2002)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2006

Author . . . .
Department of Mathematics
May 8, 2006

Certified by . . . .
Haynes R. Miller
Professor of Mathematics
Thesis Supervisor

Accepted by . . . .
Pavel Etingof
Chairman, Department Committee on Graduate Students

Noncommutative ring spectra

by

Vigleik Angeltveit

Submitted to the Department of Mathematics
on May 8, 2006, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

Let A be an A_∞ ring spectrum. We give an explicit construction of topological Hochschild homology and cohomology of A using the Stasheff associahedra and another family of polyhedra called cyclohedra. Using this construction we can then study how $THH(A)$ varies over the moduli space of A_∞ structures on A, a problem which seems largely intractible using strictly associative replacements of A. We study how topological Hochschild cohomology of any 2-periodic Morava K-theory varies over the moduli space of A_∞ structures and show that in the generic case, when a certain matrix describing the multiplication is invertible, the result is the corresponding Morava E-theory. If this matrix is not invertible, the result is some extension of Morava E-theory, and exactly which extension we get depends on the A_∞ structure.

To make sense of our constructions, we first set up a general framework for enriching a subcategory of the category of noncommutative sets over a category $\mathcal{C}$ using products of the objects of a non-Σ operad P in $\mathcal{C}$. By viewing the simplicial category as a subcategory of the category of noncommutative sets in two different ways, we obtain two generalizations of simplicial objects. For the operad given by the Stasheff associahedra we obtain a model for the 2-sided bar construction in the first case and the cyclic bar and cobar construction in the second case. Using either the associahedra or the cyclohedra in place of the geometric simplices we can define the geometric realization of these objects.

Thesis Supervisor: Haynes R. Miller
Title: Professor of Mathematics

Acknowledgments

I have a lot of people to thank for helping me complete this PhD thesis. First of all, I want to thank my advisor, professor Haynes Miller. He has been a great advisor, for several reasons. First of all, he has always taken the time to meet with me. Instead of trying to explain everything to me, he has allowed me to go exploring on my own. He has always found exactly the right questions to keep me on track, and to inspire me to keep working. And he has given me a lot of feedback on the things I have written down, including more than one draft of this document.

Next I want to thank the amazing group of algebraic topology students at MIT. I have been lucky to be at MIT at the same time as so many exceptional people. Mike Hill, who is always eager to talk about math and who always has a new spectral sequence to show me (even though he claims that all spectral sequences are really the same). He has also become a good friend, both mathematically and socially. Teena Gerhardt, for being another good friend and for voting for my suggestion to have K of K as our juvitop topic, several semesters in a row. Maybe it will happen this summer. And for amazing food for every occasion. You are spoiling us all.

I want to thank André Henriques, for realizing that the cyclohedra was the family of polyhedra I was looking for. Michael Ching, for discussions about operads and trees. Nora Ganter, for encouraging me to give talks in our seminars even before I arrived at MIT. Tyler Lawson, for answering a number of my questions. I also want to mention Chris Douglas, Francesca Deserti, Daniel Vera, John Francis and Mark Behrens, plus our honorary topologist Bridget Tenner.

I also want to thank these people for organizing several seminars every semester, for giving talks and for encouraging me to give talks. The best way to learn a subject is to agree to give a talk about it, so I have learned a lot from these seminars. I will miss our frequent meetings, and I will miss all of you.

I would also like to thank a number of people for reading various drafts of my thesis and providing valuable comments. This includes Andrej Lazarev, Andrew Baker, James Stasheff, Clemens Berger and several others. Also, Charles Rezk explained to me what a Reedy category is and directed me to Phil Hirschhorn who was also very helpful. Thanks also to Lars Hesselholt for helping me with an algebraic question, and to John Rognes for helping me use the Bökstedt spectral sequence to extract useful information.

I have discussed aspects of my thesis with a number of other people. If I were to try to mention them all here, I would surely forget someone. So I will just say a big thank you for letting me draw on the combined resources of the algebraic topology

Contents

Chapter 1

Introduction

One of the goals of this paper is to calculate topological Hochschild homology and cohomology of ring spectra such as Morava K-theory, and to do this we need to think seriously about noncommutative ring spectra. The Morava K-theory spectra are not even homotopy commutative at $p = 2$, and at odd primes there is something noncommutative about the A_p structure. Moreover, if we make Morava K-theory 2-periodic it has many different homotopy classes of multiplications, most of which are noncommutative, and all of which can be extended to A_∞ structures.

Let us write $THH(A)$ for either topological Hochschild homology or cohomology of A, while using $THH^S(A)$ for topological Hochschild homology and $THH_S(A)$ for topological Hochschild cohomology. $THH(A)$ is defined for A_∞ ring spectra, and depends on which A_∞ structure we endow A with. Usually one would first replace A with a weakly equivalent strictly associative ring spectrum $\tilde{A}$ and define topological Hochschild homology or cohomology in terms of $\tilde{A}$. This makes it hard to see how THH depends on the A_∞ structure, as the process of replacing an A_∞ ring spectrum by a strictly associative one is largely intractable. Instead we build the maps making up the A_∞ structure into the definition of $THH(A)$, using the Stasheff associahedra and another family of polyhedra called cyclohedra. The cyclohedra were first introduced by Bott and Taubes in [16], and later given the name cyclohedra by Stasheff [62]. This makes it easier to see how $THH(A)$ depends on the A_∞ structure and to study how changing the A_∞ structure might affect the result. In other words, it enables us to study how $THH(A)$ varies over the moduli space of A_∞ structures on A. Given just an A_n structure on A we can also define a kind of *partial THH*, which coincides with $sk_{n-1}THH^S(A)$ or $Tot^{n-1}THH_S(A)$ if the A_n structure is the restriction of an A_∞ structure on A.

To proceed we need a good understanding of the set, or space, of A_∞ structures on a spectrum A. The original reference for A_∞ obstruction theory for spectra as set up by Robinson in [55] implicitly assumes that the multiplication we start with is at least homotopy commutative, and he only proves that Morava K-theory is A_∞ at odd primes. We only need to modify his obstruction theory slightly to make it work for noncommutative ring specta, and then his proof that Morava K-theory is A_∞ works the same way at all primes. It is sometimes easier to assume that A is an R-module with a homotopy associative multiplication for some commutative S-algebra

R, in which case we study the space of A_∞ R-algebra structures on A. In fact, this setup allows us to prove that a large class of spectra can be given A_∞ structures. If R is even, meaning that $R_* = \pi_* R$ is concentrated in even degrees, and $A = R/I$ for a regular ideal $I \subset R_*$ then it follows almost immediately that the space of A_∞ structures on A is nonempty.

We could also set up the A_∞ obstruction theory using derived functors of derivations, this is the A_∞ version of the obstruction theory set up by Goerss-Hopkins as in [30]. With our setup, the obstructions to extending a homotopy associative multiplication on A to an A_∞ multiplication lie in what deserves to be called E_2 term of a spectral sequence converging to $\pi_* THH_R(A)$. The two approaches are related by the cofiber sequence

$$THH_R(A; M) \longrightarrow M \longrightarrow Der_R(A; M) \tag{1.0.1}$$

of spectra. Here M is any A-bimodule, and we recover the obstruction theory by setting $M = A$.

We prefer the obstruction theory as formulated in terms of associahedra and topological Hochschild cohomology, both because it is more geometric and because a closely related obstruction theory will help us calculate $\pi_* THH(A)$. To describe this obstruction theory we need to introduce another idea. We generalize the notion of a trace and formulate the dual notion of a cotrace in such a way that topological Hochschild homology corepresents traces and topological Hochschild cohomology represents cotraces. These traces and cotraces are defined in terms of the cyclohedra by viewing the collection of cyclohedra as a right module over the associahedra operad. Because this right module is also cellular, we can use obstruction theory to determine when such traces and cotraces exist. In interesting cases the obstructions to the existence of such traces and cotraces are nontrivial, and correspond to extensions in certain spectral sequences converging to $\pi_* THH(A)$.

Let us say more precisely what these spectral sequences look like when R is even and $A = R/I$ for a finitely generated regular ideal $I = (x_1, \ldots, x_n)$. Then there are spectral sequences

$$E_2^{*,*} = A_*[q_1, \ldots, q_n] \Longrightarrow \pi_* THH_R(A) \tag{1.0.2}$$

and

$$E_{*,*}^2 = \Gamma_{A_*}(\bar{q}_1, \ldots, \bar{q}_n) \Longrightarrow \pi_* THH^R(A). \tag{1.0.3}$$

Here Γ denotes a divided power algebra, though topological Hochschild homology is in general only a spectrum without any multiplicative structure, so we have to interpret the second spectral sequence additively only. By a version of the Deligne conjecture, which is now a theorem with several independent proofs, see e.g. [48], topological Hochschild cohomology admits an action by some operad equivalent to the little squares operad. In particular, $THH_R(A)$ has a homotopy commutative multiplication, and it follows that the first spectral sequence is a spectral sequence of commutative R_* algebras. There is also an action of the first spectral sequence on the second coming from the natural map $THH_R(A) \wedge_R THH^R(A) \longrightarrow THH^R(A)$, and one can check that q_i sends $\gamma_k(\bar{q}_i)$ to $\gamma_{k-1}(\bar{q}_i)$.

These spectral sequences collapse because everything is in even total degree, but

we can ask if there are additive extensions. By that we mean that an element in R_* which acts trivially on $E_2 = E_\infty$ or $E^2 = E^\infty$ acts nontrivially on the abutment. As it turns out, the trace and cotrace obstructions defined in terms of cyclohedra identify these extensions, so we have reduced the calculation of $THH^R(A)$ and $THH_R(A)$ to obstruction theory.

Now let K be Morava K-theory, either the $2(p^n - 1)$-periodic version $K(n)$ or the 2-periodic version K_n. Let E be the corresponding $K(n)$-localized Johnson-Wilson spectrum $\widehat{E(n)}$ in the first case and the corresponding Morava E-theory E_n in the second case. Then we can write $K = E/I$ as above, and use the trace and cotrace obstructions to get information about $THH^E(K)$ and $THH_E(K)$. But we are really interested in $THH^S(K)$ and $THH_S(K)$. It turns out that $THH(K)$ is independent of the ground ring, in the sense that the natural maps $THH^S(K) \longrightarrow THH^E(K)$ and $THH_E(K) \longrightarrow THH_S(K)$ are weak equivalences. Thus we are actually calculating $THH^S(K)$ and $THH_S(K)$.

In view of the equivalence $THH_E(K) \xrightarrow{\simeq} THH_S(K)$ for topological Hochschild cohomology of Morava K-theory, we get a natural map $E \longrightarrow THH_S(K)$. If we think of $THH_S(K)$ as the homotopy center of K, the fact that it receives a map from E just says that the map $E \longrightarrow K$ is central. In any case, we can ask if this map might be an equivalence, or something close to an equivalence. An affirmative answer to this question was obtained by Baker and Lazarev in [6] for $K(1) = KU/2$ at the prime 2, where they showed that in fact $THH_S(KU/2) \simeq KU_2^\wedge$.

If we stick to the 2-periodic case, then by using our obstruction theory together with the fact that there are lots of noncommutative multiplications on K_n, we show that in fact this happens generically. By this we mean that if a certain matrix which expresses the noncommutativity of the multiplication is invertible, then $THH_S(K_n) \simeq E_n$. We can obtain this result either by using our obstruction theory, or by comparing $K_n \wedge_{E_n} K_n^{op}$ with $F_{E_n}(K_n, K_n)$ and using a double centralizer theorem. The latter method was used by Baker and Lazarev to prove their result for $KU/2$. However, this method does not give any information about higher filtration extensions. In fact, this allows us to see exactly the filtration one extensions in the above spectral sequence converging to $\pi_* THH_R(A)$. This is because the A_n structure controls the filtration $n - 1$ extensions, and the comparison map $A \wedge_R A^{op} \longrightarrow F_R(A, A)$ only depends on the homotopy class of the multiplication, in other words the A_2 structure.

If the multiplication on K_n is "more commutative", the map $E_n \longrightarrow THH_S(K_n)$ is not an equivalence. The map $\pi_* E_1 \longrightarrow \pi_* THH_S(K_1)$ is always a finite extension, and we conjecture that $\pi_* E_n \longrightarrow \pi_* THH_S(K_n)$ is always a finite extension. If we instead consider the $2(p^n - 1)$-periodic Morava K-theory $K(n)$ there is less room to make noncommutative multiplications, because of the sparsity of the homotopy groups of $K(n)$. In fact, $THH_S(K(1))$ is independent of the A_∞ structure, and we conjecture that $THH_S(K(n))$ is also independent of the A_∞ structure. The degree of the extension $\pi_* \widehat{E(1)} \longrightarrow \pi_* THH_S(K(1))$ is $p - 1$, while the degree of the extension $\pi_* E_1 \longrightarrow \pi_* THH_S(K_1)$ is between 1 and $p - 1$.

To make our definition of THH, and to put ourselves in a situation where we can do calculations, we first set up a categorical framework that we hope will be of

independent interest. We can unify several constructions by considering the following setup. Let $\mathbf{\Delta\Sigma}$ denote the category of noncommutative sets [40, section 6.1] and let $\mathcal{A}$ be a subcategory of $\mathbf{\Delta\Sigma}$. If P is a non-Σ operad in some symmetric monoidal category $\mathcal{C}$ we can then define a new category $\mathcal{A}_P$ enriched over $\mathcal{C}$ using the objects in P. The simplicial category $\mathbf{\Delta}^{op}$ is a subcategory of $\mathbf{\Delta\Sigma}$ in two different ways, giving two generalizations of simplicial sets. Given a P-algebra A, we get a generalization of the 2-sided bar construction in the first case and the cyclic bar construction in the second case. If P is the operad given by the Stasheff associahedra we can define geometric realization using the associahedra instead of the standard geometric n-simplices in the first case, and this recovers Stasheff's construction of BA for an A_∞ H-space [61]. In the second case we use the cyclohedra instead of the n-simplices, and as far as we know this gives the first explicit construction of the cyclic bar construction on an A_∞ algebra.

We can think of both the 2-sided bar construction and the cyclic bar construction on an A_∞ algebra as simplicial object where the simplicial identities are allowed to hold only up to homotopy and the homotopies are required to satisfy certain coherence relations. The associahedra operad is exactly what is needed to organize the homotopies for the 2-sided bar construction, while we also need the cyclohedra for the cyclic bar construction. This extra piece of geometry is exactly what we need to see how the constructions depend on the A_∞ structure in an explicit enough way to allow us to do calculations.

This paper is divided into two parts. We set up the categorical framework we need in the first part and then apply our theory to spectra in the second. Each part has its own introduction, giving a summary of that part.

Part I

Operads and enriched categories

The construction of THH we sketched in the introduction is most naturally described in terms of certain categorical constructions, which might be of independent interest. In chapter 2 we first recall the definition of an operad in a symmetric monoidal category $\mathcal{C}$ and some objects and structures related to operads and non-Σ operads, including right modules.

In chapter 3 we consider a subcategory $\mathcal{A}$ of the category $\Delta\Sigma$ of noncommutative sets, and given a non-Σ operad P in $\mathcal{C}$ we define a new category $\mathcal{A}_P$ enriched over $\mathcal{C}$ using the objects in P. The most productive way of thinking about the category $\Delta\Sigma$ from our point of view is as having finite sets as objects, and morphisms are maps of sets $f : S \longrightarrow T$ together with a linear ordering of each inverse image $f^{-1}(t)$. This way it is clear that replacing a point $f : S \longrightarrow T$ in $Hom_{\Delta\Sigma}(S,T)$ with $\bigotimes_{t\in T} P(f^{-1}(t))$ makes sense.

We are interested in two different embeddings of Δ^{op} into $\Delta\Sigma$. The first is given by identifying Δ^{op} with a subcategory of Δ which we will denote $^{01}\Delta$ where the objects are sets of cardinality at least 2 and the maps are order-preserving maps which preserve the minimal and maximal element, and then using the natural inclusion of $^{01}\Delta$ into $\Delta\Sigma$. The second is given by looking at a certain subcategory of Connes' cyclic category ΔC, namely the category of based cyclically ordered sets and basepoint-preserving maps, which we will denote $^{0}\Delta C$.

Thus we get two generalizations of simplicial objects, either as a functor from $^{01}\Delta_P$ to $\mathcal{C}$ (or some other category) or as a functor from $^{0}\Delta C_P$ to $\mathcal{C}$. If P is the associative operad we get simplicial objects in both cases, but otherwise these two constructions are different. If A is a P-algebra and M and N are right and left A-modules, respectively, then we can generalize the 2-sided bar construction and make a functor $^{01}\Delta_P \longrightarrow \mathcal{C}$. The second generalization of the simplicial category gives a generalization of the cyclic bar construction. If A is a P-algebra and M is an A-bimodule we can make a functor $^{0}\Delta C_P \longrightarrow \mathcal{C}$ which generailizes the cyclic bar construction.

If we have a functor $F : \mathcal{A}_P \longrightarrow \mathcal{C}$ and $R : \mathcal{A}_P^{op} \longrightarrow \mathcal{C}$ we can form the coequalizer $R \otimes_{\mathcal{A}_P} F$. If P is the associative operad and R sends an n-element set to Δ^{n-1} this gives geometric realization in the usual sense, and with appropriate choices for P and R this still gives a good model for geometric realization.

We then devote chapter 4 to studying the Stasheff associahedra and the cyclohedra in some detail. We denote the associahedra operad by $\mathcal{K}$ and the cyclohedra (which do not form an operad) by $\mathcal{W}$, and prove that the cyclohedra give a functor $\mathcal{W} : \Delta C_{\mathcal{K}}^{op} \longrightarrow \mathcal{T}op$.

In chapter 5 we prove some technical results to allow us to put a model category structure on the category of functors $^{01}\Delta_{\mathcal{K}} \longrightarrow \mathcal{M}$ or $^{0}\Delta C_{\mathcal{K}} \longrightarrow \mathcal{M}$, where $\mathcal{K}$ is either the associahedra operad in simplicial sets and $\mathcal{M}$ is a simplicial model category, or $\mathcal{K}$ is the associahedra operad in topological spaces and $\mathcal{M}$ is a topological model category, and show that we get the expected spectral sequence from the skeletal filtration.

In chapter 6 we generalize the notion of a trace and introduce the dual notion of a cotrace. Given a right P-module R, Markl [42, definition 2.6] defined an R-trace on a P-module A into some other object B in $\mathcal{C}$. We generalize this by letting R be a functor $^{0}\Delta C_P^{op} \longrightarrow \mathcal{C}$, and a trace is now defined on a pair (A, M) consisting

of a P-algebra A and an A-bimodule M. An R-trace is corepresented by the cyclic bar construction while an R-cotrace is represented by what we call the cyclic cobar construction.

Chapter 2

Operads

In this chapter we recall some things about operads that we will need later, and enrich various categories of finite sets over the category our operad lives in. This is similar to [44], [64] and [59], which all consider enriching some category of finite sets using the spaces in an operad. We will focus mostly on non-Σ operads, which we will simply call operads. We will use the term Σ-operad for an operad in the usual sense, the few places where we will need operads with symmetric group actions. The original reference for operads is [45], see also [10]; for a more modern introduction the reader can see for example [49]. See also [43] for a comprehensive treatment of many topics related to operads. Our formalism is inspired by [20], though we focus mostly on non-Σ operads.

One key difference from the approach in [43] and [20] is that our operads have a zero'th space. We need a zero'th space in our operads to make sense of using an operad in $\mathcal{C}$ to enrich some category of sets over $\mathcal{C}$. Indeed, when the category of sets is the simplicial category, the maps coming from composition with a nullary operation should be thought of as face maps. One disadvantage is that cofibrant operads are much bigger in our category and our favorite operad, the Stasheff associahedra operad, is no longer cofibrant.

2.1 Sequences and symmetric sequences

Let $\mathcal{C}$ be a closed symmetric monoidal category with all countable coproducts. For the basic definitions $\mathcal{C}$ does not have to be closed, though it is a convenient technical assumption, see remark 2.1.2. Our main examples are the categories of spaces and based spaces, which we denote by $\mathcal{T}op$ and $\mathcal{T}op_*$, and the category of S-modules from [26]. By a space we mean either a compactly generated weak Hausdorff space or a simplicial set, and we will assume that all our based spaces are well pointed, i.e., that $* \longrightarrow X$ is a cofibration. We will state things in terms of (based) topological spaces, but throughout the paper (except when making references to [26]) it makes sense to use the category of (based) simplicial sets. We will denote the monoidal structure by $\otimes$ and the unit object by $*$. We will assume that $\varnothing \otimes A = \varnothing$ for all A, where $\varnothing$ is the initial object. If $\mathcal{C}$ is a specific category, we will revert to the usual notation for

that category.

We let $\mathbf{\Delta}$ denote the category of finite, nonempty, totally ordered sets and order preserving maps, and we let $\mathbf{\Delta}_+$ be the category of all finite totally ordered sets, i.e., $\mathbf{\Delta}$ together with the empty set. We define a sequence in $\mathcal{C}$ as a functor

$$P : iso(\mathbf{\Delta}_+) \longrightarrow \mathcal{C}. \tag{2.1.1}$$

The isomorphisms are required to be order-preserving, so there are no nontrivial automorphisms. We will write $P(n)$ for $P(\{1, 2, \ldots, n\})$, $n \geq 0$. We define a symmetric sequence in $\mathcal{C}$ as a functor

$$P : iso(\mathcal{F}) \longrightarrow \mathcal{C}, \tag{2.1.2}$$

where $\mathcal{F}$ is the category of finite sets. In this case the symmetric group Σ_n acts on $P(n)$. We will think of a (symmetric) sequence both as a functor from some category of finite sets and as a collection of objects indexed by the natural numbers. We will find it convenient to consider the category of all finite (totally ordered) sets when writing down coherence conditions, while (implicitly) choosing a skeleton category when taking limits. The main advantage of working with arbitrary finite sets is that we avoid constant relabeling when describing the maps in the definition of an operad etc.

To ease the notation, given a (symmetric) sequence P and a map $f : S \longrightarrow T$ in $\mathbf{\Delta}_+$ ($\mathcal{F}$) we will write $P[f]$ for $\bigotimes_{t \in T} P(f^{-1}(t))$. This notation is inspired by [43, definition 1.53] but differs from theirs in that we consider all maps and not just surjections. There is a monoidal product on sequences in $\mathcal{C}$ defined as follows:

Definition 2.1.1. *Given sequences P and Q, their composition product, which we denote by $P \circ Q$, is given by*

$$(P \circ Q)(S) = \coprod_{[f:S \to T]} P(T) \otimes Q[f] \tag{2.1.3}$$

for a finite totally ordered set S, where the coproduct runs over all isomorphism classes of totally ordered sets T and all order-preserving maps $S \longrightarrow T$.

The composition product on symmetric sequences is defined similarly, using unordered sets.

Remark 2.1.2. *This does not quite define a monoidal product unless the monoidal product in $\mathcal{C}$ distributes over coproducts, as the operation $\circ$ on sequences in $\mathcal{C}$ is not quite associative up to natural isomorphism. We refer the reader to [19], which explains how to get around this by defining $M \circ N \circ P$, which maps to both $(M \circ N) \circ P$ and $M \circ (N \circ P)$. We will ignore this technicality in this paper, since the monoidal product does distribute over coproducts when $\mathcal{C}$ is closed and all the symmetric monoidal categories we consider are closed.*

2.2 Operads and modules

Let I be the sequence with $I(1) = *$ and $I(n) = \emptyset$ for $n \neq 1$, and note that $P \circ I \cong P \cong I \circ P$.

Definition 2.2.1. *An operad is a sequence $S \mapsto P(S)$ together with a unit map $I \longrightarrow P$ and an associative and unital map*

$$P \circ P \longrightarrow P. \tag{2.2.4}$$

A right P-module is a sequence R together with an associative and unital map

$$R \circ P \longrightarrow R, \tag{2.2.5}$$

and a left P-module L is a sequence together with an associative and unital map

$$P \circ L \longrightarrow L. \tag{2.2.6}$$

A Σ-operad is defined similarly.

Thus an operad structure on P is a collection of maps $P(T) \otimes P[f] \longrightarrow P(S)$ satisfying certain conditions. Occationally it will be convenient to describe an operad P by giving composition maps $P[g] \otimes P[f] \longrightarrow P[g \circ f]$ for all $S \xrightarrow{f} T \xrightarrow{g} U$ which are associative and unital in the appropriate sense, as in [43, theorem 1.60]. If we want to recover the above definition we just restrict to the special case $U = \{1\}$.

In fact, it is enough to consider maps $S \cup_s T \xrightarrow{f} S \xrightarrow{g} \{1\}$ where $s \in S$, $S \cup_s T = (S - \{s\}) \coprod T$ ordered in the obvious way and f sends T to s, see the discussion about pseudo-operads in [43, section 1.7.1].

Notation 2.2.2. *We will denote the resulting map $P(S) \otimes P(T) \longrightarrow P(S \cup_s T)$ by $\circ_s$, and similarly for a right or left P-module.*

Giving a right P-module structure on R is equivalent to giving associative and unital maps maps $R[g] \otimes P[f] \longrightarrow R[g \circ f]$, or maps $R(T) \otimes P[f] \longrightarrow R(S)$ for each $f : S \longrightarrow T$.

Giving a left P-module structure on L is equivalent to giving associative and unital maps $P[g] \otimes L[f] \longrightarrow L[g \circ f]$, or just maps $P(T) \otimes L[f] \longrightarrow L(S)$ for each $f : S \longrightarrow T$.

We can embed the category $\mathcal{C}$ in the category of sequences in $\mathcal{C}$ in two different ways. For an object A in $\mathcal{C}$ we can either set $A(S) = \varnothing$ for $S \neq \varnothing$ and $A(\varnothing) = A$, or we can set $A(S) = A$ for all S.

We say that a P-algebra structure on A is a left P-module structure structure on either of the corresponding sequences. The two embeddings of $\mathcal{C}$ in sequences of $\mathcal{C}$ give the same notion of a P-algebra. In the first case, for a map $f : S \longrightarrow T$ of finite totally ordered sets $A[f] = \varnothing$ unless $S = \varnothing$, in which case $A[f] = A^{\otimes T}$. Then giving maps $P(T) \otimes A[f] \longrightarrow A(S)$ reduces to giving maps $P(n) \otimes A^{\otimes n} \longrightarrow A$ for $n \geq 0$ which satisfy the usual contions. For the second notion of embedding of $\mathcal{C}$ into sequences in

$\mathcal{C}$ we get $A[f] = A^{\otimes T}$ for any $f : S \longrightarrow T$, and again the maps $P(T) \otimes A[f] \longrightarrow A(S)$ reduces to maps $P(n) \otimes A^{\otimes n} \longrightarrow A$ satisfying the usual conditions.

If we have a map $f : P \longrightarrow Q$ of operads we immediately get a map

$$f^* : Q\text{-alg} \longrightarrow P\text{-alg}. \qquad (2.2.7)$$

If we introduce a simplicial direction, then we have a map f_* in the other direction given by a simplicial bar construction. What we mean by that is that if A is a P-algebra, then $B_\bullet(Q, P, A)$ is a Q-algebra, where $B_n(Q, P, A) = Q \circ P \circ \ldots \circ P \circ A$ with P repeated n times. If $\mathcal{C}$ has a notion of geometric realization we can get an honest Q-algebra $|B_\bullet(Q, P, A)|$ in $\mathcal{C}$. With some additional hypothesis it is possible to prove that f_* is a homotopy left adjoint to f^* and that if $f : P \longrightarrow Q$ is a weak equivalence of operads then $|B_\bullet(Q, P, A)|$ is weakly equivalent to A.

We say that an object M in $\mathcal{C}$ is an A-bimodule, if the sequence with A in degree 0, M in degree 1 and the initial object in all other degrees is a left P-module. Giving an A-bimodule structure on M is equivalent to giving maps

$$P(n) \otimes A^{\otimes(i-1)} \otimes M \otimes A^{\otimes(n-i)} \longrightarrow M \qquad (2.2.8)$$

for all $n \geq 1$ and $1 \leq i \leq n$ which satisfy the usual conditions. This is also sometimes called a (P, A)-module, and we might use this notation if the operad P is not clear from the context.

The notions of a left or right A-module do not fit comfortably into this framework, but we will fix that in a moment.

First we will introduce the notion of tensor product of operads. The earlies incarnation of this idea appeared in [11]. The idea is that if we have two Σ-operads P and Q, then we can study objects which are simultaneously algebras over P and Q in such a way that these two structures commute, or interchange. This type of structure is controlled by another operad, which is usually denoted $P \otimes Q$. It can be constructed by taking the free operad on the symmetric sequence $n \mapsto P(n) \coprod Q(n)$ and imposing the natural conditions coming from the operad structure on P and Q, and the interchange condition. Alternatively, it is the universal operad receiving maps from P and Q making certain diagrams which express the interchange of P and Q commute.

The tensor product of operads does not respect weak equivalences of operads. For example, it is easy to show that $Ass \otimes Ass = Comm$, because when we have two commuting multiplications which share a unit they are equal. On the other hand, Dunn [23] showed that if $\mathcal{C}_n$ denotes the little n-cubes operad, then $\mathcal{C}_m \otimes \mathcal{C}_n \simeq \mathcal{C}_{m+n}$. This is not so surprising, since we have natural inclusions of $\mathcal{C}_m \subset \mathcal{C}_{m+n}$ and $\mathcal{C}_n \subset \mathcal{C}_{m+n}$ as the *horizontal* and *vertical* cubes, respectively, and it is easy to see that defining the action of $\mathcal{C}_m$ and $\mathcal{C}_n$ on an $(m+n)$-fold loop space in this way allows for interchange. Fiedorowicz and Vogt's result [27] that also $Ass \otimes C_n \simeq C_{n+1}$ is slightly more surprising, but very useful. We will come back to this point in section 7.2.

17

Chapter 3

Subcategories of noncommutative sets

Let $\mathbf{\Delta\Sigma}$ be the category of noncommutative sets, as in [40, section 6.1] or [51]. The objects in $\mathbf{\Delta\Sigma}$ are finite sets (the empty set is allowed) and the morphisms are maps $f : S \longrightarrow T$ of finite sets together with a linear ordering of each $f^{-1}(t)$, $t \in T$. This category is noncommutative in the sense that there are 2 different maps from a set with 2 elements to a set with 1 element, and as the notation suggests a map in $\mathbf{\Delta\Sigma}$ can be factored uniquely as a permutation followed by a map in $\mathbf{\Delta}$ (if we pick a linear ordering of S and T). This is exactly the data we need so that given an associative algebra A and a morphism $f : S \longrightarrow T$ in $\mathbf{\Delta\Sigma}$, we get a map $f_* : A^S \longrightarrow A^T$ in a natural way.

3.1 Enriching subcategories of noncommutative sets

If we have a subcategory $\mathcal{A}$ of $\mathbf{\Delta\Sigma}$, then each map $f : S \longrightarrow T$ in $\mathcal{A}$ comes with a linear ordering of each $f^{-1}(t)$. Thus, given an operad P in $\mathcal{C}$, we can make sense of $P[f]$ and we can define a new category $\mathcal{A}_P$ enriched over $\mathcal{C}$ as follows:

Definition 3.1.1. *Let $\mathcal{A}$ be a subcategory of $\mathbf{\Delta\Sigma}$ and let P be an operad in $\mathcal{C}$. We define a category $\mathcal{A}_P$ enriched over $\mathcal{C}$ as follows. The objects are the same as in $\mathcal{A}$, but the Hom objects are given by*

$$Hom_{\mathcal{A}_P}(S,T) = \coprod_{f \in Hom_{\mathcal{A}}(S,T)} P[f]. \tag{3.1.1}$$

Composition is defined in terms of the operad structure in the evident way. To spell this out, given $f : S \longrightarrow T$ and $g : T \longrightarrow U$ in $\mathcal{A}$, we need to give a map $P[g] \otimes P[f] \longrightarrow P[g \circ f]$. If we write $P[f] = \bigotimes_{u \in U} \left(\bigotimes_{t \in g^{-1}(u)} P(f^{-1}(t)) \right)$, then the map is simply given by a product of the maps $P(g^{-1}(u)) \otimes \left(\bigotimes_{t \in g^{-1}(u)} P(f^{-1}(t)) \right) \longrightarrow P((g \circ f)^{-1}(u))$ given by the operad structure on P over all $u \in U$.

Example 3.1.2. *There are several natural examples one can consider; we list some of them here.*

1. *The simplicial category $\boldsymbol{\Delta}$, which we have defined to be the category of all nonempty finite totally ordered sets and order-preserving maps, or its augumentation $\boldsymbol{\Delta}_+$ which is the category $\boldsymbol{\Delta}$ together with the empty set.*

2. *The category ${}^0\boldsymbol{\Delta}$ of finite totally ordered sets with a minimal element 0 and order-preserving maps preserving the minimal element, or the category ${}^1\boldsymbol{\Delta}$ of finite totally ordered sets with a maximal element 1.*

3. *The category ${}^{01}\boldsymbol{\Delta}$ of finite totally ordered sets with both a minimal and maximal element.*

4. *Connes' cyclic category $\boldsymbol{\Delta}C$, which consists of finite cyclically ordered sets and order-preserving maps together with a linear ordering of each inverse image of an element, or its augumentation $\boldsymbol{\Delta}C_+$.*

5. *The subcategory ${}^0\boldsymbol{\Delta}C$ of $\boldsymbol{\Delta}C$ of cyclically ordered sets with a basepoint 0 and basepoint-preserving maps.*

3.2 Two generalizations of simplicial sets

Lemma 3.2.1. *The categories ${}^{01}\boldsymbol{\Delta}$ and ${}^0\boldsymbol{\Delta}C$ are isomorphic to $\boldsymbol{\Delta}^{op}$.*

Proof. (See e.g. [22, p. 621].) We construct a functor ${}^{01}\boldsymbol{\Delta} \longrightarrow \boldsymbol{\Delta}^{op}$ by sending a set with $n+2$ elements, say, $\{0, x_1, \ldots, x_n, 1\}$ to $\mathbf{n} = \{0, 1, \ldots, n\}$. The map on Hom sets is given as in the following picture:

$$\text{(3.2.2)}$$

For the second case, given $f : S \longrightarrow T$ in ${}^0\boldsymbol{\Delta}C$, the linear ordering of $f^{-1}(0)$ allows us to extend f to a map $S \coprod \{1\} \longrightarrow T \coprod \{1\}$ by sending the elements in $f^{-1}(0)$ which are greater than 0 to 0 and the elements that are less than 0 to 1. The rest of the proof is the same as for the first case. $\qquad\square$

Thus we get two generalizations of a simplicial object in $\mathcal{C}$, either as a functor ${}^{01}\boldsymbol{\Delta}_P \longrightarrow \mathcal{C}$ or a functor ${}^0\boldsymbol{\Delta}C_P \longrightarrow \mathcal{C}$. As we will see, the first generalization allows us to define the 2-sided bar construction while the second allows us to define the cyclic bar construction.

Remark 3.2.2. *By a functor $F : \mathcal{D} \longrightarrow \mathcal{D}'$ between categories two categories that are enriched over $\mathcal{C}$, we will always mean a $\mathcal{C}$-functor, i.e., F comes with maps $Hom_{\mathcal{D}}(X,Y) \longrightarrow Hom_{\mathcal{D}'}(F(X), F(Y))$ of objects in $\mathcal{C}$. For example, if $\mathcal{C} = \mathcal{T}op$ this means that we require all maps of Hom sets to be continuous.*

If $X : {}^{01}\mathbf{\Delta}_P \longrightarrow \mathcal{C}$ or ${}^0\mathbf{\Delta} C_P \longrightarrow \mathcal{C}$ we will abuse notation and denote the image of an element which corresponds to $\mathbf{n} = \{0, 1, \ldots, n\}$ in $\mathbf{\Delta}$ by X_n. If $P(0) = P(1) = *$, the injective maps in ${}^{01}\mathbf{\Delta}$ or ${}^0\mathbf{\Delta} C$, which correspond to surjective maps in $\mathbf{\Delta}$, do not change when we pass from ${}^{01}\mathbf{\Delta}$ to ${}^{01}\mathbf{\Delta}_P$ or from ${}^0\mathbf{\Delta} C$ to ${}^0\mathbf{\Delta} C_P$, and we will denote the map corresponding to s_j in $\mathbf{\Delta}^{op}$ by s_j.

This gives a new way to look at a P-algebra A, as well as right, left and bimodules over A. A P-algebra A gives a functor $(\mathbf{\Delta}_+)_P \longrightarrow \mathcal{C}$ given by $S \mapsto A^{\otimes S}$. A right A-module M gives a functor ${}^0\mathbf{\Delta}_P \longrightarrow \mathcal{C}$ given by $S \mapsto M \otimes A^{\otimes(S-\{0\})}$, a left A-module N gives a functor ${}^1\mathbf{\Delta}_P \longrightarrow \mathcal{C}$ by $S \mapsto A^{\otimes(S-\{1\})} \otimes N$, while a pair (M, N) of a right and a left A-module gives a functor ${}^{01}\mathbf{\Delta}_P \longrightarrow \mathcal{C}$ by $S \mapsto M \otimes A^{\otimes(S-\{0,1\})} \otimes N$. An A-bimodule M gives a functor ${}^0\mathbf{\Delta} C_P \longrightarrow \mathcal{C}$ by $S \mapsto M \otimes A^{\otimes(S-\{0\})}$. We will call the functor $S \mapsto M \otimes A^{\otimes(S-\{0,1\})} \otimes N$ the P-simplicial 2-sided bar construction and denote it by $B^P_\bullet(M, A, N)$ or simply $B_\bullet(M, A, N)$ if P is clear from the context. Similarly, we will call the functor $S \mapsto M \otimes A^{\otimes(S-\{0\})}$ from ${}^0\mathbf{\Delta} C_P$ to $\mathcal{C}$ the P-cyclic bar construction and denote it by $B^{cy}_\bullet(A; M)$.

We will depart from standard terminology and call the functor $S \mapsto Hom(A^{\otimes S}, M)$ the cyclic cobar construction and denote it by $C^\bullet_{cy}(A; M)$. This is a functor ${}^0\mathbf{\Delta} C^{op}_P \longrightarrow \mathcal{C}$.

Notation 3.2.3. *We will call a functor $F : \mathcal{A}_P \longrightarrow \mathcal{C}$ a left $\mathcal{A}_P$ module and a functor $R : \mathcal{A}^{op}_P \longrightarrow \mathcal{C}$ a right $\mathcal{A}_P$ module.*

Thus a P-algebra A gives rise to a left $\mathcal{A}_P$ module $S \mapsto A^{\otimes S}$ for any subcategory $\mathcal{A}$ of $\mathbf{\Delta}\Sigma$, and since a P-algebra is really a left P-module structure on a certain sequence under the composition product (definition 2.1.1), a left $\mathcal{A}_P$ module is a generalization of this which depends on $\mathcal{A}$. (A general left P-module does not give a left $\mathcal{A}_P$ module in any natural way, so one could argue that this notation is not the best, but we will use it anyway.)

The category ${}^0\mathbf{\Delta} C_P$ should be compared to the category $\bar{\mathbf{\Delta}}^{op}$ and its enrichment $\hat{P}$ from [64] (or $\hat{\mathcal{C}}$, since $\mathcal{C}$ is an operad in Thomason's paper). Indeed, his category $\bar{\mathbf{\Delta}}^{op}$ is very closely related to the category ${}^0\mathbf{\Delta} C$ of based cyclic sets. His condition in definition 1.1 that for a map $f \in \bar{\mathbf{\Delta}}^{op}$, if $f(i_0) = 0$ then either $f(i_1) > 0$ only if $i_1 > i_0$ or only if $i_1 < i_0$ guarantees that f can be lifted to a map of cyclically ordered sets. The only difference is that he only has one map $\mathbf{n} = \{0, 1, \ldots, n\} \longrightarrow \mathbf{0} = \{0\}$ instead of $n+1$ maps, so the lift is not always unique. He also works with operads (as opposed to Σ-operads), and his enriched category $\hat{P}$ is the same as our $\mathbf{\Delta} C_P$, except that he only uses the spaces $\mathcal{C}(f^{-1}(t))$ for $t \neq 0$. None of these differences matter as long as one only studies the case when $M = *$, which is in effect what he did.

We also note that a functor $R : (\mathbf{\Delta}_+)^{op}_P \longrightarrow \mathcal{C}$ is precisely a right P-module. Thus we also obtain various generalizations of right P-modules. For example, we will think of a right $(\mathbf{\Delta} C_+)_P$ module as a right P-module with some extra structure.

3.3 Geometric realization

Now let $\mathcal{D}$ be a category which is tensored and cotensored over $\mathcal{C}$. We will still call a functor $\mathcal{A}_P \longrightarrow \mathcal{D}$ a left $\mathcal{A}_P$ module etc.

Notation 3.3.1. *If we have a left $\mathcal{A}_P$ module $F : \mathcal{A}_P \longrightarrow \mathcal{D}$ and a right $\mathcal{A}_P$ module $R : \mathcal{A}_P^{op} \longrightarrow \mathcal{C}$, we will denote by $R \otimes_{\mathcal{A}_P} F$ the coequalizer*

$$\coprod_{[f:S\to T]} R(T) \otimes P[f] \otimes F(S) \rightrightarrows \coprod_{[S]} R(S) \otimes F(S) \longrightarrow R \otimes_{\mathcal{A}_P} F. \qquad (3.3.3)$$

Similarly, if F and R are both right $\mathcal{A}_P$ modules we will denote by $Hom_{\mathcal{A}_P}(R, F)$ the equalizer

$$Hom_{\mathcal{A}_P}(R, F) \longrightarrow \coprod_{[S]} Hom(R(S), F(S)) \rightrightarrows \coprod_{[f:S\to T]} Hom(R(T) \otimes P[f], F(S)).$$
$$\qquad (3.3.4)$$

If we work in $\mathcal{T}op$, $P = Ass$ and $\mathcal{A}$ is either $^{01}\mathbf{\Delta}$ or $^0\mathbf{\Delta}C$, so that $\mathcal{A}_P \cong \mathbf{\Delta}^{op}$, then $R(S) = \Delta^{|S-\{0,1\}|}$ or $R(S) = \Delta^{|S-\{0\}|}$, the standard geometric simplicies, gives a right $\mathcal{A}_P$ module. Thus we recover the usual notion of geometric realization and Tot. We can do something similar in any other category where geometric realization makes sense.

If we have a map $f : P \longrightarrow Q$ of operads, then we get a functor $f : \mathcal{A}_P \longrightarrow \mathcal{A}_Q$ between enriched categories. Thus we can pull a functor $\mathcal{A}_Q \longrightarrow \mathcal{D}$ or $\mathcal{A}_Q^{op} \longrightarrow \mathcal{D}$ back to a functor $\mathcal{A}_P \longrightarrow \mathcal{D}$ or $\mathcal{A}_P^{op} \longrightarrow \mathcal{D}$. This gives a functor $f^* : \mathcal{D}^{\mathcal{A}_Q} \longrightarrow \mathcal{D}^{\mathcal{A}_P}$. In particular, since $\mathcal{A}_{Ass} \cong \mathcal{A}$, given any A_∞ operad P and a functor $\mathcal{A} \longrightarrow \mathcal{D}$ or $\mathcal{A}^{op} \longrightarrow \mathcal{D}$ we can pull it back to a functor $\mathcal{A}_P \longrightarrow \mathcal{D}$ or $\mathcal{A}_P^{op} \longrightarrow \mathcal{D}$. With $\mathcal{A} = {}^{01}\mathbf{\Delta}_P$ or $^0\mathbf{\Delta}C_P$ we see that we can regard any simplicial object in $\mathcal{D}$ as a functor from $^{01}\mathbf{\Delta}_P$ or from $^0\mathbf{\Delta}C_P$. Similarly, any cosimplicial object can be regarded as a functor from $^{01}\mathbf{\Delta}_P^{op}$ or from $^0\mathbf{\Delta}C_P^{op}$.

Chapter 4

The associahedra and cyclohedra

The original definition of the associahedra can be found in [61]. The cyclohedra got their name from Stasheff [62], but had been considered earlier, first by Bott and Taubes in [16]. They sometimes go under the name Stasheff associahedra of type B. For an introduction to the associahedra and cyclohedra, see [42].

4.1 The associahedra

Consider all ways to parenthesize (in a meaningful way) n linearly ordered variables. The maximal number of pairs of parentheses is $n - 2$, and the Stasheff associahedron K_n has an $(n-2-i)$-cell for each way to parenthesize using i pairs of parentheses, with a face map for each way to insert an additional pair. Perhaps a more precise definition of K_n is as the cone on L_n, where L_n is the union of various copies of $K_r \times K_{n-r+1}$, as in [10, definition 1.7]. This definition also makes sense in the category of simplicial sets. For example, $K_0 = K_1 = K_2 = *$, $K_3 = I$ is an interval and K_4 is a pentagon. Figure 4-1 shows K_5.

Theorem 4.1.1. *(Stasheff [61]) For $n \geq 2$ the associahedron K_n is homeomorphic to D^{n-2}, and the sequence $\mathcal{K} = \{K_n\}_{n \geq 0}$ forms an A_∞ operad in $\mathcal{T}op$.*

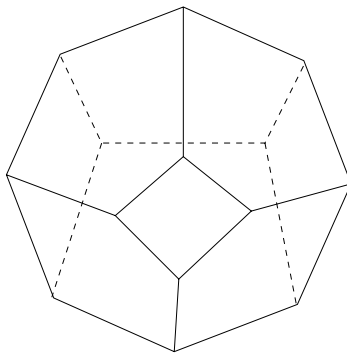

Figure 4-1: The associahedron K_5

The operad structure can be thought of as substitution of parenthesized epressions. The Boardman-Vogt W-construction, which is usually given in terms of certain metric trees, provides a cofibrant replacement of operads in a certain model category. The associahedra operad $\mathcal{K}$ (or a cubical decomposition of it) is the W-construction on the associative operad Ass [43, Example 2.22], though we have to be careful about exactly which category of operads we work in.

The model category structure on operads (in $\mathcal{T}op$) is given by levelwise weak equivalences and fibrations, while the cofibrations are what they have to be [9, theorem 3.2]. This relies on special properties of the category $\mathcal{T}op$, though it is possible to weaken the conditions necessary for getting a model category somewhat by considering reduced operads [9, theorem 3.1]. An operad P is reduced if $P(0)$ is the unit in the symmetric monoidal category. It is not clear if there is a model category structure on operads in a general symmetric monoidal model category.

But $\mathcal{K}$ is not cofibrant in the category of operads in $\mathcal{T}op$. For example, it is easy to see that there can be no map from $\mathcal{K}$ to the little intervals operad $\mathcal{C}_1$. The problem is that $* \in \mathcal{C}_1(0)$ does not act as a unit. If we perform the W-construction on Ass in this category we also get an operad with a very big first space. The solution in [43] is to consider the category of operads with $P(0) = \varnothing$. In this category one can show that there is a map from the associahedra to the little intervals operad, and the associahedra operad is indeed cofibrant in this category.

Because we want to generalize the W-construction in such a way that it produces the cyclohedra, we will sketch the details of the W-construction of Ass in the topological setting. To do this we need to discuss trees. See [20] for a much more thorough discussion of trees as related to operads. For simplicity we will not allow vertices with only one incoming edge. This means that we have to restrict the W-construction to operads with $P(1) = *$, but that is enough for our purpose. By a tree we will mean a planar directed tree where each non-leaf vertex except the root has at least two incoming edges. The root has exactly one incoming edge. A metric w on T is an assignment of a length $w(e)$ of each internal edge e in T such that $0 < w(e) \leq 1$. We topologize the space of metric trees as the quotient of the space of metric trees with $0 \leq w(e) \leq 1$ where we identify a tree with an internal edge of length 0 with the corresponding tree with that edge collapsed. Let $\mathcal{T}_n$ be the space of metric trees with n leaves. Given a tree $T \in \mathcal{T}_n$ let $vert(T)$ be the set of internal vertices, and let $In(v)$ be the set of incoming edges to v.

Definition 4.1.2. *Given an operad P in $\mathcal{T}op$ with $P(0) = \varnothing$ and $P(1) = *$, WP is the operad defined by*

$$WP(n) = \coprod_{T \in \mathcal{T}_n} \prod_{v \in vert(T)} P(In(v)) \tag{4.1.1}$$

with the natural topology.

The structure maps in WP are given by grafting trees, and assigning the length 1 to any new internal edges.

The claim in [43] is that the W-construction on Ass in this setting gives $\mathcal{K}$. For example, K_4 is given by figure 4-2.
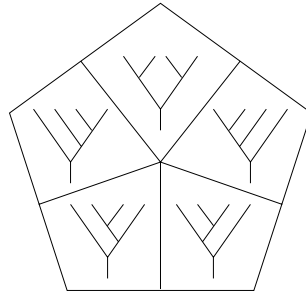
Figure 4-2: The associahedron K_4 from the W-construction

It is easy to see that $\mathcal{K}$ is cofibrant in the category of operads without zero'th space. This amounts to showing that for any trivial fibration $P \xrightarrow{\simeq} Ass$ of operads, which just amounts to requiring that each $P(n)$ is fibrant and contractible, the dotted arrow in the diagram

$$\begin{array}{ccc} & & P \\ & \nearrow & \downarrow{\scriptstyle\simeq} \\ \mathcal{K} & \longrightarrow & Ass \end{array}$$

(4.1.2)

exists. But we can construct a map like this by induction. If we are given maps $K_i \longrightarrow P(i)$ for $i < n$, the map $K_n \longrightarrow P(n)$ is determined on ∂K_n, and now we just have to solve the extension problem

$$\begin{array}{ccc} S^{n-3} & \longrightarrow & P(n) \\ \downarrow & \nearrow & \downarrow \\ D^{n-2} & \longrightarrow & * \end{array}$$

(4.1.3)

which we know we can solve because each $P(n)$ is fibrant and contractible.

Thus for any A_∞ operad P, we get a map $\mathcal{K} \longrightarrow P$ of operads without a zero'th space. If $P(0) = *$ and $P(0)$ acts as a unit in a sufficiently nice way, the map $\mathcal{K} \longrightarrow P$ can be promoted to a map of operads with a zero'th space, and we can pull back a functor $F : \mathcal{A}_P \longrightarrow \mathcal{T}op$ to a functor $\mathcal{A}_\mathcal{K} \longrightarrow \mathcal{T}op$. Thus it makes sense to concentrate on the A_∞ operad $\mathcal{K}$ as long as we are willing to restrict the kinds of operads we consider. This restriction excludes operads like the little intervals operad, so for some purposes this restriction is bad, for example if we want to consider tensor products of operads.

We note that K_n has i faces of the form $K_i \times K_{n-i+1}$ for each $2 \le i \le n-1$, given by the images of the i maps $\circ_j : K_i \times K_{n-i+1} \longrightarrow K_n$, or the i ways to put one pair of parentheses around $n - i + 1$ of the variables. In particular, K_n has $n + 1$ faces of the form K_{n-1}. As in [55], we will see that the faces $x_1(x_2 \cdots x_n)$ and $(x_1 \cdots x_{n-1})x_n$ correspond to the first and last map in the Hochschild cochain complex, while the faces $x_1 \cdots (x_i x_{i+1}) \cdots x_n$ give the rest of them.

The operad $\mathcal{K}$ has an obvious filtration, where we let $\mathcal{K}_n$ be the operad generated

24

by K_i for $i \leq n$. An algebra over $\mathcal{K}_n$ is precisely an A_n algebra as in [61], and giving an A_n algebra structure on A is equivalent to giving maps $K_i \times A^i \longrightarrow A$ for $0 \leq i \leq n$ satisfying the usual conditions.

Observe that the standard n-simplex Δ^n is the configuration space of $n+2$ points on the unit interval, where the first point is at the beginning and the last at end. Let the points be labelled by elements of a set $S \in {}^{01}\Delta$ with $|S| = n+2$, say, $S = \{0, x_1, \ldots, x_n, 1\}$. By abuse of notation, we will let x_i denote both the position of the $(i+1)$st point and an element in S. We get the standard description of Δ^n by setting $t_i = x_{i+1} - x_i$ ($t_0 = x_1$, $t_n = 1 - x_n$). The associahedron $K_{n+2} = \mathcal{K}(S)$ is also such a configuration space; it is the compactification of the configuration space of $n+2$ disinct points as above, where for each time we have a point repeated i times we use a copy of K_i instead of just a point. This is the Axelrod-Singer compactification, see [3] and also [28]. Of course, we could also describe the configuration space of distinct points on I in terms of distinct points on $\mathbb{R}$ modulo translation and dilatation. This is the point of view found in the references, and the point of view we have to take if we want to generalize to configurations on higher-dimensional manifolds, see remark 4.2.7.

If $f : S \longrightarrow T$ is a map in ${}^{01}\Delta$, we can interpret $\mathcal{K}(T) \otimes \mathcal{K}[f] \longrightarrow \mathcal{K}(S)$ in terms of the above configuration space as follows. For each $t \in T$, the map replaces the point labelled by t by points labelled by the set $f^{-1}(t)$, and the factor $\mathcal{K}(f^{-1}(t))$ tells us how. This works because $0 \in f^{-1}(0)$ and $1 \in f^{-1}(1)$, so we never remove the points at the beginning and end of the interval.

Next we compare the associahedra to the standard n-simplexes. Denote by $s^i : K_{n+2} \longrightarrow K_{n+1}$, $0 \leq i \leq n-1$, the map $K_{n+2} \cong K_{n+2} \times K_0 \longrightarrow K_{n+1}$ obtained from $\circ_{x_{i+1}}$ (notation 2.2.2). This is the same as removing the $(i+2)$nd variable in the parenthesized expression of $n+2$ variables defining K_{n+2}, and as we just saw it corresponds to removing the point marked x_{i+1} in the configuration space. We also get maps $K_{n+2} \longrightarrow K_{n+1}$ by removing 0 or 1, but these do not correspond to codegeneracy maps on Δ^n. Similarly we get maps $d^j : K_{n+2} \cong K_{n+2} \times K_2 \longrightarrow K_{n+3}$ for $0 \leq j \leq n+1$ from $\circ_{x_j}$ ($x_0 = 0$, $x_{n+1} = 1$), which correspond to replacing x_j with a double point.

As is obvious from the configuration space interpretation of K_n, there is a surjective map $K_{n+2} \longrightarrow \Delta^n$ which is a homeomorphism on the interior. The association $n \mapsto K_{n+2}$ is not quite a cosimplicial space, because some of the simplicial identities commute only up to homotopy, but the following diagrams commute:

$$
\begin{array}{ccc}
K_{n+2} \longrightarrow \Delta^n \\
\downarrow{\scriptstyle s^i} \qquad \downarrow{\scriptstyle s^i} \\
K_{n+1} \longrightarrow \Delta^{n-1}
\end{array}
\qquad
\begin{array}{ccc}
K_{n+2} \longrightarrow \Delta^n \\
\downarrow{\scriptstyle d^j} \qquad \downarrow{\scriptstyle d^j} \\
K_{n+3} \longrightarrow \Delta^{n+1}
\end{array}
\qquad (4.1.4)
$$

In particular these two diagrams show that $S \mapsto \Delta^{|S|-1}$ gives a functor ${}^{01}\Delta_{\mathcal{K}}^{op} \longrightarrow \mathcal{T}op$. We also see that the two faces of K_{n+2} coming from the inclusions $K_2 \times K_{n+1} \longrightarrow K_{n+2}$ are crushed to a point in Δ^n.

Let A be a $\mathcal{K}$-algebra, M a right A-module and N a left A-module. By regarding

$\mathcal{K}$ as a functor $^{01}\mathbf{\Delta}_\mathcal{K}^{op} \longrightarrow \mathcal{T}op$ we can now define the 2-sided bar construction as

$$B(M, A, N) = \mathcal{K} \otimes_{^{01}\mathbf{\Delta}_\mathcal{K}} B_\bullet^\mathcal{K}(M, A, N). \qquad (4.1.5)$$

We elaborate on what this definition means. The tensor product is defined as a coequalizer, which just means a quotient in $\mathcal{T}op$, so $B(M, A, N)$ is given by

$$\coprod_n K_{n+2} \times M \times A^n \times N/ \sim, \qquad (4.1.6)$$

where we identify (f^*x, y) with (x, f_*y). (The roles of f_* and f^* are reversed here because $^{01}\mathbf{\Delta}_\mathcal{K}$ has replaced $\mathbf{\Delta}^{op}$, not $\mathbf{\Delta}$.)

Proposition 4.1.3. *When $M = N = *$, the bar construction as defined above agrees with Stasheff's definition in [61]. In particular, if A is grouplike ($\pi_0 A$ is a group) then $BA = B(*, A, *)$ is a delooping of A.*

Similarly, given any functor $X : {}^{01}\mathbf{\Delta}_\mathcal{K} \longrightarrow \mathcal{T}op$ we define $|X|$ as $\mathcal{K} \otimes_{^{01}\mathbf{\Delta}_\mathcal{K}} X$, and given a functor $Y : {}^{01}\mathbf{\Delta}_\mathcal{K}^{op} \longrightarrow \mathcal{T}op$ we define $Tot(Y)$ as $Hom_{^{01}\mathbf{\Delta}_\mathcal{K}}(\mathcal{K}, Y)$.

Proposition 4.1.4. *The maps $K_{n+2} \longrightarrow \mathbf{\Delta}^n$ assemble to a natural transformation of functors from $^{01}\mathbf{\Delta}_\mathcal{K}$ to $\mathcal{T}op$.*

Proof. This follows immediately from the two commutative diagrams in equation 4.1.4 □

Proposition 4.1.5. *Let X be a simplicial space and regard X as a functor $^{01}\mathbf{\Delta}_\mathcal{K} \longrightarrow \mathcal{T}op$ via*

$$^{01}\mathbf{\Delta}_\mathcal{K} \longrightarrow {}^{01}\mathbf{\Delta}_{Ass} \overset{\cong}{\longrightarrow} \mathbf{\Delta}^{op} \overset{X}{\longrightarrow} \mathcal{T}op. \qquad (4.1.7)$$

Then the natural map

$$\mathcal{K} \otimes_{^{01}\mathbf{\Delta}_\mathcal{K}} X \longrightarrow \Delta^\bullet \otimes_{\mathbf{\Delta}^{op}} X \qquad (4.1.8)$$

is an isomorphism.

Similarly, if we regard a cosimplicial space Y as a functor $^{01}\mathbf{\Delta}_\mathcal{K}^{op} \longrightarrow \mathcal{T}op$ the natural map

$$Hom_\mathbf{\Delta}(\Delta^\bullet, Y) \longrightarrow Hom_{^{01}\mathbf{\Delta}_\mathcal{K}}(\mathcal{K}, Y) \qquad (4.1.9)$$

is an isomorphism.

Proof. Let $S = \{0, x_1, \ldots, x_n, 1\}$, $T = \{0, y_1, \ldots, y_m, 1\}$ and let $f : S \longrightarrow T$ be a map in $^{01}\mathbf{\Delta}$ which is dual to $g : \mathbf{m} \longrightarrow \mathbf{n}$ in $\mathbf{\Delta}$. Let $\bar{f}$ be a map in $^{01}\mathbf{\Delta}_\mathcal{K}$ which is in the component of f. Then $\bar{f}_* = g^* : X_n \longrightarrow X_m$, so the induced map

$$\mathcal{K}[f] \times X_n \longrightarrow X_m \qquad (4.1.10)$$

is constant in the first variable. This means that when we consider the corresponding map $\mathcal{K}(T) \times \mathcal{K}[f] \longrightarrow \mathcal{K}(S)$ which appears in the coequalizer defining $\mathcal{K} \otimes_{^{01}\mathbf{\Delta}_\mathcal{K}} X$, the image of $\{k\} \times K[f]$ in $K_{n+2} \times X_n$ is crushed to a point. Doing this for all f gives us exactly the projections $K_{n+2} \longrightarrow \Delta^n$.
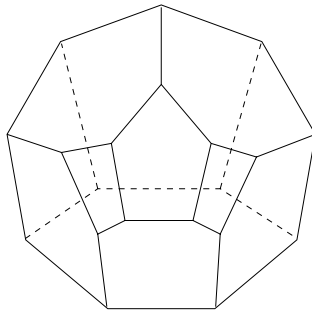
The second part is similar. □

Figure 4-3: The cyclohedron W_4

4.2 The cyclohedra

Next we consider the right module over $\mathcal{K}$ given by the cyclohedra. To define the cyclohedron W_n, we again consider all ways to parenthesize n variables, but now we let them be cyclically ordered. In this case the maximal number of pairs of parentheses is $n-1$. For example, 12 can be parenthesized as either (12) or 1)(2. The same construction as above, now with an $n-i-1$ cell for each way to parenthesize n variables using i pairs of parentheses gives the space W_n. Again a better definition of W_n might be as the cone on a union of various copies of $W_s \times K_{n-s+1}$. For example, $W_0 = W_1 = *$, $W_2 = I$ and W_3 is a hexagon. Figure 4-3 shows W_4.

Theorem 4.2.1. *For $n \geq 1$ the cyclohedron W_n is homeomorphic to D^{n-1}, and the cyclohedra assemble to a functor $\mathcal{W} : (\mathbf{\Delta} C_+)^{op}_{\mathcal{K}} \longrightarrow \mathcal{T}op$.*

Proof. It is well known ([62, section 4]) that $\mathcal{W}$ is a right $\mathcal{K}$-module, i.e., a functor $(\mathbf{\Delta}_+)^{op}_{\mathcal{K}} \longrightarrow \mathcal{T}op$, and a straightforward extension of the proof shows that it extends to a functor $(\mathbf{\Delta} C_+)^{op}_{\mathcal{K}} \longrightarrow \mathcal{T}op$.

$$(\mathbf{\Delta}_+)^{op}_{\mathcal{K}} \longrightarrow \mathcal{T}op \qquad\qquad (4.2.11)$$
$$\downarrow \qquad \nearrow$$
$$(\mathbf{\Delta} C_+)^{op}_{\mathcal{K}}$$

$\square$

This result is related to Markl's result [42, theorem 2.12] that if we consider the Σ-operad with $n \mapsto \Sigma_n \times K_n$ then the symmetric sequence $n \mapsto C_n \backslash \Sigma_n \times W_n$ is a right module over this operad.

Next we describe the version of the Boardman-Vogt W-construction which gives us $\mathcal{W}$. The idea is to change the definition of a tree slightly, in a way that is similar to [20, definition 7.3]. First of all, we allow the root to have more than one incoming edge. We also require the leaves to come with a cyclic ordering, and we identify trees that differ by a cyclic permutation of the root edges. We also assign a length to the root edges. Let $\mathcal{T}C_n$ be the space of such trees with leaves labelled by the cyclically ordered set $\{1, 2, \ldots, n\}$.
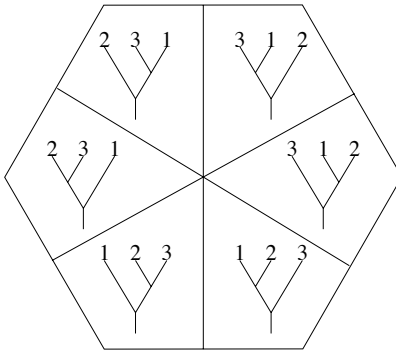
27

Figure 4-4: The cyclohedron W_3 from a relative W-construction

Definition 4.2.2. *Given an operad P in $\mathcal{T}op$ with $P(0) = \varnothing$ and $P(1) = *$, and a functor $R : {}^0\Delta C_P \longrightarrow \mathcal{T}op$, $W(R, P)$ is the functor ${}^0\Delta C_{WP} \longrightarrow \mathcal{T}op$ given by*

$$W(R, P)(n) = \coprod_{T \in \mathcal{T}C_n} R(In(r)) \times \Big(\prod_{v \in vert(T)} P(In(v)) \Big) \qquad (4.2.12)$$

with the natural topology. Here r is the root vertex, and $vert(T)$ is the set of internal vertices of T.

Proposition 4.2.3. *The cyclohedra can be obtained as $\mathcal{W} = W(R, Ass)$, where $R : \Delta C_{Ass} \longrightarrow \mathcal{T}op$ sends any set to $*$.*

Proof. This follows almost immediately. The relative W-construction gives an $(n-1)$-cube for each binary tree. We can think of each vertex of W_n as a binary tree where all the internal edges have lenght 1. By subdividing W_n as in figure 4-4 we see that W_n can be decomposed as a union of $(n-1)$-cubes, one for each binary tree. $\qquad \square$

The cyclohedra, regarded as a functor $\Delta C_{\mathcal{K}} \longrightarrow \mathcal{T}op$, also has a universal lifting property. For any $R : \Delta C_{\mathcal{K}} \longrightarrow \mathcal{T}op$ with each $R(S) \simeq *$, we can construct a natural transformation $\mathcal{W} \longrightarrow R$ by lifting one cell at a time.

Remark 4.2.4. *We could consider this construction for other subcategories of $\Delta\Sigma$. For example, if we consider $R : \Delta\Sigma \longrightarrow \mathcal{T}op$ given by $R(S) = *$ for all S we find that $W(R, Ass)(3)$ is two copies of W_3 joined at the center. More generally, $W(R, Ass)(n)$ is $(n-1)!$ copies of W_n joined at a codimension 2 subspace.*

The functor $\mathcal{W}$ has a filtration which is compatible with the filtration of $\mathcal{K}$. We let $\mathcal{W}_n$ be the functor $\mathcal{W}_n : (\Delta C_+)^{op}_{\mathcal{K}_n} \longrightarrow \mathcal{T}op$ generated W_i for $i \le n$.

Again we can relate this to configuration spaces. We can also consider the n-simplex Δ^n as the configuration space of $n+1$ points on S^1 labelled by elements of some $S \in {}^0\Delta C$ with $|S| = n+1$, say, $S = \{0, x_1, \ldots, x_n\}$, with 0 at the basepoint. Then W_{n+1} is the Axelrod-Singer compactification of the interior of this space where we use a copy of K_i instead of a point every time we have a point repeated i times.

28

Again we have maps

$$s^i : W_{n+1} \longrightarrow W_n \qquad (4.2.13)$$

for $0 \leq i \leq n-1$, which can be interpreted as removing the point x_{i+1} in the above configuration space, and maps $d^j : W_{n+1} \longrightarrow W_{n+2}$ for $0 \leq j \leq n+1$ which can be interpreted as replacing the point x_i with a double point. Here d^0 and d^{n+1} both replace 0 by a double point, either with $\{0, x_1\}$ or $\{x_{n+1}, 0\}$. Again we have a surjective map $W_{n+1} \longrightarrow \Delta^n$, which gives commutative diagrams

$$
\begin{array}{ccc}
W_{n+1} \longrightarrow \Delta^n \\
\downarrow{s^i} \qquad \downarrow{s^i} \\
W_n \longrightarrow \Delta^{n-1}
\end{array}
\qquad
\begin{array}{ccc}
W_{n+1} \longrightarrow \Delta^n \\
\downarrow{d^j} \qquad \downarrow{d^j} \\
W_{n+2} \longrightarrow \Delta^{n+1}
\end{array}
\qquad (4.2.14)
$$

Again we see that we also get a map $W_{n+1} \longrightarrow W_n$ from removing the point 0, but this does not correspond to a codegeneracy map on Δ^n.

If we have a functor $X : {}^0\Delta C_{\mathcal{K}} \longrightarrow \mathcal{T}op$, we define the geometric realization $|X|$ as $\mathcal{W} \otimes_{{}^0\Delta C_{\mathcal{K}}} X$, and if we have a functor $Y : {}^0\Delta C_{\mathcal{K}}^{op} \longrightarrow \mathcal{T}op$ we define the total object $Tot(Y)$ as $Hom_{{}^0\Delta C_{\mathcal{K}}}(\mathcal{W}, Y)$. In particular, given an A_∞ space A and an A-bimodule M we can define the cyclic bar construction $B^{cy}(A; M)$ and the cyclic cobar construction $C_{cy}(A; M)$ this way.

Proposition 4.2.5. *The maps $W_{n+1} \longrightarrow \Delta^n$ assemble to a natural transformation of functors from ${}^0\Delta C_{\mathcal{K}}$ to $\mathcal{T}op$.*

It is clear that the analog of proposition 4.1.5 holds:

Proposition 4.2.6. *Let X be a simplicial space and regard X as a functor ${}^0\Delta C_{\mathcal{K}} \longrightarrow \mathcal{T}op$ via*

$$
{}^0\Delta C_{\mathcal{K}} \longrightarrow {}^0\Delta C_{Ass} \xrightarrow{\cong} \Delta^{op} \xrightarrow{X} \mathcal{T}op. \qquad (4.2.15)
$$

Then the natural map

$$
\mathcal{W} \otimes_{{}^0\Delta C_{\mathcal{K}}} X \longrightarrow \Delta^\bullet \otimes_{\Delta^{op}} X \qquad (4.2.16)
$$

is an isomorphism.

Similarly, if we regard a cosimplicial space Y as a functor ${}^0\Delta C_{\mathcal{K}}^{op} \longrightarrow \mathcal{T}op$ the natural map

$$
Hom_{\Delta}(\Delta^\bullet, Y) \longrightarrow Hom_{{}^0\Delta C_{\mathcal{K}}}(\mathcal{W}, Y) \qquad (4.2.17)
$$

is an isomorphism.

We can also consider functors defined only on ${}^0\Delta C_{\mathcal{K}_n}$. If $X : {}^0\Delta C_{\mathcal{K}_n} \longrightarrow \mathcal{T}op$, the expression $\mathcal{W}_n \otimes_{{}^0\Delta C_{\mathcal{K}_n}} X$ makes sense, and by abuse of notation we will denote it by $sk_{n-1}|X|$, because if X is the restriction of a functor from ${}^0\Delta C_{\mathcal{K}}$ then this does give the $(n-1)$-skeleton. Similarly we will denote $Hom_{{}^0\Delta C_{\mathcal{K}_n}}(\mathcal{W}_n, Y)$ by $Tot^{n-1}(Y)$ for a functor $Y : {}^0\Delta C_{\mathcal{K}_n}^{op} \longrightarrow \mathcal{T}op$. In particular, given a pair (A, M) consisting of an A_n algebra A and an A-bimodule M we can define $sk_{n-1}B^{cy}(A; M)$ and $Tot^{n-1}C_{cy}(A; M)$.

Remark 4.2.7. *A natural generalization is to consider the configuration space of points in $\mathbb{R}^m$ modulo translation and dilatation, and the resulting Σ-operad F_m we*

29

get from the Axelrod-Singer compactification of this space. If we have a parallelizable m-manifold M, the compactified configuration space of points on M is naturally a right module over F_m. It might be natural to consider a pointed manifold and based configurations if we want to consider a right F_m-algebra A and an (F_m, A)-module. If M is not parallelizable one has to consider a framed version of this, we refer to [41] for the details.

Chapter 5

A Reedy model category structure

We need conditions which guarantee that a functor $^{01}\Delta_{\mathcal{K}} \longrightarrow \mathcal{T}op$ or $^{0}\Delta C_{\mathcal{K}} \longrightarrow \mathcal{T}op$ is well behaved. In particular, we want to know when the skeletal filtration gives a spectral sequence converging to the homology of the geometric realization. To accomplish this we set up a Reedy model category structure on the category of such functors and their natural transformations, and identify the conditions we need to say that such a functor is cofibrant.

This chapter draws heavily on Chapter 15 in Hirschhorn's book [31].

5.1 Reedy categories

We start by recalling a number of things from [31].

Definition 5.1.1. *A Reedy category is a small category $\mathcal{A}$ together with two subcategories $\overrightarrow{\mathcal{A}}$ (the direct subcategory) and $\overleftarrow{\mathcal{A}}$ (the inverse subcategory), both of which contain all the objects of $\mathcal{A}$, together with a degree function assigning a nonnegative integer to each object in $\mathcal{A}$, such that*

1. *Every non-identity morphism of $\overrightarrow{\mathcal{A}}$ raises degree.*

2. *Every non-identity morphism of $\overleftarrow{\mathcal{A}}$ lowers degree.*

3. *Every morphism $g : S \longrightarrow T$ in $\mathcal{A}$ has a unique factorization*

$$S \xrightarrow{\overleftarrow{g}} U \xrightarrow{\overrightarrow{g}} T \tag{5.1.1}$$

   with $\overleftarrow{g}$ a morphism in $\overleftarrow{\mathcal{A}}$ and $\overrightarrow{g}$ a morphism in $\overrightarrow{\mathcal{A}}$.

The canonical example of a Reedy category is the simplicial indexing category $\boldsymbol{\Delta}$, or rather a skeleton of $\boldsymbol{\Delta}$. In this case $\overrightarrow{\boldsymbol{\Delta}}$ is the subcategory of injective maps and $\overleftarrow{\boldsymbol{\Delta}}$ is the subcategory of surjective maps.

Now let $\mathcal{M}$ be a model category, and suppose X is a functor $\mathcal{A} \longrightarrow \mathcal{M}$.

Definition 5.1.2. *Let S be an object in $\mathcal{A}$. The latching object $L_S X$ is the colimit*

$$L_S X = \varinjlim_{\partial(\vec{\mathcal{A}}/S)} X, \tag{5.1.2}$$

where $\vec{\mathcal{A}}/S$ is the category of objects over S and $\partial(\vec{\mathcal{A}}/S)$ is the full subcategory containing all the objects except the identity on S.

The matching object $M_S X$ is the limit

$$M_S X = \varprojlim_{\partial(S/\overleftarrow{\mathcal{A}})} X, \tag{5.1.3}$$

where $S/\overleftarrow{\mathcal{A}}$ is the category of objects under S and $\partial(S/\overleftarrow{\mathcal{A}})$ is the full subcategory containing all the objects except the identity on S.

By construction there are maps $L_S X \longrightarrow X_S$ and $X_S \longrightarrow M_S X$.

Definition 5.1.3. *Let X and Y be functors $\mathcal{A} \longrightarrow \mathcal{M}$, and let $f : X \longrightarrow Y$ be a natural transformation.*

1. *The map f is a Reedy weak equivalence if each*

$$f_S : X_S \longrightarrow Y_S \tag{5.1.4}$$

   is a weak equivalence.

2. *The map f is a Reedy cofibration if each*

$$X_S \cup_{L_S X} L_S Y \longrightarrow Y_S \tag{5.1.5}$$

   is a cofibration.

3. *The map f is a Reedy fibration if each*

$$X_S \longrightarrow Y_S \times_{M_S Y} M_S X \tag{5.1.6}$$

   is a fibration.

We recall the following theorem, which is due to Dan Kan, from [31, theorem 15.3.4]:

Theorem 5.1.4. *Let $\mathcal{A}$ be a Reedy category and let $\mathcal{M}$ be a model category. Then the category $\mathcal{M}^{\mathcal{A}}$ of functors from $\mathcal{A}$ to $\mathcal{M}$ with the Reedy weak equivalences, Reedy cofibrations and Reedy fibrations is a model category.*

5.2 Enriched Reedy categories

Next we do the same for enriched categories. Let $\mathcal{C}$ be a closed symmetric monoidal model category ([33, definition 4.2.6]), which will be either simplicial sets or spaces in

our applications, and let $\mathcal{M}$ be a $\mathcal{C}$-model category. By this we mean a model category which is enriched, tensored, and cotensored over $\mathcal{C}$, and satisfies the following axioms: (compare [31, definition 9.1.6])

M6. For every two objects X and Y in $\mathcal{M}$ and every object K in $\mathcal{C}$ there are natural isomorphisms

$$Hom(X \otimes K, Y) \cong Hom(K, Hom(X, Y)) \cong Hom(X, Y^K) \qquad (5.2.7)$$

of objects in $\mathcal{C}$.

M7. If $i : A \longrightarrow B$ is a cofibration in $\mathcal{M}$ and $p : X \longrightarrow Y$ is a fibration in $\mathcal{M}$, then the map

$$Hom(B, X) \overset{i^* \times p_*}{\longrightarrow} Hom(A, X) \times_{Hom(A,Y)} Hom(B, X) \qquad (5.2.8)$$

is a fibration in $\mathcal{C}$ that is trivial if either i or p is.

If $\mathcal{C}$ is the category of simplicial sets then these are the extra axioms that make a model category which is enriched, tensored, and cotensored over $\mathcal{C}$ into a simplicial model category. If $\mathcal{C}$ is topological spaces, by which we mean compactly generated weak Hausdorff spaces, then $\mathcal{C}$ is a symmetric monoidal model category ([33, proposition 4.2.11]) and a $\mathcal{C}$-category is sometimes called a topological category.

These axioms have some immediate consequences. For example (compare [31, proposition 9.3.9]) it follows that if $A \longrightarrow B$ is a (trivial) cofibration in $\mathcal{M}$ then so is $A \otimes K \longrightarrow B \otimes K$ for any K in $\mathcal{C}$. Similarly, if $X \longrightarrow Y$ is a (trivial) fibration then so is $X^K \longrightarrow Y^K$ for any K in $\mathcal{C}$.

Definition 5.2.1. *An enriched Reedy category is a small category $\mathcal{A}$ enriched over $\mathcal{C}$ together with an underlying Reedy category $\mathcal{A}_0$ with the same objects as $\mathcal{A}$ and a decomposition of Hom objects in $\mathcal{A}$ as*

$$Hom_{\mathcal{A}}(S, T) = \coprod_{g \in Hom_{\mathcal{A}_0}(S,T)} Hom_{\mathcal{A}}(S, T)_g \qquad (5.2.9)$$

such that if $g = \overrightarrow{g} \overleftarrow{g}$ is the unique factorization of a map g in $\mathcal{A}_0$, then there is a natural isomorphism

$$Hom_{\mathcal{A}}(S, T)_g \cong Hom_{\mathcal{A}}(U, T)_{\overrightarrow{g}} \otimes Hom_{\overleftarrow{g}}(S, U). \qquad (5.2.10)$$

Even though $\mathcal{A}$ has a discrete object set, the same is not true for $\partial(\overrightarrow{\mathcal{A}}/S)$ and $\partial(S/\overleftarrow{\mathcal{A}})$. Thus when defining the latching object $L_S X$, we are forced to take a colimit over a category where both the objects and morphisms are objects in $\mathcal{C}$. See [35, chapter 3] for the general theory of enriched limits and colimits.

Definition 5.2.2. *Let $X : \mathcal{A} \longrightarrow \mathcal{M}$ be a $\mathcal{C}$-functor. The latching object $L_S X$ is the coequalizer*

$$\coprod_{T,U<S} Hom(U, S) \otimes Hom(T, U) \otimes X_T \rightrightarrows \coprod_{T<S} Hom(T, S) \otimes X_T \longrightarrow L_S X, \quad (5.2.11)$$

*where one map is given by the composition $Hom(U, S) \otimes Hom(T, U) \longrightarrow Hom(T, S)$
and the other is given by $Hom(T, U) \otimes X_T \longrightarrow X_U$.*

The matching object $M_S X$ is the equalizer

$$M_S X \longrightarrow \prod_{T < S} Hom(Hom(T, S), X_T) \rightrightarrows \prod_{T, U < S} Hom(Hom(U, S) \otimes Hom(T, U), X_T).$$

$$(5.2.12)$$

The category $\mathcal{A}$ has an obvious filtration, where $F^n \mathcal{A}$ is the full subcategory of $\mathcal{A}$ whose objects have degree less than or equal to n.

Lemma 5.2.3. *([31, theorem 15.2.1, remark 15.2.10]) Suppose X is a functor $F^{n-1} \mathcal{A}$ $\longrightarrow \mathcal{M}$. Extending X to a functor $F^n \mathcal{A} \longrightarrow \mathcal{M}$ is equivalent to choosing, for each object S of degree n, an object X_S and a factorization $L_S X \longrightarrow X_S \longrightarrow M_S X$ of the natural map $L_S X \longrightarrow M_S X$.*

Proof. This uses the unique factorization in the definition of a Reedy category, in the same way as in the proof of [31, theorem 15.2.1]. $\square$

Lemma 5.2.4. *Suppose that for every object T of $\mathcal{A}$ of degree less than S, the map $X_T \cup_{L_T X} L_T Y \longrightarrow Y_T$ is a (trivial) cofibration. Then $L_S X \longrightarrow L_S Y$ is a (trivial) cofibration.*

Similarly, suppose that for every object T of $\mathcal{A}$ of degree less than S the map $X_T \longrightarrow Y_T \times_{M_T Y} M_T X$ is a (trivial) fibration. Then $M_S X \longrightarrow M_S Y$ is a (trivial) fibration.

Proof. We will do the case where each $X_T \cup_{L_T X} L_T Y \longrightarrow Y_T$ is a trivial cofibration, the other cases are similar. Let $E \longrightarrow B$ be a fibration. We have to show that any diagram

$$
\begin{array}{ccc}
L_S X & \longrightarrow & E \\
\downarrow & \nearrow & \downarrow \\
L_S Y & \longrightarrow & B
\end{array}
$$

$$(5.2.13)$$

has a lift. Classically we had to construct a map $Y_T \longrightarrow E$ for each object $T \longrightarrow S$ in $\partial(\vec{\mathcal{A}}/S)$ by induction on the degree of T. We need to make sure that these maps are compatible, so in our case we need to construct a map $Hom(T, S) \otimes Y_T \longrightarrow E$.

We proceed by induction. Suppose we have constructed a map $Hom(U, S) \otimes Y_U \longrightarrow E$ for all U of degree less than T. We then have maps

$$Hom(T, S) \otimes Hom(U, T) \otimes Y_U \longrightarrow Hom(U, S) \otimes Y_U \longrightarrow E \qquad (5.2.14)$$

for each U of degree less than T. These maps assemble to a map $Hom(T, S) \otimes L_T Y \longrightarrow E$. We also have maps $Hom(T, S) \otimes X_T \longrightarrow E$, so we get a diagram

$$
\begin{array}{ccc}
Hom(T, S) \otimes (X_T \cup_{L_T X} L_T Y) & \longrightarrow & E \\
\downarrow & \nearrow & \downarrow \\
Hom(T, S) \otimes Y_T & \longrightarrow & B
\end{array}
$$

$$(5.2.15)$$

By assumption, each map $X_T \cup_{L_T X} L_T Y \longrightarrow Y_T$ is a trivial cofibration, and since we are in a $\mathcal{C}$-model category this remains true after tensoring with $Hom(T, S)$, so we have a lift. $\square$

Lemma 5.2.5. *A map $X \longrightarrow Y$ is a trivial Reedy cofibration if and only if each $X_S \cup_{L_S X} L_S Y \longrightarrow Y_S$ is a trivial cofibration.*

Similarly, $X \longrightarrow Y$ is a trivial Reedy fibration if and only if each $X_S \longrightarrow Y_S \times_{M_S Y} M_S X$ is a trivial fibration.

Proof. Recall that the pushout of a trivial cofibration is a trivial cofibration. Suppose that $f : X \longrightarrow Y$ is a trivial Reedy cofibration. By the previous lemma each $L_S X \longrightarrow L_S Y$ is a trivial cofibration, so when we take the pushout over the map $L_S X \longrightarrow X_S$ we find that the map $X_S \longrightarrow X_S \cup_{L_S X} L_S Y$ is a trivial cofibration. By assumption the composite $X_S \longrightarrow X_S \cup_{L_S X} L_S Y \longrightarrow Y_S$ is a weak equivalence, so by the two out of three axiom so is $X_S \cup_{L_S X} L_S Y \longrightarrow Y_S$.

The other case is similar. $\square$

Theorem 5.2.6. *Suppose $\mathcal{M}$ is a $\mathcal{C}$-model category. Then $\mathcal{M}^{\mathcal{A}}$ with the Reedy weak equivalences, cofibrations and fibrations, is a model category.*

Proof. If we have a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & Y \end{array} \qquad (5.2.16)$$

where $i : A \longrightarrow B$ is a Reedy cofibration and $p : X \longrightarrow$ is a Reedy fibration, with either i or p a weak equivalence, we need to constuct a lift. We can do this by induction on the degree, using the diagrams

$$\begin{array}{ccc} A_S \cup_{L_S A} L_S B & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ B_S & \longrightarrow & Y_S \times_{M_S Y} M_S X \end{array} \qquad (5.2.17)$$

and the previous lemma. $\square$

5.3 The Reedy category $\mathcal{A}_P$

Now let $\mathcal{A} \subset \mathbf{\Delta\Sigma}$ be a subcategory of the category of noncommutative sets which is also a Reedy category, and let P be an operad in $\mathcal{C}$ with $P(0) = P(1) = *$.

Proposition 5.3.1. *The category $\mathcal{A}_P$ is a Reedy category enriched over $\mathcal{C}$.*

Proof. The decomposition of $Hom_{\mathcal{A}_P}(S, T)$ as a coproduct over $Hom_{\mathcal{A}}(S, T)$ is the obvious one. The condition $P(0) = P(1) = *$ ensures that the direct subcategory $\overrightarrow{\mathcal{A}}_P$ is in fact equal to $\overrightarrow{\mathcal{A}}$, and it is easy to see that $\mathcal{A}_P$ satisfies the *unique factorization* condition. $\square$

Corollary 5.3.2. *Let $\mathcal{M}$ be a simplicial model category. Then the category of left $\mathcal{A}_P$ modules $F : \mathcal{A}_P \longrightarrow \mathcal{M}$ is a model category.*

From the proof of proposition 5.3.1 we observe the following:

Observation 5.3.3. *The definition of latching objects for a left $\mathcal{A}$ module or $\mathcal{A}_P$ module, which only depends on the direct limit category, does not change when we pass from $\mathcal{A}$ to $\mathcal{A}_P$. In particular, if $\mathcal{A}$ is either $^{01}\boldsymbol{\Delta}$ or $^0\boldsymbol{\Delta}$ and X is a left $\mathcal{A}_P$ module, the usual description of latching objects in a simplicial category in terms of a coequalizer*

$$\coprod_{0 \le i < j \le n-1} X_{n-2} \rightrightarrows \coprod_{0 \le i \le n-1} X_{n-1} \longrightarrow L_n X \tag{5.3.18}$$

as in [31, proposition 15.2.6] is still valid.

Dually, the usual description of matching objects for a right $\mathcal{A}$ module or $\mathcal{A}_P$ module does not change when we pass from $\mathcal{A}$ to $\mathcal{A}_P$.

Next we show that when $\mathcal{A}_P$ is either $^{01}\boldsymbol{\Delta}_{\mathcal{K}}$ or $^0\boldsymbol{\Delta}C_{\mathcal{K}}$ then the skeletal filtration gives an appropriate spectral sequence.

Theorem 5.3.4. *Let $\mathcal{C}$ be either simplicial sets or topological spaces, and let $\mathcal{K}$ be the associahedra operad in $\mathcal{C}$. Let $\mathcal{M}$ be a pointed $\mathcal{C}$-model category, let $X : {}^{01}\boldsymbol{\Delta}_{\mathcal{K}} \longrightarrow \mathcal{M}$ or $^0\boldsymbol{\Delta}C_{\mathcal{K}} \longrightarrow \mathcal{M}$ be Reedy cofibrant and let E be a homology theory. Then the skeletal filtration gives a spectral sequence*

$$E^2_{p,q} = H_p(E_q(X)) \Longrightarrow E_{p+q}|X|. \tag{5.3.19}$$

Proof. We filter $|X|$ by letting $F^n|X|$ be the image of $\coprod_{i \le n} K_{i+2} \otimes X_i$ or $\coprod_{i \le n} W_{i+1} \otimes X_i$ in $|X|$. As in the classical case, the cofibrancy of X implies that the filtration quotients look like

$$F^n|X|/F^{n-1}|X| \cong (K_{n+2}/\partial K_{n+2}) \wedge X_n \cong \Sigma^n X_n, \tag{5.3.20}$$

and the rest is standard. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

There is also a dual setup for Reedy fibrant right modules.

Theorem 5.3.5. *Let Y be a Reedy fibrant right $^{01}\boldsymbol{\Delta}_{\mathcal{K}}$ module or $^0\boldsymbol{\Delta}C_{\mathcal{K}}$ module, and let E be a homology theory. Then the total space filtration gives a spectral sequence*

$$E_2^{p,q} = H^p(E_q(Y)) \Longrightarrow E_{q-p}Tot(Y). \tag{5.3.21}$$

While the spectral sequence coming from the skeletal filtration usually has good convergence properties, we need additional conditions to guarantee convergence of the spectral sequence coming from the total object filtration. See for example [17] for details.

Chapter 6

Traces and cotraces

6.1 Traces over a P-algebra A

We start by recalling Markl's definition of a trace ([42, definition 2.6]), adapted to right modules over operads (as opposed to Σ-operads). As usual, let P be an operad and let A be a P-algebra. Let $\mathcal{E}_A$ be the endomorphism operad for A, with $\mathcal{E}_A(S) = Hom(A^{\otimes S}, A)$, and let $\mathcal{E}_{A,B}$ be the sequence given by $\mathcal{E}_{A,B}(S) = Hom(A^{\otimes S}, B)$. Then $\mathcal{E}_{A,B}$ is a right $\mathcal{E}_A$-module, and by using the map $P \longrightarrow \mathcal{E}_A$ defining the P-algebra structure on A, a right P-module. Given another right P-module R, we can ask for a map $R \longrightarrow \mathcal{E}_{A,B}$ of right P-modules. Markl defines an R-trace over A (into B) as such a map. This is equivalent to giving maps $R(S) \otimes A^{\otimes S} \longrightarrow B$ for each finite totally ordered set S, such that the diagram

$$
\begin{array}{ccc}
R(T) \otimes P[f] \otimes A^{\otimes S} & \longrightarrow & R(T) \otimes A^{\otimes T} \\
\downarrow & & \downarrow \\
R(S) \otimes A^{\otimes S} & \longrightarrow & B
\end{array}
\tag{6.1.1}
$$

commutes for all maps $f : S \longrightarrow T$ in $\mathbf{\Delta}_+$. Thus an R-trace is simply a map

$$
R \otimes_{(\mathbf{\Delta}_+)_P} A^\bullet \longrightarrow B,
\tag{6.1.2}
$$

where $A^\bullet$ is the functor $S \mapsto A^{\otimes S}$.

6.2 Traces over a pair (A, M)

We will modify this construction so that it applies to the situation where we have a pair (A, M) consisting of a P-algebra A and an A-bimodule M, and a functor $R : {}^0\mathbf{\Delta}C_P^{op} \longrightarrow \mathcal{C}$. Let $S \in {}^0\mathbf{\Delta}C$, and let $\mathcal{E}_{A,M,B}$ be the functor ${}^0\mathbf{\Delta}C_P^{op} \longrightarrow \mathcal{C}$ defined by $\mathcal{E}_{A,M,B}(S) = Hom(M \otimes A^{\otimes S - \{0\}}, B)$. Then we can ask for a natural transformation $R \longrightarrow \mathcal{E}_{A,M,B}$ of functors from ${}^0\mathbf{\Delta}C_P^{op}$ to $\mathcal{C}$.

Definition 6.2.1. *Let A be a P-algebra, M an A-bimodule and $R : {}^0\mathbf{\Delta}C_P^{op} \longrightarrow \mathcal{C}$.*

An R-trace over (A, M) into B is a natural transformation $R \longrightarrow \mathcal{E}_{A,M,B}$ of functors.

As in the classical case, it is easy to find the object corepresenting R-traces. Giving an R-trace over (A, M) into B is equivalent to giving a map

$$R \otimes_{{}^0\boldsymbol{\Delta}C_P} B_\bullet^{cy}(A; M) \longrightarrow B. \tag{6.2.3}$$

In particular, we have the following:

Observation 6.2.2. *With $P = \mathcal{K}_n$, $1 \leq n \leq \infty$, a $\mathcal{W}_n$-trace is corepresented by the partial cyclic bar construction $sk_{n-1}B_\bullet^{cy}(A; M)$ (which exists even if A is only A_n). In particular, a $\mathcal{W}$-trace is corepresented by $B^{cy}(A; M)$.*

Giving an R-trace is equivalent to giving maps $R(S) \otimes M \otimes A^{\otimes S - \{0\}} \longrightarrow B$ for each $S \in {}^0\boldsymbol{\Delta}C$ such that the diagram

$$R(T) \otimes P[f] \otimes M \otimes A^{\otimes S - \{0\}} \longrightarrow R(T) \otimes M \otimes A^{\otimes T - \{0\}} \tag{6.2.4}$$

$$R(S) \otimes M \otimes A^{\otimes S - \{0\}} \longrightarrow B$$

commutes for all $f : S \longrightarrow T$ in ${}^0\boldsymbol{\Delta}C$.

Notation 6.2.3. *If $R(1) = *$, we will call an R-trace over (A, M) into B which restricts to $f : R(1) \otimes M \cong M \longrightarrow B$ an R-trace extending f. If $M = B$ and f is the identity map, we will call an R-trace over (A, M) into M extending f an R-trace on M. If $P = \mathcal{K}$, we say that an A_∞ structure on A together with an A-bimodule structure on M admits a $\mathcal{W}_n$-trace if the identity map on M extends to a $\mathcal{W}_n$-trace.*

We can interpret R in terms of trees in the following way. We picture a map $R(S) \otimes A^{\otimes S - \{0\}} \otimes M \longrightarrow B$ as a tree

$$
\begin{array}{ccc}
A_2 & M & A_1 \\
& \boxed{P(T)} & \\
& B &
\end{array}
\tag{6.2.5}
$$

where the leaves are cyclically ordered and the leaf labelled M is the basepoint. Here A_i is the copy of A labelled by i in $S = \{0, 1, \ldots, n\}$.

Given a map $f : S \longrightarrow T$ in ${}^0\Delta C$, the commutativity of diagram 6.2.4 says that

$$(6.2.6)$$

6.3   Cotraces

We can also reverse the role of B and M, and consider the functor $\tilde{\mathcal{E}}_{B,A,M} : {}^0\Delta C_P^{op} \longrightarrow \mathcal{C}$ defined by $S \mapsto Hom(B \otimes A^{\otimes S - \{0\}}, M)$. In this case, we interpret the operation corresponding to a tree by first rerooting the tree, making the basepoint leaf becomes the new root. For example, after rerooting the tree on the left hand side of diagram 6.2.6, it looks like

$$(6.3.7)$$

Note that the cyclic ordering in diagram 6.2.6 has been replaced by a linear ordering of the A-factors.

Definition 6.3.1. *An R-cotrace of B into (A, M) is a natural transformation $R \longrightarrow \tilde{\mathcal{E}}_{B,A,M}$ of functors.*

A cotrace is represented by a certain object. This is dual to the notion of a trace, but we have to use an extra adjunction, so we present it as a lemma.

Lemma 6.3.2. *R-cotraces are represented by $Hom_{{}^0\Delta C_P}(R, C_{cy}^\bullet(A; M))$.*

Proof. Giving an R-cotrace of B into (A, M) is equivalent to giving maps $R(S) \otimes B \otimes A^{\otimes S - \{0\}} \longrightarrow M$ which satisfy certain coherence relations. But giving maps $R(S) \otimes B \otimes A^{\otimes S - \{0\}} \longrightarrow M$ is equivalent to giving maps $B \longrightarrow Hom(R(S) \otimes A^{\otimes S - \{0\}}, M)$, and the coherence conditions translate into the conditions for equalizing the maps defining $Tot(C_{cy}^\bullet(A; M)) = Hom_{{}^0\Delta C_P}(R, C_{cy}^\bullet(A; M))$. $\square$

Again we single out the associahedra and cyclohedra case:

Observation 6.3.3. *With $P = \mathcal{K}_n$, $1 \leq n \leq \infty$, a $\mathcal{W}_n$-cotrace is represented by the partial cyclic cobar construction $Tot^{n-1}C_{cy}(A; M)$ (which exists even if A is only A_n).*

Giving an R-cotrace is equivalent to giving maps $R(S) \otimes B \otimes A^{\otimes S - \{0\}} \longrightarrow M$ for each $S \in {}^0\Delta C$ such that the diagram

$$
\begin{array}{ccc}
R(T) \otimes P[f] \otimes B \otimes A^{\otimes S - \{0\}} & \longrightarrow & P(f^{-1}(0)) \otimes R(T) \otimes B \otimes A^{\otimes T - \{0\}} \otimes A^{\otimes f^{-1}(0) - \{0\}} \\
\downarrow & & \downarrow \\
& & P(f^{-1}(0)) \otimes M \otimes A^{\otimes f^{-1}(0) - \{0\}} \\
& & \downarrow \\
R(S) \otimes B \otimes A^{\otimes S - \{0\}} & \longrightarrow & M
\end{array}
$$

(6.3.8)

commutes for any $f : S \longrightarrow T$ in ${}^0\Delta C$. Here the top horizontal map is obtained by writing $A^{\otimes S - \{0\}}$ as $A^{\otimes S - f^{-1}(0) - \{0\}} \otimes A^{\otimes f^{-1}(0) - \{0\}}$ and then using using the maps $P(f^{-1}(t)) \otimes A^{\otimes f^{-1}(t)} \longrightarrow A$ for each $t \neq 0$. The fact that this diagram has an extra term corresponds to the fact that rerooting a tree with 2 levels yields a tree with 3 levels, as in diagram 6.3.7.

The example we have in mind is $P = \mathcal{K}$ and $R = \mathcal{W}$, or perhaps $P = \mathcal{K}_n$ and $R = \mathcal{W}_n$. In this case a $\mathcal{W}_n$-trace, $1 \leq n \leq \infty$, is a collection of maps $W_i \times M \times A^{i-1} \longrightarrow B$ for $1 \leq i \leq n$ making diagram 6.2.4 commute for all f. Note that $W_1 = *$, so the starting point is a map $f : M \longrightarrow B$.

If we restrict the trace map $W_n \times M \times A^{n-1} \longrightarrow B$ to one of the n faces of W_n of the form K_n, we get a map

$$
K_n \times M \times A^{n-1} \xrightarrow{1 \times t} K_n \times A^{i-1} \times M \times A^{n-i} \longrightarrow M \xrightarrow{f} B, \qquad (6.3.9)
$$

where t is the cyclic permutation of $M \times A^{n-1}$ which puts the first $n - i$ factors of A at the end and the second map is one of the maps defining the bimodule structure on M. In particular, the existence of a $\mathcal{W}_n$-trace extending the identity map on M says, loosely speaking, that the maps $(m, a_1, \ldots, a_{n-1}) \mapsto a_{n-i+1} \cdots a_{n-1} m a_1 \cdots a_{n-i}$ for $1 \leq i \leq n$ are homotopic in a coherent way.

A $\mathcal{W}_n$-cotrace is a collection of maps $W_i \times B \times A^{i-1} \longrightarrow M$ for $1 \leq i \leq n$ making diagram 6.3.8 commute. The restriction of $W_n \times B \times A^{n-1} \longrightarrow M$ to one of the K_n-faces is given by

$$
K_n \times B \times A^{n-1} \xrightarrow{1 \times \tilde{t}} K_n \times A^{i-1} \times B \times A^{n-i} \xrightarrow{1 \times f \times 1} K_n \times A^{i-1} \times M \times A^{n-i} \longrightarrow M, \quad (6.3.10)
$$

where $\tilde{t}$ is the permutation of $B \times A^{n-1}$ placing B in the i'th position. Note that in this case there is no cyclic permutation of the factors, the cyclic ordering of $M \times A^{n-1}$ has been replaced by a linear ordering of the A-factors in $B \times A^{n-1}$.

In particular, the existence of a $\mathcal{W}_n$-cotrace extending the identity map on M says, loosely speaking, that the maps $(a_1, \ldots, a_{n-1}, m) \mapsto a_1 \cdots a_{i-1} m a_i \cdots a_{n-1}$ for $1 \leq i \leq n$ are homotopic in a coherent way.

Part II

Topological Hochschild homology and cohomology

In chapter 7 we use the machinery from part I to define of THH of an A_∞ ring spectrum with coefficients in a bimodule as the cyclic bar or cobar construction. In particular, we show that our construction is homotopy-invariant, and that our definition is equivalent to the usual definition found for example in [26, chapter IX]. We then use the fact that $THH^R(A; M)$ corepresents traces of (A, M) to give a characterization of when the canonical map $M \longrightarrow THH^R(A; M)$ splits. Dually, cotraces into (A, M) give a characterization of when the canonical map $THH_R(A; M) \longrightarrow M$ splits.

Next we study different kinds of duality between topological Hochschild homology and cohomology. If A is commutative, or even just E_2, then the canonical map $A \longrightarrow THH^R(A)$ makes $THH^R(A)$ into an A-module, and if M is a symmetric A-bimodule we show that there is a duality

$$THH_R(A; M) \simeq F_A(THH^R(A), M). \tag{6.3.11}$$

If A is not at least E_2, then such a statement does not make sense, but we sometimes have a different kind of duality. If $DM \simeq \Sigma^d M$ as an A-bimodule we find that

$$THH_R(A; M) \simeq \Sigma^{-d} F_R(THH^R(A; M), R). \tag{6.3.12}$$

In chapter 8, we improve Robinson's obstruction theory for A_∞ ring spectra [55] to include noncommutative ring spectra. In our setup, the obstructions to an A_∞ structure on A live in what becomes the E_2-term of the spectral sequence converging to $\pi_* THH_R(A)$ associated to the Tot-filtration of $THH_R(A)$ if A turns out to be A_∞. In fact, this setup gives a spectral sequence converging to the space of A_∞ structures on A with a fixed A_n structure for some $n \geq 2$ which looks like a truncated version of the spectral sequence converging to $\pi_* THH_R(A)$ if $THH_R(A)$ exists.

Using this obstruction theory we prove that if R is even and I is a regular ideal, then any homotopy-associative multiplication on $A = R/I$ can be extended to an A_∞ multiplication. We also set up an obstruction theory for extending a map to a $\mathcal{W}$-trace or cotrace.

In chapter 9 we study the moduli space of A_∞ structures more closely. Using results from Strickland [63] we show that the obstruction to the existence of a $\mathcal{W}_2$-trace or cotrace on $A = R/I$ is given by a power operation related to $\mathbb{R}P^2$. In particular this shows that Morava K-theory $K(n)$ does not admit a $\mathcal{W}_2$-trace or cotrace at $p = 2$. We also show that $K(n)$ does not admit a $\mathcal{W}_p$-trace or cotrace for p odd, and identify obstructions that will allow us to say something about $THH_{\widehat{E(n)}}(K(n))$ later.

In chapter 10, we try to calculate $THH(A)$ for $A = R/I$ with R even and I a finitely generated regular ideal, for any A_∞ structure on A, in terms of obstructions to certain cotraces. For example, we find that for any such A, $THH_R(A) \simeq R_A$, the A-localization of R, if and only if a certain matrix which expresses the noncommutativity of the multiplication (A_2 structure) on A is invertible. In many other cases, $\pi_* THH_R(A)$ is a finite extension of $\pi_* R_A$.

We then use the $\mathcal{W}$-cotrace obstructions to calculate THH of Morava K-theory K_1 and $K(1)$, and to make conjectures about $THH(K(n))$ and $THH(K_n)$ for $n > 1$.

We find that while $THH_{E_n}(K_n)$ varies over the moduli space of A_∞ structures on K_n, the map $\pi_* E_n \longrightarrow \pi_* THH_{E_n}(K_n)$ is always injective. We conjecture that this extension is always finite. We also conjecture that $\pi_* \widehat{E(n)} \longrightarrow \pi_* THH_{\widehat{E(n)}}(K(n))$ is a finite, tamely ramified, extension of $\widehat{E(n)}$ which does not depend on the A_∞ structure. These conjectures are all theorems for $n = 1$.

Finally, in chapter 11 we prove that when $R = \widehat{E(n)}$ and $A = K(n)$ or $R = E_n$ and $A = K_n$, the canonical maps

$$THH^S(A) \longrightarrow THH^R(A) \tag{6.3.13}$$

and

$$THH_R(A) \longrightarrow THH_S(A) \tag{6.3.14}$$

are weak equivalences. This tells us that as far as Morava K-theory is concerned, Morava E-theory (or the localized Johnson-Wilson spectrum) is 'close' to the sphere spectrum. This is a manifestation of the Devinatz-Hopkins theorem which says that $L_{K(n)}S \simeq E_n^{h\mathbb{G}_n}$ [21] for the extended Morava stabilizer group $\mathbb{G}_n$, or the reinterpretation of this result as saying that the map $L_{K(n)}S \longrightarrow E_n$ is a $K(n)$-local pro-Galois extension [58].

Chapter 7

THH of A_∞ ring spectra

We are now ready to define topological Hochschild homology and cohomology of an A_∞ ring spectrum. Let R be a commutative S-algebra as in [26]. We will work in the category of R-modules, so all smash products and function spectra will be over R unless the notation suggests otherwise. We will denote the n-fold smash product of A with itself by $A^{(n)}$ and the function spectrum from A to B by $F(A, B)$. We will assume that all R-modules are q-cofibrant.

7.1 The definition of THH

Recall from notation 3.3.1 the definition of $R \otimes_{\mathcal{A}_P} F$ and $Hom_{\mathcal{A}_P}(R, F)$ for functors $F : \mathcal{A}_P \longrightarrow \mathcal{D}$ or $\mathcal{A}_P^{op} \longrightarrow \mathcal{D}$ and $R : \mathcal{A}_P^{op} \longrightarrow \mathcal{C}$. In particular, we can make sense of this when $\mathcal{C} = \mathcal{T}op_*$ and $\mathcal{D}$ is the category of R-modules.

We defined the associahedra (cyclohedra) as an operad (right module) in unbased spaces, but of course we can just define $\mathcal{K}_+$ by $\mathcal{K}_+(S) = \mathcal{K}(S)_+$ and similarly for $\mathcal{W}_+$. Thus we can define the geometric realization of the $\mathcal{K}$-simplicial cyclic bar construction of an A_∞ spectrum in the same way as in part I.

Definition 7.1.1. *Let A be an A_∞ ring spectrum and M an A-bimodule. We define topological Hochschild homology of A with coefficients in M as*

$$THH^R(A; M) = \mathcal{W}_+ \otimes_{{}_0\mathbf{\Delta}C\mathcal{K}_+} B_\bullet^{cy}(A; M), \qquad (7.1.1)$$

where $B_\bullet^{cy}(A; M)(S) = M \wedge A^{(S - \{0\})}$.

Similarly, we define topological Hochschild cohomology of A with coefficients in M as

$$THH_R(A; M) = Hom_{{}_0\mathbf{\Delta}C\mathcal{K}_+}(\mathcal{W}, C_{cy}^\bullet(A; M)), \qquad (7.1.2)$$

where $C_{cy}^\bullet(A; M)(S) = F_R(A^{(S - \{0\})}, M)$.

Our first order of business is to check that our definition is homotopy invariant and compare it to the standard definition.

Proposition 7.1.2. *If A is strictly associative, then our definition of THH agrees with the one given in [26, chapter IX]. Moreover, THH is homotopy invariant in the*

sense that if $A \longrightarrow A'$ is a map of A_∞ ring spectra which is a weak equivalence then $THH(A) \simeq THH(A')$.

Proof. By proposition 4.2.6 our definition agrees with the usual cyclic bar or cobar construction if A is strictly associative, and this is the second definition of THH given in [26, chapter IX]. Recall from [26, theorem VII.4.4] that the category of S-modules, and hence the category of R-modules, is a topological model category. By theorem 5.3.4 and theorem 5.3.5 we get the usual spectral sequences (proposition 7.1.4 below), provided we can show that $B^{cy}_\bullet(A; M)$ is Reedy cofibrant and $C^\bullet_{cy}(A; M)$ is Reedy fibrant. But the conditions for $B^{cy}_\bullet(A; M)$ being cofibrant are the same as in the classical case, because the latching objects are the same (observation 5.3.3), so this follows in the same way as in the classical situation (remember that we assume A is q-cofibrant). The fibrancy of $C^\bullet_{cy}(A; M)$ is similar, using that the matching objects in the opposite category are the same as in the classical case.

Once we have these spectral sequences, a weak equivalence $A \longrightarrow A'$ gives an isomorphism between E_2 terms and thus a weak equivalence between $THH(A)$ and $THH(A')$. $\qquad\square$

Remark 7.1.3. *If A is not q-cofibrant, the correct way to define $THH^R(A)$ is as the geometric realization of a cofibrant replacement of $B^{cy}_\bullet(A; M)$ in the model category structure on functors $^0\Delta C_\mathcal{K} \longrightarrow$ spectra given by theorem 5.2.6. If we use this definition then we always get the spectral sequences for topological Hochschild homology below.*

Similarly, the correct way to define $THH_R(A)$ is as Tot of a fibrant replacement.

For ease of reference, we recall the standard spectral sequences used to calculate the homotopy or homology groups of THH.

Proposition 7.1.4. *([26, chapter IX]) There are spectral sequences*

$$E^2_{s,t} = Tor^{\pi_*(A\wedge_R A^{op})}_{s,t}(A_*, M_*) \Longrightarrow \pi_{s+t}THH^R(A; M), \qquad (7.1.3)$$

$$E^{s,t}_2 = Ext^{s,t}_{\pi_*(A\wedge_R A^{op})}(A_*, M_*) \Longrightarrow \pi_{t-s}THH_R(A; M). \qquad (7.1.4)$$

If E is a commutative R-algebra, or if $E_*(A \wedge_R A^{op})$ is flat over $\pi_*(A \wedge_R A^{op})$, then there are spectral sequences

$$E^2_{s,t} = Tor^{E_*(A\wedge_R A^{op})}_{s,t}(E^R_* A, E^R_* M) \Longrightarrow E_{s+t}THH^R(A; M), \qquad (7.1.5)$$

$$E^{s,t}_2 = Ext^{s,t}_{E_*(A\wedge_R A^{op})}(E^R_* A, E^R_* M) \Longrightarrow E_{t-s}THH_R(A; M). \qquad (7.1.6)$$

Here $E^R_* X$ means $\pi_*(E \wedge_R X)$.

Under reasonable finiteness conditions on each group these spectral sequences converge strongly, see [12, theorem 6.1 and 7.1].

The spectral sequence

$$E^2_{*,*} = Tor^{H_*(A\wedge_S A^{op};\mathbb{F}_p)}_{*,*}(H_*(A; \mathbb{F}_p); H_*(M; \mathbb{F}_P)) \Longrightarrow H_*(THH^S(A; M); \mathbb{F}_p) \quad (7.1.7)$$

is called the Bökstedt spectral sequence, after Marcel Bökstedt who first defined topological Hochschild homology ([14], [15]). We will use the Bökstedt spectral sequence in section 9.2 to identify some $\mathcal{W}$-trace and cotrace obstructions which will help us calculate $THH(K(n))$. For more calculations using the Bökstedt spectral sequence, see for example [47] or [1].

The inclusion of the 0-skeleton gives a map $M \longrightarrow THH^R(A; M)$, and it is well known that when A is an E_∞ R-algebra and $A = M$, then the map $A \longrightarrow THH^R(A)$ is split. This is easy to prove. For example, it follows immediately from the Hopf-algebra strucure on $THH^R(A)$, see [1]. Similarly, there is a map $THH_R(A; M) \longrightarrow M$, and when A is E_∞ and $M = A$, then this map also splits.

Commutativity is a much too strong condition to put on A and M in order to get a splitting, what is needed is a trace or cotrace on M (notation 6.2.3).

Observation 7.1.5. *A splitting of $M \longrightarrow THH^R(A; M)$ amounts to a $\mathcal{W}$-trace on M. More generally, a factorization of $f : M \longrightarrow B$ through $THH^R(A; M)$ amounts to a $\mathcal{W}$-trace from (A, M) to B extending f.*

Observation 7.1.6. *A splitting of $THH_R(A; M) \longrightarrow M$ amounts to a $\mathcal{W}$-cotrace on M. More generally, a factorization of $f : B \longrightarrow M$ through $THH_R(A; M)$ amounts to a $\mathcal{W}$-cotrace from B to (A, M) extending f.*

By a splitting we simply mean a splitting of R-modules. We do not claim that $THH^R(A; M)$ or $THH_R(A; M)$ is an A-module, or that a splitting $A \longrightarrow THH_R(A)$ makes $THH_R(A)$ into one.

7.2 Duality between topological Hochschild homology and cohomology

This section is not strictly necessary for the rest of the paper, but we include it here for completeness.

While the Deligne conjecture implies that topologcial Hochschild cohomology increases the coherence of the multiplication on A from A_∞ to E_2, topological Hochschild homology decreases the coherence from E_n to E_{n-1}. This is due independently by Basterra-Mandell [8] and Fiedorowicz-Vogt [27]. It is not surprising that we often have some kind of duality between topological Hochschild homology and cohomology. In this section we investigate when we get various kinds of duality.

If A is a strictly commutative R-algebra, and M is a symmetric A-bimodule, then it is easy to see that there is a duality

$$THH_R(A; M) \simeq F_A(THH^R(A), M). \tag{7.2.8}$$

Indeed, since $THH^R(A)$ is the geometric realization of a simplicial spectrum with $n \mapsto A^{(n+1)}$, and the levelwise left A-module action on the first factor commutes with the face and degeneracy maps, this gives a left A-module structure on $THH^R(A)$. Thus $F_A(THH^R(A), M)$ is Tot of a cosimplicial spectrum with $n \mapsto F_A(A^{(n+1)}, M) \simeq F_R(A^{(n)}, M)$.

A priori the cosimplicial structure is different than for $THH_R(A; M)$ because the former uses only the left A-module structure on M while the latter uses both the left and right A-module structure. This is why we need M to be a symmetric bimodule.

Let us look at an example of this kind of duality, between $THH^S(H\mathbb{Z}/p)$ and $THH_S(H\mathbb{Z}/p)$. We will present the calculations for p odd, the $p = 2$ case is similar but easier, since there are no differentials in the spectral sequences. We can run the spectral sequences from equation 7.1.3 and 7.1.4 to calculate the homotopy groups in each case. First of all, $\pi_*(H\mathbb{Z}/p \wedge H\mathbb{Z}/p^{op}) \cong A_*$, the dual Steenrod algebra with

$$A_* = \mathbb{Z}/p[\bar{\xi}_1, \bar{\xi}_2, \ldots] \otimes \Lambda(\bar{\tau}_0, \bar{\tau}_1, \ldots), \tag{7.2.9}$$

where we use the conjugate classes because they behave better for this purpose. The E^2-term of the topological Hochschild homology spectral sequence, which is Tor over A_*, is a tensor product of an exterior algebra and a divided power algebra,

$$E^2_{**} = \Lambda(\sigma\bar{\xi}_1, \sigma\bar{\xi}_2, \ldots) \otimes \Gamma(\sigma\bar{\tau}_0, \sigma\bar{\tau}_1, \ldots). \tag{7.2.10}$$

The Bökstedt spectral sequence has our E^2-term tensored with A_*. By Bökstedt's calculations there is a d^{p-1}-differential $d^{p-1}(\gamma_p(\sigma\bar{\tau}_i)) = \sigma\bar{\xi}_{i+1}$, killing all the exterior generators and leaving a truncated polynomial algebra

$$E^p_{**} = E^\infty_{**} = \mathbb{Z}/p[\sigma\bar{\tau}_0, \sigma\bar{\tau}_1, \ldots]/(\sigma\bar{\tau}_i)^p. \tag{7.2.11}$$

The p'th power of $\sigma\bar{\tau}_i$ is only zero modulo lower filtration, and there is a hidden extension $(\sigma\bar{\tau}_i)^p = \sigma\bar{\tau}_{i+1}$. Thus we find that $\pi_* THH^S(H\mathbb{Z}/p) \cong \mathbb{Z}/p[x_2]$ for a class in degree 2. This is Bökstedt's result [15].

By the above duality we then have $\pi_* THH_S(H\mathbb{Z}/p) \cong \Gamma(y_{-2})$, a divided power algebra on a class in degree -2, because the dual of a polynomial algebra is a divided power algebra. We could also run the topological Hochschild cohomology spectral sequence, which has E_2-term

$$E_2^{**} = \Lambda(\delta\bar{\xi}_1, \delta\bar{\xi}_2, \ldots) \otimes \mathbb{Z}/p[\delta\bar{\tau}_0, \delta\bar{\tau}_1, \ldots]. \tag{7.2.12}$$

This time there is a d_{p-1}-differential $d_{p-1}(\delta\bar{\xi}_i) = (\delta\bar{\tau}_{i-1})^p$ killing all the p'th powers of the polynomial generators. The result is a truncated polynomial algebra on classes y_{-2p^i}, which is the same as a divided power algebra on y_{-2}.

This result is somewhat surprising from the algebraic viewpoint, because the polynomial and divided power algebras have switched places. If R is a (differential graded) ring and we define Hochschild homology and cohomology of a projective differential graded R-algebra in the obvious way, then if we agree to write $\mathbb{Z}/p$ for a differential graded projective $\mathbb{Z}$-algebra with homology $\mathbb{Z}/p$ concentrated in degree zero (for example $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ with the obvious multiplication) it is not hard to show that $HH^{\mathbb{Z}}_*(\mathbb{Z}/p) \cong \Gamma(x_2)$ and $HH_{\mathbb{Z}}^*(\mathbb{Z}/p) \cong \mathbb{Z}/p[y_{-2}]$. This is also the result we obtain if we calculate the homotopy groups of $THH^{H\mathbb{Z}}(H\mathbb{Z}/p)$ and $THH_{H\mathbb{Z}}(H\mathbb{Z}/p)$.

If A is not strictly commutative, it is less clear how to make $THH^R(A)$ into an A-module. Indeed, if A is only associative it cannot be done at all. But if A is at least

E_2, and M is an (E_2, A)-module (see equation 2.2.8), then this can be done. To make $THH^R(A)$ into an A-module we assume that A is an algebra over $\mathcal{C}_1 \otimes Ass$, which is an E_2 Σ-operad by [27]. Then we use the multiplication in the Ass-direction to define the simplicial spectrum $B_\bullet^{cy}(A)$, while we use the multiplication in the $\mathcal{C}_1$-direction to define the left A-module structure.

It follows as above that if M is a symmetric A-bimodule we still have the above duality between topological Hochschild homology and cohomology. In particular, equation 7.2.8 still holds with $A = M$ for an E_2 algebra A, because the A-bimodule structure on A is equivalent to a symmetric A-bimodule structure. We will omit the proof.

The other kind of duality we want to consider is a kind of Gorenstein algebra condition. Recall from [25] that one of the conditions for A to be a Gorenstein R-algebra is that the dual $DA = F_R(A, R)$ is equivalent to a suspended copy of A as a left A-module. We need this equivalence between A and $\Sigma^d DA$ to be a duality of A-bimodules. We might as well make our definition for an A-bimodule M instead of just A itself.

Definition 7.2.1. *Let A be an A_∞ R-algebra and let M be an A-bimodule which is dualizable as an R-module. We say that M is a self-dual A-bimodule of dimension d if there is an equivalence $M \longrightarrow \Sigma^d DM$ of A-bimodules, where DM has the A-bimodule structure dual to the A-bimodule structure on M.*

Here is the observation that makes this interesting:

Proposition 7.2.2. *If M is a self-dual A-bimodule of dimension d, then there is a duality*

$$THH_R(A; M) \simeq \Sigma^d F_R(THH^R(A; M), R). \tag{7.2.13}$$

Proof. The point is that both sides are Tot of the same cosimplicial spectrum. The left hand sides is given by $n \mapsto F_R(A^{(n)}, M)$, while the right hand side is given by $n \mapsto \Sigma^d F_R(A^{(n)} \wedge M, R) \simeq F_R(A^{(n)}, \Sigma^d DM) \simeq F_R(A^{(n)}, M)$. It is straightforward to check that the coface and codegeneracy maps match up as long as the equivalence $\Sigma^d DM \simeq M$ is one of A-bimodules. $\square$

Here is the main reason we are interested in this definition. Unfortunately we have not yet been able to prove this, but all our calculations confirm it:

Conjecture 7.2.3. *If $A = R/I$ with R even commutative and I regular, then A is a self-dual bimodule over itself.*

7.3 A circle action on topological Hochschild homology

In this section we sketch two proofs that if A is a $\mathcal{K}$-algebra, then the geometric realization $B^{cy}(A) = B^{cy}(A; A)$ defined using the cyclohedra has a natural S^1-action. While any proof of this result in the classical setting should carry over, we outline

two proofs here. The first is based on Loday's book [40], while the second elaborates on an idea by Drinfeld [22] of writing the geometric realization of a cyclic set as a colimit over finite subsets of S^1. For simplicity we will present the proofs for functors to $\mathcal{T}op$.

We follow [40, section 7.1], adapted to our setting. There is a forgetful fuctor

$$U : \mathcal{C}^{\boldsymbol{\Delta} C_P} \longrightarrow \mathcal{C}^{0\boldsymbol{\Delta} C_P}, \tag{7.3.14}$$

and it has a left adjoint $F : \mathcal{C}^{0\boldsymbol{\Delta} C_P} \longrightarrow \mathcal{C}^{\boldsymbol{\Delta} C_P}$ given by $F(X)_n = C_{n+1} \times X_n$, where we write X_n for $X(S)$ with $|S| = n + 1$ and C_{n+1} is the cyclic group of order $n + 1$. Now let $P = \mathcal{K}$ and let $|X| = \mathcal{W} \otimes_{0\boldsymbol{\Delta} C_\mathcal{K}} X$, and suppose that $\mathcal{C} = \mathcal{T}op$. Let $\mathbf{C}$ denote the functor $\boldsymbol{\Delta} C_\mathcal{K} \longrightarrow \mathcal{T}op$ given by $\mathbf{C}(S) = S$, or in other words $\mathbf{C}_n = C_{n+1}$. By proposition 4.2.6, $|C| \cong S^1$, and as in [40] one can prove the following lemma:

Lemma 7.3.1. *The map* $(pr_1, pr_2) : |F(X)| \longrightarrow |\mathbf{C}| \times |X| = S^1 \times |X|$ *is a homeomorphism.*

Proof. We only sketch the proof. The hard part is showing that $|\mathbf{C} \times X| \cong |\mathbf{C}| \times |X|$. For this we need some kind of Eilenberg-Zilber theorem. If $A = A_{*,*}$ is a *generalized bisimplicial space*, the idea is to show that the total complex $Tot(A)$ is equivalent to the diagonal $d(A)$. This can be accomplished by setting up a spectral sequence with E_2 term

$$E^2_{s,t} = \pi_s(\pi_t A_{*,*}) \tag{7.3.15}$$

converging to both $\pi_{s+t}Tot(A)$ and $\pi_{s+t}d(A)$. This is rather technical, and we omit the details. $\square$

Then, if $X : \boldsymbol{\Delta} C_\mathcal{K} \longrightarrow \mathcal{T}op$, we have a map $ev : \mathbf{C} \times X \longrightarrow X$, and we can define the circle action as $\zeta = |ev| \circ (pr_1, pr_2)^{-1}$. The following theorem follows in the same way as in [40]:

Theorem 7.3.2. *Suppose X is a functor $\boldsymbol{\Delta} C_\mathcal{K} \longrightarrow \mathcal{T}op$. Then*
i) $|X|$ is endowed with a canonical action of S^1.
ii) $X \mapsto |X|$ is a functor from $\mathcal{T}op^{\boldsymbol{\Delta} C_\mathcal{K}}$ to S^1-spaces.

For the second proof, let F be a finite subset of S^1 together with a multiplicity of each point in F, and define $\pi_0(S^1 - F)$ as the cyclically ordered set with one element for each connected component of $S^1 - F$ plus $n - 1$ elements for each multiplicity n point in F. We think of a point with multiplicity n as n points that are very close together, so there should be $n - 1$ connected components between the n points. We have a map $F \longrightarrow F'$ if F' is obtained from F by either adding points or reducing the multiplicity of a multiple point, or if F' is obtained from F by a rotation of S^1. If X is a functor $\boldsymbol{\Delta} C \longrightarrow \mathcal{T}op$, we define the geometric realization of X as

$$|X| = \varinjlim_F X(\pi_0(S^1 - F)). \tag{7.3.16}$$

Note that Drinfeld [22] does not allow multiple points, so his definition of $|X|$ does not involve the face maps on X. As a result he only obtains $|X|$ as a set, and he has

to use some trick to put in the topology. With multiple points we avoid this, and the topology on $|X|$ comes from the topology on the direct limit system.

Now we can generalize this by letting F be a finite subset of S^1 together with a multiplicity of each point and an element of K_n for each n-tuple point. Thus F runs over $\coprod_n S^1 \times W_n$. We have a map $F \longrightarrow F'$ if F' is obtained from F by either adding points or reducing the multiplicity of a point in F according to the element in the given associahedron. If X is a functor $\mathbf{\Delta} C_{\mathcal{K}} \longrightarrow \mathcal{T}op$, then we get an alternative description of $|X|$ as

$$|X| = \varinjlim_{F} X(\pi_0(S^1 - F)). \tag{7.3.17}$$

With this description of $|X|$ it is now obvious that X has an action of S^1, and in fact an action of the group of orientation-preserving homeomorphisms of S^1.

We should note that the usual problem with this kind of definition of topological Hochschild homology persists, in that we do not get a cyclotomic spectrum [13] out of our construction. To get a cyclotomic spectrum one is still forced to do something more sophisticated.

Chapter 8

A_∞ obstruction theory

In this chapter we set up an obstruction theory for extending a homotopy associative multiplication on a spectrum to an A_∞ multiplication. The original reference for this is [55], but Robinson implicitly assumes that the multiplication is homotopy commutative. There is also an issue with unitality, which is addressed in Robinson's more recent paper [56]. Other works on the same subject, such as [29], also assume that the ring spectrum in question is homotopy commutative.

The classification of A_∞ structures comes in two flavors: One where we fix some initial data such as the A_2 structure and build the A_∞ structure one step at a time, and another where we allow ourselves to conjugate the A_∞ structure on A by an appropriately defined automorphism of $\bigvee_n (K_n)_+ \wedge A^{(n)}$. We will not attempt a classification of A_∞ structures up to conjugation, though the reader is encouraged to look at Lazarev's paper [37] to see how this works algebraically.

8.1 The obstructions

Suppose that we have an A_{n-1} structure on a spectrum A, $n \geq 4$, and we want to extend it to an A_n structure. Then we need a map

$$(K_n)_+ \wedge A^{(n)} \longrightarrow A \tag{8.1.1}$$

which is compatible with the A_{n-1} structure. Because all the faces of K_n are products of associahedra of lower dimension, the map $(K_n)_+ \wedge A^{(n)} \longrightarrow A$ is determined on $\partial K_n \wedge A^{(n)} \simeq \Sigma^{n-3} A^{(n)}$. Thus the obstruction to extending the multiplication from an A_{n-1} structure to an A_n structure lies in

$$[\Sigma^{n-3} A, A] = A^{3-n}(A^{(n)}). \tag{8.1.2}$$

The unitality condition on the A_n structure also fixes the map on $(K_n)_+ \wedge s_j(A^{(n-1)})$ for $0 \leq j \leq n-1$, where $s_j : A^{(n-1)} \longrightarrow A^{(n)}$ is given by the unit $S \longrightarrow A$ on the appropriate factor. (This does not quite make sense, as $s_j(A^{(n-1)})$ is not a subset of $A^{(n)}$; what we mean is that the corresponding diagram is required to commute.) Also note that the set of A_{n-1} structures on A, with a given A_{n-2} structure, is isomorphic

to $A^{3-n}(A^{(n-1)})$ as a set, though this set has no group structure. We define a bigraded group $E_1^{*,*}$ by

$$E_1^{s,t} = A^{-t}(A^{(s)}). \tag{8.1.3}$$

There are $s + 2$ maps $A^t(A^{(s)}) \longrightarrow A^t(A^{(s+1)})$, which we denote by d^i for $0 \le i \le s + 1$. Let ϕ denote the A_2 structure. Then d^0 sends $f : A^{(s)} \longrightarrow A$ to $A^{(s+1)} \xrightarrow{1 \wedge f} A^{(2)} \xrightarrow{\phi} A$, d^i sends f to $A^{(s+1)} \xrightarrow{1^{i-1} \wedge \phi \wedge 1^{s-i}} A^{(s)} \xrightarrow{f} A$ for $1 \le i \le s$ and d^{s+1} sends f to $A^{(s+1)} \xrightarrow{f \wedge 1} A^{(2)} \xrightarrow{\phi} A$.

Adding the obvious codegeneracy maps, this structure makes $E_1^{*,*}$ into a graded cosimplicial group, and the obstruction to extending the given A_{n-1} structure lies in the associated normalized cochain complex. Note that using the normalized cochain complex captures the unitality condition.

Using the same geometric argument that Robinson used in [55] we get the following theorem:

Theorem 8.1.1. *Let $n \ge 4$, and suppose we have an A_{n-2} structure on A which can be extended to an A_{n-1} structure in at least one way. Then the obstruction to extending the A_{n-2} structure to an A_n structure, while allowing the A_{n-1} structure to vary, lies in $E_2^{n,n-3}$.*

If the A_{n-2} structure can be extended to an A_n structure, the set of A_{n-1} structures which can be extended to A_n have a free transitive action of $E_2^{n-1,n-3}$.

Proof. The argument in [55] shows that the obstruction to extending the A_{n-1} structure to an A_n structure, which lives in $E_1^{n,n-3}$, maps to zero under d. Similarly, if we change the A_{n-1} structure by an element $f \in E_1^{n-1,n-3}$, Robinson's argument shows that the obstruction changes by df. The last part is similar. $\square$

While this theorem is sufficient for our purpose, an easy extension of the above argument shows that if we fix the A_{n-i-1} structure and allow the A_{n-i} structure to change in such a way that the obstructions to an A_m structure for $n - i < m < n$ remain unchanged, this changes the obstruction to an A_n structure by what we should interpret as a d_i differential, at least as long as $n - i \ge 3$. Moreover, while $E_2^{n,2-n}$ gives the set of connected components in the space of A_n structures which extend to A_{n+1}, $E_2^{n,i+2-n}$ gives π_i of this space.

Under some very reasonable conditions on A, we can then identify $E_2^{s,t}$ with $Ext_{\pi_* A \wedge A^{op}}^{s,t}(A_*, A_*)$. If A is A_∞, then this is the E_2-term of the spectral sequence converging to $\pi_{t-s}THH_S(A)$.

There is a similar story for the obstructions to extending a map $f : A \longrightarrow B$ to a map of A_∞ ring spectra. To be precise, we fix the A_∞ structure on B and allow the the A_∞ structure on A to vary only up to homotopy. Then we need the diagram

$$
\begin{array}{ccc}
(K_n)_+ \wedge A^{(n)} & \longrightarrow & (K_n)_+ \wedge B^{(n)} \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}
\tag{8.1.4}
$$

to commute for some map $(K_n)_+ \wedge A^{(n)} \longrightarrow A$ homotopic to the original A_n structure, or equivalently the original diagram to commute up to homotopy. The two maps $(K_n)_+ \wedge A^{(n)} \longrightarrow B$ agree on ∂K_n, so this gives an obstruction in

$$[\Sigma^{n-2} A^{(n)}, B] \cong B^{2-n}(A^{(n)}), \tag{8.1.5}$$

and one can show that the obstructions really lie in $E_2^{n,n-2}$, where $E_1 = B^{-t}(A^{(s)})$ and the d_1-differential is given in the same way as before.

This obstruction theory also works for an R-module A, where R is a commutative S-algebra, by smashing over R everywhere and replacing $A^*(A^{(n)})$ with $A_R^*(A^{(n)}) = \pi_* F_R(A^{(n)}, A)$ etc.

Before turning to our main application of this obstruction theory, we include a proof that the mod p Moore spectrum is A_{p-1} but not A_p. This result seems to be known by the experts in the field, but there is no proof in the literature except in the cases $p = 2$ and $p = 3$ [50]. See [60] for a comment about this, and an alternative approach that also leads to a proof. First we need a couple of lemmas:

Lemma 8.1.2. *Suppose that A is (-1)-connected, and that A is an A_n ring spectrum. Then the map*

$$A \longrightarrow H\pi_0 A \tag{8.1.6}$$

is a map of A_n ring spectra.

Proof. Using the obstruction theory for maps above we see that the obstructions lie in

$$[\Sigma^{i-2} A^{(i)}, H\pi_0 A] \cong H^{2-i}(A^{(i)}, \pi_0 A) \tag{8.1.7}$$

for $3 \le i \le n$. These groups are all trivial, so there are no obstructions to extending the map to a map of A_n ring spectra. $\square$

Remark 8.1.3. *It is also true that if A is (-1)-connected and an E_n ring spectrum for some $1 \le n \le \infty$, the map $A \longrightarrow H\pi_0 A$ is a map of E_n ring spectra.*

Lemma 8.1.4. *In the dual Steenrod algebra A_* there is a p-fold Massey product*

$$\langle \bar{\tau}_i, \ldots, \bar{\tau}_i \rangle = -\bar{\xi}_{i+1} \tag{8.1.8}$$

defined with no indeterminacy.

Proof. We need two ingredients for this proof. Kochman showed [36, corollary 20] that in the mod p homology of a triple loop space, the p-fold Massey product on a class x in dimension $2n - 1$ is given by $-\beta Q^n(x)$. While his proof does not apply directly to E_∞ (or E_3) ring spectra, the result is still true.

One way to show this is to look at the universal example. Most of the details can be found in [34]. Given a spectrum X, let $\mathbb{P}X$ denote the free commutative ring spectrum on X as in [26, construction II.4.4]. Then the universal example lives in $H_*(\mathbb{P}S^{2n-1}; \mathbb{Z}/p)$. By [18, theorem IX.2.1] (or [39, proposition VII.3.5] and the

corresponding calculation of the homology of QS^{2n-1} in spaces), this homology is given by

$$H_*(\mathbb{P}S^{2n-1}; \mathbb{Z}/p) = F(Q^I \iota), \qquad (8.1.9)$$

where I runs over admissible sequences with excess less than $2n-1$ and $F(S)$ denotes a polynomial algebra on the even-dimensional classes in S and an exterior algebra on the odd-dimensional generators.

We can read off Massey products as differentials in the appropriate spectral sequence coming from the bar construction [46, theorem 8.31], which in this case is the spectral sequence

$$E^2_{*,*} = Tor^{H_*(\mathbb{P}S^{2n-1}; \mathbb{Z}/p)}(\mathbb{Z}/p, \mathbb{Z}/p) \Longrightarrow H_*(\mathbb{P}S^{2n}; \mathbb{Z}/p). \qquad (8.1.10)$$

A counting argument, which is carried out in [34], shows that we do have a differential $d^{p-1}([\iota|\dots|\iota]) = c\beta Q^n(\iota)$ for some $c \neq 0$. If we want to determine the constant c, we can do that by comparing this with the isomorphic spectral sequence $Tor^{H_*(QS^{2n-1}; \mathbb{Z}/p)}(\mathbb{Z}/p, \mathbb{Z}/p) \Longrightarrow H_*(QS^{2n}; \mathbb{Z}/p)$ and use Kochman's result, which applies because this is happening in infinite loop spaces.

This gives us $\langle \bar{\tau}_i, \dots, \bar{\tau}_i \rangle = -\beta Q^{p^i}(\bar{\tau}_i)$. Steinberger calculated how the Dyer-Lashof operations act on the dual Steenrod algebra in [18, theorem III.2.3]. In particular, Steinberger's result says that $\beta Q^{p^i}(\bar{\tau}_i) = \bar{\xi}_{i+1}$, and the lemma follows. $\square$

Theorem 8.1.5. *The mod p Moore spectrum M_p is A_{p-1} but not A_p.*

Proof. First of all, the obstruction to extending an A_{n-1} structure to an A_n structure in $[\Sigma^{n-3} M_p^{(n)}, M_p]$ has to factor through projection onto the top cell in $\Sigma^{n-3} M_p^{(n)}$ by unitality, so the obstruction really lies in $[S^{2n-3}, M_p] = \pi_{2n-3} M_p$. But the homotopy groups of M_p start out with a $\mathbb{Z}/p$ in degree 0, and the next non-zero homotopy group is $\pi_{2p-3} M_p \cong \mathbb{Z}/p$. Thus the obstructions vanish for $2 \leq n \leq p-1$, and the next obstruction group is $\mathbb{Z}/p$, generated by

$$\Sigma^{p-3} M_p^{(p)} \longrightarrow S^{2p-3} \xrightarrow{\alpha_1} S^0 \longrightarrow M_p. \qquad (8.1.11)$$

However, calculating the obstruction inside this obstruction group is more difficult. Instead we consider the map $M_p \longrightarrow H\mathbb{Z}/p$. Suppose that M_p is A_p. According to lemma 8.1.2 this map is a map of A_p ring spectra, and as such it commutes with p-fold Massey products. Taking mod p homology of this map we get a map

$$\Lambda_{\mathbb{F}_p}(\bar{\tau}_0) \longrightarrow A_* \qquad (8.1.12)$$

but this map cannot commute with p-fold Massey products as $\langle \bar{\tau}_0, \dots, \bar{\tau}_0 \rangle$ is visibly zero in $\Lambda_{\mathbb{F}_p}(\bar{\tau}_0)$ and nonzero in A_* by lemma 8.1.4. $\square$

8.2 Quotients of even commutative S-algebras

Now suppose that R is an even commutative S-algebra. By even we mean that R_* is concentrated in even degrees. In this section we will again work in the category

of R-modules [26], and as in chapter 7 we will take all smash products and function spectra to be over R.

Let I be a finitely generated regular ideal in R_*, generated by $(x_1, \ldots, x_n)$ with $|x_i| = d_i$, and let $A = R/I$ as in [63]. Thus $A = R/x_1 \wedge_R \ldots \wedge_R R/x_n$, $A_* = R_*/I$ and A is a cell R-module with 2^n cells. If I is not finitely generated, say, $I = (x_1, x_2, \ldots)$ then we can still define R/I as the telescope of $R/x_i \wedge \ldots \wedge R/x_n$ as in [63, p. 2581] and most of the results in this section go through. Here R/x is defined as the cofiber

$$\Sigma^d R \xrightarrow{x} R \longrightarrow R/x. \tag{8.2.13}$$

Let $\beta_i : R/x_i \longrightarrow \Sigma^{d_i+1} R$ be the next map in the cofiber sequence defining R/x_i, and define $Q_i : R/x_i \longrightarrow \Sigma^{d_i+1} R/x_i$ as the composition of β_i and the unit map $R \longrightarrow R/x_i$.

Remark 8.2.1. *The bockstein $\beta_i : R/x_i \longrightarrow \Sigma^{d_i+1} R$ is not an invariant of the spectrum R/x_i, because it depends on x_i. For example, if we replace x_i by $-x_i$, we also need to replace β_i by $-\beta_i$. The spectrum R/x_i is a 2-cell R-module with cells in degree 0 and $d_i + 1$, so it is dualizable and $D(R/x_i) = F_R(R/x_i, R)$ is also a 2-cell R-module, now with cells in degree 0 and $-d_i - 1$. In fact, $D(R/x_i) \simeq \Sigma^{-d_i-1} R/x_i$, but the equivalence depends on x_i. Since R/x_i is defined by the cofiber sequence*

$$\Sigma^{d_i} R \xrightarrow{x_i} R \longrightarrow R/x_i \longrightarrow \Sigma^{d_i+1} R \xrightarrow{x_i} \Sigma R, \tag{8.2.14}$$

the dual spectrum $D(R/x_i)$ can be defined by the cofiber sequence

$$\Sigma^{-1} R \xrightarrow{x_i} \Sigma^{-d_i-1} R \longrightarrow D(R/x_i) \longrightarrow R \xrightarrow{x_i} \Sigma^{-d_i} R \tag{8.2.15}$$

and we see that there is a map from the $(d_i + 1)$-fold desuspension of the first cofiber sequence to the second one. In fact, using that R/x_i is dualizable as an R-module, we can identify β_i as being adjoint to a map $R \longrightarrow D(R/x_i) \wedge \Sigma^{d_i+1} R \xrightarrow{\simeq} R/x_i$, and this is in fact just the canonical R-module map $R \longrightarrow R/x_i$.

By abuse of notation, let Q_i also denote the corresponding map $A \longrightarrow \Sigma^{d_i+1} A$. It is clear that $Q_i^2 = 0$, and the following result is standard:

Lemma 8.2.2. *([63, proposition 4.15]) The ring $\pi_* F_R(K, K)$ (with composition as the product) is an exterior algebra*

$$\pi_* F_R(K, K) \cong \Lambda_{K_*}(Q_1, \ldots, Q_n), \tag{8.2.16}$$

with $|Q_i| = -d_i - 1$.

Remark 8.2.3. *If I is not finitely generated the above result is still true if we use the completed exterior algebra.*

The following result can also be found in [5, theorem 3.3] and [38, lemma 2.6]:

Proposition 8.2.4. *Given any homotopy associative multiplication on $A = R/I$ with R even and $I = (x_1, x_2, \ldots)$ a regular ideal, $\pi_* A \wedge_R A^{op}$ is given by*

$$\pi_* A \wedge_R A^{op} = \Lambda_{A_*}(\alpha_1, \alpha_2, \ldots) \tag{8.2.17}$$

as a ring. Here $|\alpha_i| = d_i + 1$.

Proof. The proofs in [5] and [38] both use that $\pi_* F_R(A, A)$ is a (completed) exterior algebra together with a Kronecker pairing. Here we present a different proof:

There is a multiplicative Künneth spectral sequence (see [7])

$$E^2_{*,*} = Tor^{R_*}_{*,*}(A_*, A^{op}_*) \Longrightarrow \pi_* A \wedge_R A^{op}. \tag{8.2.18}$$

By using a Koszul resolution of $A_* = R_*/I$ it is easy to see that $E^2_{*,*} = \Lambda_{A_*}(\alpha_1, \alpha_2, \ldots)$ with α_i in bidegree $(1, d_i)$. The spectral sequence collapses, so all we have to do is to show that there are no multiplicative extensions. Because α_i^2 is well defined up to *lower* filtration and $E^2_{1,*}$ is concentrated in odd total degree, it follows that $\alpha_i^2 \in A_* \otimes_{R_*} A^{op}_* \cong A_*$ in $\pi_* A \wedge_R A^{op} = A_* A^{op}$. Now there are several ways to show that $\alpha_i^2 = 0$. If we denote the map $A_* A^{op} \longrightarrow A_*$ by ϵ, it is enough to show that $\epsilon(\alpha^2) = 0$ since ϵ gives an isomorphism from filtration 0 in the spectral sequence to A_*. For example, we can use that A is an $A \wedge_R A^{op}$-module and study the two maps $A_* A^{op} \otimes A_* A^{op} \otimes A_* \longrightarrow A_*$. One sends $\alpha_i \otimes \alpha_i \otimes 1$ to $\epsilon(\alpha_i^2)$, the other one sends it to 0. $\qquad\square$

An extension of the argument in the proof shows that there cannot even be any Toda brackets in $\pi_* A \wedge_R A^{op}$, by comparing brackets formed in $(A \wedge_R A^{op})^{(n)}$ and $(A \wedge_R A^{op})^{(n-1)} \wedge A$.

The above result is *not* true for $A \wedge_R A$, in which case α_i might very well square to something non-zero.

Corollary 8.2.5. *Any homotopy associative multiplication on $A = R/I$ can be extended to an A_∞ structure.*

Proof. Using theorem 8.1.1, the relevant obstructions lie in $Ext^{n,n-3}_{\pi_* A \wedge_R A^{op}}(A_*, A_*)$. In particular the obstructions are in odd total degree. But $\pi_* A \wedge_R A^{op} \cong \Lambda_{A_*}(\alpha_1, \alpha_2, \ldots)$ with $|\alpha_i| = |d_i| + 1$, so Ext over it is a polynomial algebra

$$Ext^{*,*}_{\pi_* A \wedge_R A^{op}}(A_*, A_*) \cong A_*[q_1, q_2, \ldots] \tag{8.2.19}$$

with $|q_i| = (d_i + 1, -1)$, which is concentrated in even total degree, and there can be no obstructions. $\qquad\square$

In particular, this settles [6, conjecture 2.16], where Baker and Lazarev conjecture that any homotopy associative multiplication on R/x can be extended to an A_∞ multiplication.

This also shows that while there are no obstructions to the existence of an A_∞ structure on R/I, the A_∞ structure might not be unique. For example, if A is 2-periodic then the proof shows that we have a power series worth of A_∞ structures.

We go on to list some ring spectra that are automatically A_∞ by corollary 8.2.5. Most of these were already known to be A_∞. For example, with $R = MU$ or $MU_{(p)}$, regular quotients include ku, $H\mathbb{Z}$, $H\mathbb{Z}/p$, BP, $P(n)$, $k(n)$ and $BP\langle n \rangle$.

We can also invert elements in R_* [26, proposition V.2.3], so we can add KU, $B(n)$, $K(n)$ and $E(n)$ to the list. For the homotopy groups of these spectra, see for example [63, p. 2573-2574]

We will also be interested in 2-periodic Morava K-theory. Recall from [54] that given a formal group Γ of height n over a perfect field k of characteristic p, there is a ring spectrum $E_n = E_{(k,\Gamma)}$ with homotopy

$$\pi_* E_n = \mathbb{W}k[[u_1, \ldots, u_{n-1}]][u, u^{-1}], \tag{8.2.20}$$

where $|u_i| = 0$ and $|u| = 2$. Here $\mathbb{W}k$ denotes the Witt ring on k. Also recall from [30] that E_n has the structure of a commutative S-algebra. We define 2-periodic Morava K-theory K_n (we apologize for using the same notation for Morava K-theory and the n'th associahedron) as $K_n = E_n/I$, where $I = (p, u_1, \ldots, u_{n-1})$. Thus

$$\pi_* K_n = k[u, u^{-1}] \tag{8.2.21}$$

and K_n is also A_∞ by corollary 8.2.5. We will see in chapter 11 that we get the same answer when calculating $THH(K_n)$ over the sphere spectrum or over E_n.

It will also be convenient to exhibit $K(n)$ as a quotient of a variant of the Johnson-Wilson spectrum $E(n)$. Let $\widehat{E(n)}$ be the $K(n)$-localization of $E(n)$. This localization has the effect of completing the homotopy of $E(n)$ at the ideal $I = (p, v_1, \ldots, v_{n-1})$, so

$$\pi_* \widehat{E(n)} = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n, v_n^{-1}]_I^{\wedge}. \tag{8.2.22}$$

A variation of the obstruction theory in [30], for example the one based on Γ-cohomology in [57] shows that $\widehat{E(n)}$ is also a commutative S-algebra. Thus we find that $K(n) = \widehat{E(n)}/I$ is a quotient of an even commutative S-algebra by a finitely generated regular ideal. In this case we find that we get the same answer for $THH(K(n))$ when using the sphere spectrum or $\widehat{E(n)}$ as the ground ring.

As in the proof of corollary 8.2.5 we find that if $A = R/I$ with R even commutative and $I = (x_1, \ldots, x_n)$ a finitely generated regular ideal, the spectral sequence from proposition 7.1.4 converging to $\pi_* THH_R(A)$ collapses at the E_2 term and looks like

$$E_2^{*,*} = A_*[q_1, \ldots, q_n] \implies \pi_* THH_R(A). \tag{8.2.23}$$

If $|x_i| = d_i$, then q_i is in bidegree $(d_i + 1, -1)$ and contributes to $\pi_{-d_i-2} THH_R(A)$.

The spectral sequence converging to $\pi_* THH^R(A)$ is similar, with

$$E_{*,*}^2 = \Gamma_{A_*}(\bar{q}_1, \ldots, \bar{q}_n) \implies \pi_* THH^R(A). \tag{8.2.24}$$

Here $\bar{q}_i$ is in bidegree $(d_i + 1, 1)$ and contributes to $\pi_{d_i+2} THH^R(A)$. Because topological Hochschild homology of an A_∞ ring spectrum is in general just a spectrum without a multiplication, this has to be interpreted additively only. There is also

an action of the first spectral sequence on the second coming from the natural map $THH_R(A) \wedge_R THH^R(A) \longrightarrow THH^R(A)$, and one can check that q_i sends $\gamma_k(\bar{q}_i)$ to $\gamma_{k-1}(\bar{q}_i)$.

8.3 Trace and cotrace obstructions

In this section we connect the obstructions to a $\mathcal{W}$-trace or cotrace extending a map $M \longrightarrow B$ or $B \longrightarrow M$ with the A_∞ structure on A. We will characterize the cotrace obstructions first, and then use duality to calculate the trace obstructions in the cases we are interested in.

Suppose we have a $\mathcal{W}_{m-1}$-cotrace extending $f : B \longrightarrow M$, i.e., a factorization of $f : B \longrightarrow M$ as

$$B \longrightarrow Tot^{m-2}THH_R(A; M) \qquad (8.3.25)$$
$$\searrow \quad \downarrow$$
$$M$$

If we want to lift f to a $\mathcal{W}_m$-cotrace we have to give a map $B \longrightarrow F((W_m)_+ \wedge A^{(m-1)}, M)$, or equivalently a map $(W_m)_+ \wedge B \wedge A^{(m-1)} \longrightarrow M$ which is compatible with the $\mathcal{W}_{m-1}$-cotrace. This compatibility determines the map on $\partial W_m \cong S^{m-2}$. Thus the obstruction lies in

$$[\Sigma^{m-2} B \wedge A^{(m-1)}, M] \cong M^{2-m}(B \wedge A^{(m-1)}). \qquad (8.3.26)$$

We let $\mathcal{E}_1^{s,t} = M^{-t}(B \wedge A^{(s)})$, so the obstruction lies in $\mathcal{E}_1^{m-1,m-2}$. The set of possible extensions has, if it is nonempty, a free transitive action of $\mathcal{E}_1^{m-1,m-1}$. Again there is a unitality condition, and as a result the obstructions really lie in the associated normalized cochain complex.

The cyclohedron W_m has m copies of W_{m-1} on its boundary, so changing the $\mathcal{W}_{m-1}$-cotrace by a map

$$f : \Sigma^{m-2} B \wedge A^{(m-2)} \longrightarrow M \qquad (8.3.27)$$

changes the obstruction by the sum of m terms. We let the differential $d : \mathcal{E}_1^{s-1,t} \longrightarrow \mathcal{E}_1^{s,t}$ be given by $d = \sum_{i=0}^{s} d^i f$, where d^0 and d^s use the left, respectively right, A-module structure on M and $d^1, \ldots, d^{s-1}$ are given by precomposing with the multiplication of two adjacent copies of A and define $\mathcal{E}_2^{*,*}$ as the homology of d. $\mathcal{E}_1^{*,t}$ can also be made into a cosimplicial group, by adding the obvious codegeneracy maps.

Next we introduce an assumption which holds in all the cases we will consider, that M^*B is flat over A_*. This implies that we have a Künneth isomorphism

$$M^*B \otimes_{A_*} A^*(A^{(m)}) \xrightarrow{\cong} M^*(B \wedge A^{(m)}) \qquad (8.3.28)$$

sending $(B \xrightarrow{f} M, A^{(m)} \xrightarrow{g} A)$ to $B \wedge A^{(m)} \xrightarrow{f \wedge g} M \wedge A \longrightarrow M$. Then $\mathcal{E}_1^{*,*} \cong M^*B \otimes_{A_*} E_1^{*,*}$, where $E_1^{*,*}$ is the E_1 term of the spectral sequence where the obstructions to an

A_∞ structure on A lives, and also the spectral sequence converging to $\pi_* THH_R(A)$ since we have assumed that A is A_∞. The d_1-differential on $\mathcal{E}_1$ is induced up from the d_1-differential on E_1, and it follows that

$$\mathcal{E}_2^{*,*} \cong M^* B \otimes_{A_*} E_2^{*,*}. \tag{8.3.29}$$

Now let $A = M = R/I$ with R even commutative and $I = (x_1, \ldots, x_n)$ a finitely generated regular ideal, and let $B = R/x_i$. Then $A^* B \cong \Lambda_{A_*}(Q_i)$, and we will write this as $A_* \{1, dq_i\}$. Thus

$$\mathcal{E}_2^{*,*} = A_*[q_1, \ldots, q_n]\{1, dq_i\}. \tag{8.3.30}$$

Here $|q_i| = -d_i - 2$ as before and $|dq_i| = -d_i - 1$.

The obstruction to a $\mathcal{W}_m$-cotrace lies in odd total degree, so we conclude that it looks like

$$r_i(q_1, \ldots, q_n)dq_i \tag{8.3.31}$$

for some homogeneous polynomial r_i of degree $m - 1$ in the q_i's and total degree -1.

Thus the obstructions to a $\mathcal{W}_m$-cotrace extending the natural map $R/x_1 \vee \ldots \vee R/x_n \longrightarrow A$ looks like

$$\sum r_i(q_1, \ldots, q_n)dq_i \tag{8.3.32}$$

for some polynomials $r_i(q_1, \ldots, q_n)$ as above.

It is enough to find a $\mathcal{W}$-cotrace extending each $R/x_i \longrightarrow A$ in the sense that the following holds:

Lemma 8.3.1. *If the natural maps $R/x_i \longrightarrow A$ extend to a $\mathcal{W}$-cotrace for each i, then the identity map on A also extends to a $\mathcal{W}$-cotrace.*

Proof. This relies on the fact that $THH_R(A)$ is an A_∞ ring spectrum, so we can define the map $A = R/x_1 \wedge \ldots \wedge R/x_n \longrightarrow THH_R(A)$ as $R/x_1 \wedge \ldots \wedge R/x_n \longrightarrow THH_R(A) \wedge \ldots \wedge THH_R(A) \longrightarrow THH_R(A)$. $\square$

We also want to see how the obstruction changes if we change the A_m structure on A. Because W_m has m copies of K_m on its boundary, changing the A_m structure changes the obstruction by a sum of m terms.

Lemma 8.3.2. *Fix an A_{m-1} structure on A and a $\mathcal{W}_{m-1}$-cotrace extending $R/x_i \longrightarrow A$. If we change the A_m structure on A by $f(q_1, \ldots, q_n)$ for some polynomial f of degree m in the q_i's and total degree -2, the obstruction to extending the $\mathcal{W}_{m-1}$-cotrace to a $\mathcal{W}_m$-cotrace changes by $\frac{d}{dx_i}f$.*

Proof. Suppose $f(q_1, \ldots, q_n) = aq_1^{j_1} \cdots q_n^{j_n}$. Then changing the A_m structure by f corresponds to changing the map $(K_m)_+ \wedge A^{(m)} \longrightarrow A$ by $\Sigma^{m-2} A^{(m)} \xrightarrow{Q_1^{(j_1)} \wedge \ldots \wedge Q_n^{(j_n)}} \Sigma^{|a|} A^{(m)} \xrightarrow{a} A$. The cotrace obstruction is a map

$$(\partial W_m)_+ \wedge R/x_i \wedge A^{(m-1)} \longrightarrow A, \tag{8.3.33}$$

59

and restricted to one of the faces of the the form K_n this map looks like

$$(K_m)_+ \wedge A^{(k-1)} \wedge R/x_i \wedge A^{(m-k)} \longrightarrow A. \tag{8.3.34}$$

This map only changes if R/x_i is in a position where we apply Q_i to it, and it follows that the obstruction changes on j_i of the faces, which is what we needed to prove. $\quad\square$

Using this lemma it is easy to prove the following:

Theorem 8.3.3. *If $A = R/I$ as above and i is invertible in A_* for $i < p$, then there exists a unique A_{p-1} structure on A such that the natural map $R/x_1 \vee \ldots \vee R/x_n \longrightarrow A$ admits a $\mathcal{W}_{p-1}$-cotrace.*

Next we do the same for $\mathcal{W}$-traces. Given a $\mathcal{W}_{m-1}$-trace on a map $M \longrightarrow B$ we would like to extend it to a $\mathcal{W}_m$-trace. Thus we need a map

$$(W_m)_+ \wedge M \wedge A^{(m-1)} \longrightarrow B \tag{8.3.35}$$

which is determined on $\partial W_m \cong S^{m-2}$ by the $\mathcal{W}_{m-1}$-trace, so the obstruction lies in

$$[\Sigma^{m-2} M \wedge A^{(m-1)}, B] \cong B^{2-m}(M \wedge A^{(m-1)}). \tag{8.3.36}$$

We can make this into a graded simplicial group, and the d_1-differential is given by B-cohomology of the Hochschild homology differential, but in the cases we care about we can actually translate these obstructions into cotrace obstructions.

Let $A = M = R/I$ as above, and let $B = R/x_i$. There is no degree 0 map $M \longrightarrow B$, but by using the bocksteins we get a map

$$R/I \xrightarrow{\beta_{I-i}} \Sigma^{(d_1+\ldots+d_n)-d_i+n-1} R/x_i. \tag{8.3.37}$$

Here β_{I-i} means $\beta_n \circ \ldots \circ \beta_{i+1} \circ \beta_{i-1} \circ \ldots \circ \beta_1$, i.e., we use all the bocksteins except β_i. Using that R/x_i and R/I are dualizable as R-modules, and in fact self-dual up to a dimension shift, this map is adjoint to the natural map

$$R/x_i \longrightarrow R/I. \tag{8.3.38}$$

Thus if we let $B = \Sigma^{(d_1+\ldots+d_n)-d_i+n-1} R/x_i$ we can translate the obstruction to a $\mathcal{W}_m$-trace on the above map into an obstruction living in

$$[\Sigma^{m-2} B \wedge A^{(m-1)}, M]. \tag{8.3.39}$$

We get the following:

Proposition 8.3.4. *Let $A = R/I$ as above. Suppose the canonical map $R/x_i \longrightarrow A$ admits a $\mathcal{W}_{m-1}$-cotrace. Then the corresponding map $A \longrightarrow \Sigma^{(d_1+\ldots+d_n)-d_i+n-1} R/x_i$ admits a $\mathcal{W}_{m-1}$-trace. Conversely, if $A \longrightarrow \Sigma^{(d_1+\ldots+d_n)-d_i+n-1} R/x_i$ admits a $\mathcal{W}_{m-1}$-trace then $R/x_i \longrightarrow A$ admits a $\mathcal{W}_{m-1}$-cotrace.*

The obstruction to extending the $\mathcal{W}_{m-1}$-cotrace to a $\mathcal{W}_m$-cotrace and the obstruction to extending the $\mathcal{W}_{m-1}$-trace to a $\mathcal{W}_m$-trace map to each other under the duality

$$[R/x_i, A] \cong [A, \Sigma^{(d_1 + \ldots + d_n) - d_i + n - 1} R/x_i] \tag{8.3.40}$$

described above.

We record the analog of theorem 8.3.3:

Theorem 8.3.5. *If $A = R/I$ as above and i is invertible in A_* for $i < p$, then there exists a unique A_{p-1} structure on A which admits a $\mathcal{W}_{p-1}$-trace.*

Chapter 9

Classifying the A_∞ structures

In this section we focus on the case $A = R/I$ with R even commutative and I regular. As we can see from the previous section, modulo the first few terms there is a power series worth of A_∞ structures on A under some equivalence relation which fixes the A_n structure for some $n \geq 2$. We need some more detailed information about the set of A_∞ structures. First of all, we need to know when an A_n structure allows a $\mathcal{W}_n$-trace or cotrace. We give a satisfactory answer of this in the first subsection for $n = 2$ using work of Strickland [63].

If A_* is p-local and $n < p$, then theorem 8.3.3 and 8.3.5 say that there is a unique A_n structure with this property. In the second subsection we study the A_p structures in characteristic p. While we have no good general answer, we can still say what happens when A is Morava K-theory, based on a delicate analysis of the Bökstedt spectral sequence for the connective case.

9.1 Homotopy classes of multiplications

In this section we will only deal with homotopy classes of maps, and our goal is to classify the possible A_2 structures on A. We will call an A_2 structure on A a product, to distinguish it from an A_∞ multiplication. Most of this section takes place in the homotopy category, and we are still working with R as the ground ring. Here we are building on work by Strickland [63].

We start by studying products on R/x, where R is an even commutative S-algebra and x is regular. The results we need from Strickland can be summed up in [63, proposition 3.1].

Given a product ϕ on R/x, Strickland shows that ϕ is always homotopy associative, so by corollary 8.2.5 ϕ can be extended to an A_∞ multiplication. By unitality ϕ and ϕ^{op} agree on the bottom 3 cells of $R/x \wedge R/x$ regarded as a 4-cell R-module, so $\phi^{op} - \phi$ factors through projection onto the top cell in $R/x \wedge_R R/x$. Following [63] (and [5]) we define $c(\phi) \in (R/x)_{2d+2}$ by the following equation:

$$\phi^{op} - \phi = c(\phi) \circ (\beta \wedge \beta). \tag{9.1.1}$$

Here $c(\phi)$ depends on x, see remark 8.2.1. We will find it convenient to write ϕ^{op} as

$\phi\big(1 \wedge 1 + c(\phi)Q \wedge Q\big)$. (Recall that Q is the composite $R/x \xrightarrow{\beta} \Sigma^{d+1}R \longrightarrow \Sigma^{d+1}R/x$.)

If $\phi' = \phi\big(1 \wedge 1 + vQ \wedge Q\big)$ for some $v \in (R/x)_{2d+2}$, then $\phi'^{op} = \phi^{op}\big(1 \wedge 1 - vQ \wedge Q\big)$ and we find that $c(\phi') = c(\phi) - 2v$. It follows that if 2 is a unit in $(R/x)_*$, then R/x has a unique commutative product.

If 2 is not invertible, then we can still determine whether or not R/x has a commutative product. Because R is E_∞ (we only need E_2 for this), the map $S^d \wedge S^d \xrightarrow{x \wedge x} R \wedge R \xrightarrow{\phi} R$ factors through $\mathbb{R}P_{2d-1}^{2d+2} \simeq \Sigma^{2d}\mathbb{R}P^2$ and gives a class $P(x) \in R^{-2d}(\mathbb{R}P^2) \cong R_{2d} \oplus R_{2d+2}/2$. Here $\mathbb{R}P_a^b$ denotes a stunted projective space with cells in dimension $a + 1$ through b. Following Strickland, we define $\tilde{P}(x)$ as the projection of $P(x)$ on the second factor (or summand).

Proposition 9.1.1. *([63, proposition 3.1 part 5]) For any product ϕ on R/x, we have*

$$c(\phi) \equiv \tilde{P}(x) \mod (2, x). \tag{9.1.2}$$

Part 2 of [63, proposition 3.1] says that the set of A_2 structures on R/x has a free transitive action of $\pi_{2d+2}R/x$, so the enumeration of the possible A_n structures on $A = R/x$ in terms of power series from the previous section, which a priori only held for $n \geq 3$, is also valid for $n = 2$.

Now let $A = R/I = R/x_1 \wedge_R \ldots \wedge_R R/x_n$ for a regular ideal $I = (x_1, \ldots, x_n)$. By choosing a product ϕ_i on each R/x_i we get a product $\phi = \phi_1 \wedge \ldots \wedge \phi_n$ on A, but we also have *mixed* products, i.e., products that cannot be obtained by smashing together products on each R/x_i. The point is that if ϕ is some product on A, then so is

$$\phi' = \phi\big(1 \wedge 1 + v_{ij}Q_i \wedge Q_j\big) \tag{9.1.3}$$

for any $v_{ij} \in A_{d_i+d_j+2}$.

Theorem 9.1.2. *Fix an associative and unital product ϕ_0 on A. Given any other associative and unital product ϕ on A, it can be written uniquely as*

$$\phi = \phi_0 \prod_{i,j} \big(1 \wedge 1 + v_{ij}Q_i \wedge Q_j\big) \tag{9.1.4}$$

for some $v_{ij} \in \pi_{d_i+d_j+2}A$, where the product denotes composition (which can be taken in any order, because all the factors are even). Moreover, all such products are associative and unital.

Proof. Associativity is some kind of cocycle condition, and one could imagine a simple proof based on this. However, the relevant maps $A^0(A \wedge A) \longrightarrow A^0(A \wedge A \wedge A)$ are not linear, and this complicates things.

We use the Künneth isomorphism

$$A^*A \cong Hom_{A_*}(A_*A, A_*) \tag{9.1.5}$$

and similar formulas for $A^*(A^{(2)})$ and $A^*(A^{(3)})$. These isomorphisms depend on a choice of multiplication, and we will use ϕ_0 for each of them. For example, the map $A^*A \longrightarrow Hom_{A_*}(A_*A, A_*)$ is given by sending $A \xrightarrow{f} A$ to $A_*A \xrightarrow{A_*f} A_*A \xrightarrow{\phi_0} A_*$.

Let $\epsilon : A_*A \longrightarrow A_*$ be the map induced by ϕ_0. To check if ϕ is associative, it is enough to check whether or not the diagram

$$\begin{array}{ccc}
A_*A \otimes_{A_*} A_*A \otimes_{A_*} A_*A & \xrightarrow{\;\phi \wedge 1\;} & A_*A \otimes_{A_*} A_*A \\
{\scriptstyle 1 \wedge \phi}\Big\downarrow & & \Big\downarrow {\scriptstyle \phi} \\
& & A_*A \\
& & \Big\downarrow {\scriptstyle \epsilon} \\
A_*A \otimes_{A_*} A_*A & \xrightarrow{\;\phi\;} A_*A \xrightarrow{\;\epsilon\;} & A_*
\end{array} \qquad (9.1.6)$$

commutes.

Recall that $A^*A \cong \Lambda_{A_*}(Q_1, \ldots, Q_n)$ and that $A_*A \cong \Lambda_{A_*}(\alpha_1, \ldots, \alpha_n)$, at least additively. Under the Künneth isomorphism Q_i corresponds to the map sending α_i to 1.

Now suppose that ϕ is some unital product on A. We can write

$$\phi = \phi_0 \prod_{I,J} \left(1 \wedge 1 + v_{IJ} Q_I \wedge Q_J\right), \qquad (9.1.7)$$

where I and J run over indexes $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_s)$ with $i_1 < \ldots < i_r$ and $j_1 < \ldots < j_s$, $Q_I = Q_{i_1} \cdots Q_{i_r}$ and $Q_J = Q_{j_1} \cdots Q_{j_s}$. Let $|I|$ denote the number of indices in I. By unitality we have $|I| > 0$ and $|J| > 0$, and because A_* is even $|I| + |J|$ has to be even.

If $\phi = \phi_0(1 \wedge 1 + v_{ij} Q_i \wedge Q_j)$, then we can calculate $\phi(\phi \wedge 1)$ and $\phi(1 \wedge \phi)$ using diagram 9.1.6. For example, $\phi(\phi \wedge 1)$ and $\phi(1 \wedge \phi)$ both send $\alpha_i \otimes \alpha_j \otimes 1$ to v_{ij}, as we see by following diagram 9.1.6 around both ways. Similarly, they send $\alpha_i \otimes 1 \otimes \alpha_j$ and $1 \otimes \alpha_i \otimes \alpha_j$ to v_{ij}, and they send $\alpha_i \otimes \alpha_i \alpha_j \otimes \alpha_j$ to $-v_{ij}^2$. Those are all the relevant terms, and shows that

$$\phi(1 \wedge \phi) = \phi(\phi \wedge 1) = \phi_0(\phi_0 \wedge 1) \circ$$
$$\left(v_{ij}(Q_i \wedge Q_j \wedge 1 + Q_i \wedge 1 \wedge Q_j + 1 \wedge Q_i \wedge Q_j) - v_{ij} Q_i \wedge Q_{ij} \wedge Q_j\right). \quad (9.1.8)$$

This shows that any ϕ as in the theorem is associative.

To show that none of the other products are associative, it is enough to show that

$$\phi = \phi_0(1 \wedge 1 + v_{IJ} Q_I \wedge Q_J) \qquad (9.1.9)$$

is not associative for any I, J with $|I| + |J| > 2$. For example, if

$$\phi = \phi_0(1 \wedge 1 + v Q_{ij} \wedge Q_{kl}) \qquad (9.1.10)$$

then $\phi(1 \wedge \phi)$ sends $\alpha_i \alpha_j \otimes \alpha_k \otimes \alpha_l$ to v but $\phi(\phi \wedge 1)$ sends it to zero. $\qquad \square$

Remark 9.1.3. *Alternatively, we can say that given an associative product ϕ on A,*

it can be written as

$$\phi = \phi_0 \prod_{i \neq j} \left(1 \wedge 1 + v_{ij} Q_i \wedge Q_j \right) \tag{9.1.11}$$

for a unique ϕ_0 which comes from products on each R/x_i.

Definition 9.1.4. *We define an $n \times n$ matrix $C(\phi)$ with coefficients in A_* by $C_{ii} = c(\phi_i)$ and $C_{ij} = -v_{ij} - v_{ji}$, where we write ϕ as in remark 9.1.3 and $\phi_0 = \phi_1 \wedge \ldots \wedge \phi_n$. Note that $C(\phi) = 0$ if and only if ϕ is commutative.*

If $f : A \longrightarrow A$ is an automorphism, we get a new product ϕ^f on A by

$$A \wedge A \xrightarrow{f \wedge f} A \wedge A \xrightarrow{\phi} A \xrightarrow{f^{-1}} A. \tag{9.1.12}$$

If $f = 1 + w_{ij} Q_i Q_j$, with inverse $f^{-1} = 1 - w_{ij} Q_i Q_j$, then for $\phi = \prod(1 \wedge 1 + v_{ij} Q_i \wedge Q_j)$ as above, the expression for ϕ^f is given by replacing v_{ij} by $v_{ij} + w_{ij}$ and v_{ji} by $v_{ji} - w_{ij}$. Expanding this argument shows that for any automorphism $f \in Aut_R(A)$ we get $C(\phi^f) = C(\phi)$.

Proposition 9.1.5. *If $\phi' = \phi^f$, then $C(\phi') = C(\phi)$. Conversely, if*

$$\phi = \phi_0 \prod_{i \neq j} \left(1 \wedge 1 + v_{ij} Q_i \wedge Q_j \right) \tag{9.1.13}$$

and

$$\phi' = \phi_0 \prod_{i \neq j} \left(1 \wedge 1 + v'_{ij} Q_i \wedge Q_j \right) \tag{9.1.14}$$

with the same ϕ_0 and $C(\phi) = C(\phi')$, then $\phi' = \phi^f$ for some automorphism f of A.

This proposition almost says that two multiplications are equivalent if and only if they have the same matrix C, but not quite. For example, if ϕ and ϕ^{op} are the two multiplications on $KU/2$, then $c(\phi) = c(\phi^{op}) = u \in \pi_2 KU/2$, but ϕ and ϕ^{op} are not equivalent (in the category of KU-ring spectra, that is). If 2 is invertible, then this problem goes away:

Proposition 9.1.6. *If 2 is invertible in A_*, then the matrix $C(\phi)$ determines ϕ up to conjugation by an automorphism of A.*

The following calculation, which is a special case of [63, proposition 6.2], is essential to Baker and Lazarev's calculation of $THH_{KU}(KU/2)$:

Proposition 9.1.7. *For $R = KU$, $\tilde{P}(2) = u$.*

This shows that neither of the two products on $KU/2$ are commutative, and this is what Baker and Lazarev used to show that $THH_{KU}(KU/2) \simeq KU_2^\wedge$ in [6]

9.2 A_p structures

We now have a good understanding of the number of A_n structures on $A = R/I$ for any n, and if n is invertible in A_* for $n < p$, then we also know that there exists an A_n structure that admits a $\mathcal{W}_n$-trace and cotrace. If $p > 2$ and p is not invertible in A_*, we would like to characterize the A_p structures on A in the same way as we did for $p = 2$ in the previous section. Unfortunately, we have not yet been able to connect the obstruction to the existence of a *good* A_p structure on A with a power operation, but we make the following conjecture.

Conjecture 9.2.1. *Suppose that $2(p-1)|d$. Given an A_{p-1} structure on $A = R/x$ which admits a $\mathcal{W}_{p-1}$-cotrace and an A_p structure ϕ_p extending the A_{p-1} structure, let $c(\phi_p)$ be the obstruction to the existence of a $\mathcal{W}_p$-cotrace. Then we have*

$$c(\phi_p) \equiv \tilde{P}(x) \mod (p, x), \tag{9.2.15}$$

where $\tilde{P}$ is the power operation obtained by extending the map $S^{pd} \xrightarrow{x \wedge \ldots \wedge x} R^{(p)} \longrightarrow R$ over $(B\Sigma_p)_{pd-1}^{pd+2(p-1)} \simeq \Sigma^{pd}(B\Sigma_p)^{2p-2}$, the suspension of the $(2p-2)$-skeleton of $B\Sigma_p$.

In particular, this would show that there is no $\mathcal{W}_p$-trace or cotrace on $K(n)$. But we can show this in another way, by finding A_* comodule extensions in the Bökstedt spectral sequence converging to $H_*(THH^S(k(n)); \mathbb{F}_p)$. This will also supply us with the obstructions to $\mathcal{W}$-cotraces extending the natural maps $\widehat{E(n)}/v_i \longrightarrow K(n)$ for $0 \leq i \leq n-1$ ($v_0 = p$). We will give the argument for odd primes, the $p = 2$ case is similar, after making the usual changes in the notation. Consider the connective Morava K-theory spectrum $k(n)$. From [4] we know that

$$H_*(k(n); \mathbb{F}_p) = P(\bar{\xi}_i | i \geq 1) \otimes E(\bar{\tau}_i | i \neq n). \tag{9.2.16}$$

The calculation of the E^2-term of the Bökstedt spectral sequence converging to $H_*(THH^S(k(n)); \mathbb{F}_p)$ is similar to the calculations found for example in [1, section 5], and we get

$$E^2_{**} = H_*(k(n); \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_i | i \geq 1) \otimes \Gamma(\sigma\bar{\tau}_i | i \neq n). \tag{9.2.17}$$

Because there is no multiplication on $THH^S(k(n))$ this has to be interpreted additively only.

This E^2-term injects into the E^2-term for the corresponding spectral sequence for $H\mathbb{Z}/p$, so the differentials are induced by the corresponding differentials for $H\mathbb{Z}/p$. Thus there is a differential $d^{p-1}(\gamma_p(\sigma\bar{\tau}_i)) = \sigma\bar{\xi}_{i+1}$, and the E^p term looks like

$$E^p_{**} = H_*(k(n); \mathbb{F}_p) \otimes E(\sigma\bar{\xi}_{n+1}) \otimes P_p(\sigma\bar{\tau}_i | i \neq n). \tag{9.2.18}$$

At this point the map of spectral sequences stops being injective, so we can not use this argument to say that the spectral sequence collapses. But we can say that there are no more differentials in low degrees:

Proposition 9.2.2. *The Bökstedt spectral sequence converging to $H_*(k(n); \mathbb{F}_p)$ has no d^n differentials for $n \geq p$ in degree less than $2p^{n+1} - 1$.*

Proof. By comparing with the Bökstedt spectral sequence for $H\mathbb{Z}/p$, any differential has to hit something something which is in the kernel of the map $E^p_{**}(k(n)) \longrightarrow E^p_{**}(H\mathbb{Z}/p)$. The first element in the kernel is $\sigma\bar{\xi}_{n+1}$, which has degree $2p^{n+1} - 1$. $\square$

In particular, this shows that $\bigotimes_{0 \leq i < n} P_p(\sigma\bar{\tau}_i)$ survives to E^∞.

Remark 9.2.3. *If $n = 1$, then one can show ([2]) that the spectral sequence does collapse, by using that the map $\ell \longrightarrow k(1)$ makes the Bökstedt spectral sequence for $k(1)$ into a module spectral sequence over the Bökstedt spectral sequence for ℓ, and using that $E^p_{**}(k(1))$ is generated as a module over $E^p_{**}(\ell)$ by classes in filtration $0 \leq i \leq p - 1$. One could imagine a similar argument with $E^*_{**}(k(n))$ as a module over $E^*_{**}(BP\langle n \rangle)$ if $BP\langle n \rangle$ is at least E_2, though in this case the module generators are in filtration $0 \leq i \leq n(p - 1)$.*

Using that the Bökstedt spectral sequence is a spectral sequence of A_* comodules also restricts the possible differentials. If $d^n(x) = y \neq 0$ with $|x|$ minimal, then y has to be A_* comodule primitive. But this is not enough to show that the spectral sequence collapses in this case.

It might also be possible to use the corresponding spectral sequence converging to $H_*(THH_S(k(n)); \mathbb{F}_p)$. This spectral sequence is better behaved because $THH_S(k(n))$ is an E_2 ring spectrum, and if we can show that it collapses at E_p it might be possible to use the pairing $E^{**}_p \otimes E^p_{**} \longrightarrow E^p_{**}$ coming from $THH_S(k(n)) \wedge THH^S(k(n)) \longrightarrow THH^S(k(n))$ to show that the latter spectral sequence collapses.

Recall that in the corresponding Bökstedt spectral sequence for $H\mathbb{Z}/p$ there are multiplicative extensions $(\sigma\bar{\tau}_i)^p = \sigma\bar{\tau}_{i+1}$. Thus we find that $\sigma(\bar{\tau}_{n-1}(\sigma\bar{\tau}_{n-1})^{p-1}) = \sigma\bar{\tau}_n$ in $H_*(THH^S(H\mathbb{Z}/p), \mathbb{F}_p)$, and more generally $\sigma(\bar{\tau}_i(\sigma\bar{\tau}_i)^{p-1} \cdots (\sigma\bar{\tau}_{n-1})^{p-1}) = \sigma\bar{\tau}_n$. We use this to prove that there are A_* comodule extensions in the Bökstedt spectral sequence for $k(n)$.

Proposition 9.2.4. *Let $x_i = (\sigma\bar{\tau}_i)^{p-1} \cdots (\sigma\bar{\tau}_{n-1})^{p-1}$. Then the A_* comodule action on $\bar{\tau}_i x_i$ is given by*

$$\nu(\bar{\tau}_i x_i) = 1 \otimes \bar{\tau}_i x_i + \sum \bar{\tau}_j \otimes \bar{\xi}^{p^j}_{i-j} x_i - \sum \bar{\tau}_j \otimes \bar{\xi}^{p^j}_{n-j}. \qquad (9.2.19)$$

All the classes in $\bigotimes_{0 \leq i < n} P_p(\sigma\bar{\tau}_i)$ are A_* comodule primitive, and together with the natural A_* comodule structure on $H_*(k(n))$ this determines the A_* comodule structure on $H_*(THH(k(n)); \mathbb{F}_p)$ up to dimension $2p^{n+1} - 1$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} H_*(THH^S(k(n)); \mathbb{F}_p) & \xrightarrow{\ \sigma\ } & H_*(THH^S(k(n)); \mathbb{F}_p) \\ \downarrow & & \downarrow \\ H_*(THH^S(H\mathbb{Z}/p); \mathbb{F}_p) & \xrightarrow{\ \sigma\ } & H_*(THH^S(H\mathbb{Z}/p); \mathbb{F}_p) \end{array} \qquad (9.2.20)$$

The classes in question all survive to E^∞_{**} by proposition 9.2.2, and because $\sigma(\bar{\tau}_i x_i) = 0$ in $H_*(THH^S(k(n)); \mathbb{F}_p)$, the image of $\bar{\tau}_i x_i$ in $H_*(THH^S(H\mathbb{Z}/p); \mathbb{F}_p)$ has to be in the

67

kernel of σ. But $\sigma(\bar{\tau}_i x_i) = \sigma\bar{\tau}_n$ in $H_*(THH^S(H\mathbb{Z}/p); \mathbb{F}_p)$, and the image is given by the element with the same name in the Bökstedt spectral sequence for $H\mathbb{Z}/p$ modulo lower filtration. Thus $\bar{\tau}_i x_i$ in $H_*(THH^S(k(n)); \mathbb{F}_p)$ has to map to $\bar{\tau}_i x_i$ minus something in lower filtration which also maps to $\sigma\bar{\tau}_n$ under σ. The only elements in lower filtration that map to $\sigma\bar{\tau}_n$ are $\bar{\tau}_n$ and $\bar{\tau}_j x_j$ for $j > i$. If necessary we can adjust $\bar{\tau}_i x_i$ by adding elements in lower filtration in the Bökstedt spectral sequence for $k(n)$ so $\bar{\tau}_i x_i$ maps to $\bar{\tau}_i x_i - \bar{\tau}_n$.

Because the map from $H_*(THH^S(k(n)); \mathbb{F}_p)$ to $H_*(THH^S(H\mathbb{Z}/p); \mathbb{F}_p)$ is a map of A_* comodules, it follows that the A_* comodule action on $\bar{\tau}_i x_i$ is as claimed. The claim about all the classes in $\bigotimes_{0 \le i < n} P_p(\sigma\bar{\tau}_i)$ being primitive follows immediately by using that $H_*(THH^S(k(n)); \mathbb{F}_p) \longrightarrow H_*(THH^S(H\mathbb{Z}/p); \mathbb{F}_p)$ is injective in low degrees. $\quad\square$

Now let us see what happens in the Adams spectral sequence with E_2-term $Ext_{A_*}(\mathbb{F}_p, H_*(THH^S(k(n)); \mathbb{F}_p))$ converging to $\pi_* THH^S(k(n))$. (Here Ext means Ext of comodules, as opposed to in the spectral sequence from equation 7.2.12.) Because everything is concentrated in Adams filtration 0 and 1 in degrees less than the degree of v_n^2, we can run the whole Adams spectral sequence in low degrees.

Theorem 9.2.5. In $\pi_* THH^S(k(n))$ there is a relation

$$v_i(\sigma\bar{\tau}_i)^{p-1} \cdots (\sigma\bar{\tau}_{n-1})^{p-1} = v_n. \tag{9.2.21}$$

Proof. First of all, since all the classes in $\bigotimes_{0 \le i < n} P_p(\sigma\bar{\tau}_i)$ are primitive, we get corresponding classes in filtration 0 in the Adams spectral sequence. Also, since $\bar{\tau}_n$ is missing from $H_*(k(n); \mathbb{F}_p)$ we get a class v_n in filtration 1. There are no classes in higher filtration in these degrees, so the classes $\bigotimes_{0 \le i < n} P_p(\sigma\bar{\tau}_i)$ and v_n all survive to $\pi_* THH^S(k(n))$.

Recall, e.g. from [52, p. 63] that v_n is represented by $-\sum \bar{\tau}_i \otimes \bar{\xi}_{n-i}^{p^i}$ in the cobar complex for p odd, with a similar formula for $p = 2$. This also implies that $v_i(\sigma\bar{\tau}_i)^{p-1} \cdots (\sigma\bar{\tau}_{n-1})^{p-1}$ is represented by $-\sum \bar{\tau}_j \otimes \bar{\xi}_{i-j}^{p^j}(\sigma\bar{\tau}_i)^{p-1} \cdots (\sigma\bar{\tau}_{n-1})^{p-1}$.

From the A_* comodule structure we found in proposition 9.2.4, we find that the expressions representing $v_i(\sigma\bar{\tau}_i)^{p-1} \cdots (\sigma\bar{\tau}_{n-1})^{p-1}$ and v_n are homologous, so the two expressions have to be equal in $\pi_* THH^S(k(n))$. $\quad\square$

Chapter 10

Calculations of $THH(A)$ for $A = R/I$

We will continue to work in the category of R-modules. As we have seen, $\pi_* A \wedge_R A^{op}$ is an exterior algebra, so we get a spectral sequence

$$E_2^{*,*} = A_*[q_1, \ldots, q_n] \implies \pi_* THH_R(A). \tag{10.0.1}$$

This spectral sequence collapses, and there can be no multiplicative extensions, but there can be additive extensions. By that we mean that an element in R_* which acts trivially on the E_∞ term actually acts nontrivially on $\pi_* THH_R(A)$.

10.1 What we can say from the A_2 structure

In [6], Baker and Lazarev were able to prove that when $R = KU$ and $A = KU/2 = K(1)$, (with any A_∞ multiplication on $K(1)$) there is such an additive extension, and in fact uq (which is in degree 0) represents multiplication by 2 in $\pi_* THH_{KU}(KU/2)$. This shows that

$$THH_{KU}(KU/2) \simeq KU_2. \tag{10.1.2}$$

Their primary tool was the map $A \wedge_R A^{op} \longrightarrow F_R(A, A)$ adjoint to the action of $A \wedge_R A^{op}$ on A. They used the following piece of Morita theory, a kind of double centralizer theorem which is an easy consequence of the theory developed in [24]:

Theorem 10.1.1. *([6]) For a finite cell R-module A, the map*

$$R \longrightarrow F_{F_R(A,A)}(A, A) \tag{10.1.3}$$

is an A-localization.

Corollary 10.1.2. *If the map $A \wedge_R A^{op} \longrightarrow F_R(A, A)$ is an equivalence, then*

$$THH_R(A) \simeq F_{F_R(A,A)}(A, A) \simeq R_A. \tag{10.1.4}$$

It is plausible that $A \wedge_R A^{op}$ and $F_R(A, A)$ can be equivalent when $A = R/I$ with R even and I finitely generated regular, as the homotopy is an exterior algebra over

A_* on n generators in both cases. (Though the exterior generators are in different degrees.) An extension of the argument in [6] gives the following:

Theorem 10.1.3. *If the matrix $C(\phi)$ from definition 9.1.4 is invertible, we get*

$$THH_R(A) \simeq R_A. \tag{10.1.5}$$

Proof. The map $A \wedge_R A^{op} \longrightarrow F_R(A, A)$ sends α_i to $\sum_j C_{ij}(\phi)Q_j$, so we get an equivalence $A \wedge_R A^{op} \simeq F_R(A, A)$ of R-ring spectra if and only if $C(\phi)$ is invertible. The result then follows from the double commutant formula. $\square$

Corollary 10.1.4. *If A is 2-periodic and either I has at least two generators or 2 is invertible in A_*, then there exists a product ϕ on A with $THH_R(A) \simeq R_A$.*

Proof. The conditions guarantee that there exists a ϕ with $C(\phi)$ invertible in A_*. $\square$

We can say something similar about $THH^R(A)$. While this is not a ring spectrum, we can describe $\pi_* THH^R(A)$ as a module over R_*.

Theorem 10.1.5. *If $C(\phi)$ is invertible, we get*

$$\pi_* THH^R(A) \cong R_*/(x_1^\infty, \ldots, x_n^\infty). \tag{10.1.6}$$

This is all we can say using only the A_2 structure. The A_2 structure tells us what the extensions that increase or decrease the filtration by 1 are, but that is all.

10.2 Extensions to higher filtration

To detect additive extensions that increase the filtration by more than 1, we use the obstruction theory for $\mathcal{W}_m$-cotraces from section 8.3. A factorization of R/x_i through filtration $m-1$ of the Tot-tower for $THH_R(A)$ is the same as a $\mathcal{W}_m$-cotrace, but we can say something slightly stronger:

Proposition 10.2.1. *Suppose $R/x_i \longrightarrow A$ extends to a $\mathcal{W}_{m-1}$-cotrace, and that the obstruction to a $\mathcal{W}_m$-cotrace is*

$$r_i(q_1, \ldots, q_n)dq_i. \tag{10.2.7}$$

Then the extension of x_i to filtration $m-1$ in the spectral sequence converging to $\pi_* THH_R(A)$ is precisely $r_i(q_1, \ldots, q_n)$.

Proof. A $\mathcal{W}_m$-cotrace fits as the dotted arrow in the diagram

$$\Sigma^{d_i} R \xrightarrow{\ x_i\ } R \longrightarrow R/x_i \tag{10.2.8}$$

$$\downarrow \quad \nearrow$$

$$Tot^{m-1}$$

The $\mathcal{W}_m$ obstruction is the obstruction to extending the map $R \longrightarrow Tot^{m-1}$ over the top cell in R/x_i, which is precisely $x_i \in [\Sigma^{d_i} R, Tot^{m-1}] = \pi_{d_i} Tot^{m-1}$. $\square$

70

Corollary 10.2.2. *If A_* is p-local and $R/x_i \longrightarrow A$ extends to a W_{m-1}-cotrace for some $m < p$, then for any homogeneous polynomial $r_i(q_1, \ldots, q_n) \in A_*[q_1, \ldots, q_n]$ of degree $m - 1$ in the q_i's and total degree d_i, there exists an A_m structure on A such that the extension of x_i in the spectral sequence converging to $\pi_* THH_R(A)$ is $r_i(q_1, \ldots, q_n)$, modulo filtration m.*

Thus we can get any extension we want in the spectral sequence, as long as we stay in low filtration.

Something similar holds for topological Hochschild homology, though in that case the extensions lower the filtration by $m - 1$.

10.3   The Morava K-theories

We start by connecting the relation

$$v_i(\sigma\bar{\tau}_i)^{p-1} \cdots (\sigma\tau_{n-1}^-)^{p-1} = v_n \tag{10.3.9}$$

in $\pi_* THH^S(k(n))$ from theorem 9.2.5 to the $\mathcal{W}$-cotrace obstruction theory. Recall that we have a spectral sequence

$$E_{*,*}^2 = \Gamma_{\mathbb{Z}/p}(\bar{q}_0, \ldots, \bar{q}_{n-1})[v_n, v_n^{-1}] \Longrightarrow \pi_* THH^{\widehat{E(n)}}(K(n)). \tag{10.3.10}$$

Proposition 10.3.1. *Under the natural map $THH^S(k(n)) \longrightarrow THH^S(K(n)) \longrightarrow THH^{\widehat{E(n)}}(K(n))$, the class $\sigma\bar{\tau}_i$ in $\pi_* THH^S(k(n))$ maps to $\bar{q}_i$ in $\pi_* THH^{\widehat{E(n)}}(K(n))$.*

Proof. The key fact is that the exterior generator α_i in $\pi_* K(n) \wedge_{\widehat{E(n)}} K(n)^{op}$ which gives rise to $\bar{q}_i$ in the spectral sequence converging to $\pi_* THH^{\widehat{E(n)}}(K(n))$ also lives in $\pi_* k(n) \wedge_S k(n)^{op}$. The rest is a simple matter of comparing two ways to calculate $\pi_* THH^S(k(n))$ in low degrees, either by first running the Bökstedt spectral sequence and then the Adams spectral sequence or by running the Künneth spectral sequence. $\square$

Corollary 10.3.2. *There are additive extensions*

$$v_i\bar{q}_i^{p-1} \cdots \bar{q}_{n-1}^{p-1} = v_n \tag{10.3.11}$$

in the spectral sequence converging to $\pi_* THH^{\widehat{E(n)}}(K(n))$.

Thus multiplication by v_i acts nontrivially on $\pi_* THH^{\widehat{E(n)}}(K(n))$. This means that the canonical map $K(n) \longrightarrow \Sigma^{2(p^n-1)/(p-1)-n-2p^i-1}\widehat{E(n)}/v_i$ (the composite of all the bocksteins except β_i) extends to an $(n-i)(p-1)$-trace but not an $(n-i)(p-1)+1$-trace. By proposition 8.3.4 we can translate this into cotrace obstructions.

Corollary 10.3.3. *The canonical map $\widehat{E(n)}/v_i \longrightarrow K(n)$ extends to a $\mathcal{W}_{(n-i)(p-1)}$-cotrace, and the obstruction to a $\mathcal{W}_{(n-i)(p-1)+1}$-cotrace is*

$$(-1)^{n-i}v_n q_i^{p-1} \cdots q_{n-1}^{p-1}. \tag{10.3.12}$$

71

Proof. We only need to know that there are no obstructions to a $\mathcal{W}_m$-cotrace for $m \leq (n-1)(p-1)$. This follows for degree reasons; there are no possible obstructions to a $\mathcal{W}_m$-cotrace for $m \leq (n-i)(p-1)$. The extra sign is there because q_i^{p-1} is dual to $\gamma_{p-1}(\bar{q}_i)$, which is $-\bar{q}_i^{p-1}$. $\qquad\qquad\square$

By proposition 10.2.1 we then have an extension $v_n q_i^{p-1} \cdots q_{n-1}^{p-1} = (-1)^{n-i} v_i$ in filtration $(n-i)(p-1)$ in the spectral sequence converging to $\pi_* THH_{\widehat{E(n)}}(K(n))$.

We have not yet addressed what happens in higher filtration. We only know that v_i is congruent to $(-1)^{n-i} v_n q_i^{p-1} \cdots q_{n-1}^{p-1}$ modulo higher filtration. At least in the case $n = 1$ this is all we need to know to write down a precise answer. We will do this first, before trying to calculate $\pi_* THH_{\widehat{E(n)}}(K(n))$ for $n > 1$.

When $n = 1$ we have a spectral sequence

$$E_2^{*,*} = \mathbb{Z}/p[v_n, v_n^{-1}][q] \implies \pi_* THH_{\widehat{E(1)}}(K(1)). \qquad (10.3.13)$$

If we have hidden extensions where $p = f(q) = -v_1 q^{p-1} + \ldots$, we get

$$\pi_* THH_{\widehat{E(1)}}(K(1)) \cong \mathbb{Z}_p[v_1, v_1^{-1}][[q]]/(p - \tilde{f}(q)), \qquad (10.3.14)$$

where $\tilde{f}$ is any lift of f to $\mathbb{Z}_p[v_1, v_1^{-1}][[q]]$.

Theorem 10.3.4. *For any A_∞ structure on $K(1)$, the homotopy groups of topological Hochschild cohomology of $K(1)$ are given by*

$$\pi_* THH_{\widehat{E(1)}}(K(1)) \cong \widehat{E(1)}_*[[q]]/(p + v_1 q^{p-1}), \qquad (10.3.15)$$

which is a tamely ramified extension of $\widehat{E(1)}_*$ of degree $p - 1$.

Proof. By corollary 10.3.3 and the following discussion,

$$\pi_* THH_{\widehat{E(1)}}(K(1)) \cong \widehat{E(1)})_*[[q]]/(p - \tilde{f}(q)), \qquad (10.3.16)$$

where $\tilde{f}$ is a lift of some power series $f(q) = -v_1 q^{p-1} + \ldots$ to characteristic zero. The result now follows by the Weierstrass preparation theorem. $\qquad\square$

This recovers Baker and Lazarev's result about $THH_{\widehat{E(1)}}(K(1))$ for $p = 2$ as a special case.

If we instead consider topological Hochschild homology we get the following:

Theorem 10.3.5. *The homotopy groups of topological Hochschild homology of $K(1)$ are given by $\pi_* THH^{\widehat{E(1)}}(K(1)) \cong \mathbb{Z}/p^\infty$ for i even and zero for i odd.*

For general n we have n power series $f_0(q_0, \ldots, q_{n-1}), \ldots, f_{n-1}(q_0, \ldots, q_{n-1})$ in $K(n)_*[[q_0, \ldots, q_{n-1}]]$. We know that

$$\pi_* THH_{\widehat{E(n)}}(K(n)) \cong \widehat{E(n)}_*[[q_0, \ldots, q_{n-1}]]/(p - \tilde{f}_0, \ldots, v_{n-1} - \tilde{f}_{n-1}) \qquad (10.3.17)$$

72

for lifts $\tilde{f}_0, \ldots, \tilde{f}_{n-1}$ of power series $f_0, \ldots, f_{n-1}$ to $\widehat{E(n)}_*[[q_0, \ldots, q_{n-1}]]$. Here f_i has leading term $(-1)^{n-i} v_n q_i^{p-1} \cdots q_{n-1}^{p-1}$, but we do not know enough of the coefficients of the f_i's to identify this ring precisely.

We conjecture that things work the same way as for $n = 1$, in the following sense:

Conjecture 10.3.6. *The ring $\pi_* THH_{\widehat{E(n)}}(K(n))$ is independent of the A_∞ structure on $K(n)$ and is a finite extension of $\widehat{E(n)}_*$.*

We know enough to say that after inverting the maximal ideal in $\pi_* \widehat{E(n)}$ we get a finite extension, because we can rearrange the relations to read $v_i = -v_{i+1} q_i^{p-1} + \ldots$. If the rest of the coefficients in the f_i's are generic enough, for example if we operate under the assumption that any extension that can happen does happen, then we do get a finite extension.

The topological Hochschild homology calculations are similar. In particular, we find $\pi_* THH^{\widehat{E(n)}}(K(n))$ is always infinitely divisible by $p, v_1, \ldots, v_{n-1}$.

The spectrum $THH_{\widehat{E(1)}}(K(1))$ cannot be E_∞, as one can see by considering suitable power operations in $K(1)$-local E_∞ ring spectra. Recall, e.g. from [53] that a $K(1)$-local E_∞ ring spectrum T (which has to satisfy a technical condition which we do not have to worry about here) has power operations ψ and θ such that (in particular) ψ is a ring homomorphism and

$$\psi(x) = x^p + p\theta(x) \qquad (10.3.18)$$

for $x \in T^0 X$. Now, if T_* has an i'th root of some multiple of p, say, $\zeta^i = ap$ for a unit a and $i > 1$, then we get

$$ap = \psi(\zeta)^i = (\zeta^p + p\theta(\zeta))^i, \qquad (10.3.19)$$

and the right hand side is divisible by p^2 while the left hand side is not. In particular, we can apply this to $T = THH_{\widehat{E(1)}}(K(1))$ as above to show that this cannot be an E_∞ ring spectrum.

We believe that a similar argument shows that $THH_{E(n)}(K(n))$ or $THH_{E_n}(K_n)$ can never be E_∞, except in the cases when $THH_{E_n}(K_n) \simeq E_n$.

If we want to calculate $THH_{E_n}(K_n)$, the result depends on the A_∞ multiplication.

Theorem 10.3.7. *For any p and n there exists A_∞ structures on K_n such that*

$$THH_{E_n}(K_n) \simeq E_n. \qquad (10.3.20)$$

Proof. This follows by corollary 10.1.4, except if $n = 1$ and $p = 2$. This last case is covered by theorem 10.3.4. $\square$

This is the generic result, since it happens whenever the matrix $C(\phi)$ is invertible. If we choose a more *commutative* multiplication on K_n, we get

$$\pi_* THH_{E_n}(K_n) \cong (E_n)_*[[q_0, \ldots, q_{n-1}]]/(p - \tilde{f}_0, \ldots, u_{n-1} - \tilde{f}_{n-1}) \qquad (10.3.21)$$

73

for some $\tilde{f}_i \in (E_n)_*[[q_0, \ldots, q_{n-1}]]$ (with leading term of degree at least 1 in the q_i's). By comparing the A_∞ structures on K_n with those on $K(n)$ we also know that the reduction of $\tilde{f}_i$ to $(K_n)_*[[q_0, \ldots, q_{n-1}]]$ has a term $(-1)^{n-i} u^{(n-i)(p-1)} q_i^{p-1} \cdots q_{n-1}^{p-1}$.

In the case $n = 1$ we can describe $THH_{E_1}(K_1)$ more explicitly. If the A_2 structure on K_1 is noncommutative, we get E_1. If the A_2 structure is commutative but the A_∞ structure does not admit a $\mathcal{W}_3$-cotrace, we get a degree 2 extension, $\pi_* THH_{E_1}(K_1) = (E_1)_*[[q]]/(p - au^2q^2)$ for some a. If the A_∞ structure admits a $\mathcal{W}_3$-cotrace but not a $\mathcal{W}_4$-cotrace, we get a degree 3 extension, and so on. But we never have a $\mathcal{W}_p$-cotrace, so the degree of the extension is always between 1 and $p - 1$.

The corresponding statement for $THH^{E_1}(K_1)$ is that $\pi_i THH^{E_1}(K_1)$ is some number of copies of $\mathbb{Z}/p^\infty$ for i even, and the number of copies is always between 1 and $p - 1$.

Chapter 11

THH of Morava K-theory over S

In this section we prove that $THH(K(n))$ and $THH(K_n)$ do not depend on the ground ring. By this we mean that the canonical maps

$$THH^S(K_n) \longrightarrow THH^{E_n}(K_n) \tag{11.0.1}$$

and

$$THH_{E_n}(K_n) \longrightarrow THH_S(K_n) \tag{11.0.2}$$

are weak equivalences, and similarly for $K(n)$ using $\widehat{E(n)}$ instead of E_n. The earliest incarnation of this equivalence can be found in [55], where Robinson observed that for p odd the t_i's in

$$\pi_*(K(n) \wedge_S K(n)) \cong K(n)_*[\alpha_0, \ldots, \alpha_{n-1}, t_1, t_2, \ldots]/(\alpha_i^2, v_n t_i^{p^n} - v_n^{p^i} t_i) \tag{11.0.3}$$

do not contribute to the *Ext* groups $Ext^{**}_{\pi_* K(n) \wedge_S K(n)}(K(n)_*, K(n)_*)$. Something similar is true at $p = 2$ if we use $K(n) \wedge_S K(n)^{op}$. While α_i squares to t_{i+1} instead of 0 in this case, the *Ext* calculation is still valid.

Much of the material in this section comes from [54], where Rezk does something similar to show that certain derived functors of derivations vanish. We have also used ideas from [32].

We expect THH to be invariant under change of ground ring from S to E_n, or the other way around, because something similar holds algebraically.

Lemma 11.0.8. *Let $R \longrightarrow R'$ be a Galois extension of rings and suppose A is an R' algebra. Then the canonoical maps*

$$HH_*^R(A) \longrightarrow HH_*^{R'}(A) \tag{11.0.4}$$

and

$$HH_{R'}^*(A) \longrightarrow HH_R^*(A) \tag{11.0.5}$$

are isomorphisms

Proof. Recall from [65] that Hochschild homology satisfies étale descent and Galois descent. Étale descent shows that $HH_*^{R'}(A) \cong HH_*^R(A)$ when $A = A' \otimes_R R'$, and then

Galois descent shows that it holds for any A. The cohomology case is similar. □

Now, if E_n is the Morava E-theory associated to the Honda formal group over F_{p^n}, Rognes describes [58, section 5.4] how the unit map $S \longrightarrow E_n$ is a $K(n)$-local (or K_n-local) pro-Galois extension with Galois group $\mathbb{G}_n$, the extended Morava stabilizer group. Similarly, $S \longrightarrow \widehat{E(n)}$ is a $K(n)$-local Galois extension with the slightly smaller Galois group $\mathbb{G}_n/K$ for $K = \mathbb{F}_{p^n}^\times \times Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$, so we expect the result, if not the proof, to carry over.

11.1 Perfect algebras

Let A and B be commutative $\mathbb{F}_p$-algebras, and suppose $i : A \longrightarrow B$ is an algebra map. There is a Frobenius map F sending x to x^p on each of these $\mathbb{F}_p$-algebras. Let A^F denote A regarded as an A-algebra using the Frobenius F. Now we can define a relative Frobenius $F_A : A^F \otimes_A B \longrightarrow B$ as $F_A(a \otimes b) = i(a)b^p$ on decomposable tensor factors:

$$
\begin{array}{ccc}
A & \overset{i}{\longrightarrow} & B \\
\scriptstyle F \downarrow & & \downarrow \\
A & \longrightarrow & A^F \otimes_A B \overset{F}{\searrow} \\
& \overset{i}{\searrow} & \searrow \\
& & B
\end{array}
\tag{11.1.6}
$$

Definition 11.1.1. *We say that $i : A \longrightarrow B$ is* perfect *if $F_A : A^F \otimes_A B \longrightarrow B$ is an isomorphism.*

This definition specializes to the usual definition of a perfect $\mathbb{F}_p$-algebra when $A = \mathbb{F}_p$.

Now suppose that $i : A \longrightarrow B$ has an augmentation $\epsilon : B \longrightarrow A$. Let $I = ker(\epsilon)$ be the augmentation ideal, so that $B \cong A \oplus I$ additively.

Lemma 11.1.2. *For $i \geq 0$ and any B-module M we have*

$$
Tor_i^B(I, M) \cong Tor_{i+1}^B(A, M) \tag{11.1.7}
$$

and

$$
Ext_B^i(I, M) \cong Ext_B^{i+1}(A, M). \tag{11.1.8}
$$

Proof. This follows by choosing a resolution like

$$
A \longleftarrow A \oplus I \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \dots, \tag{11.1.9}
$$

of A, where $P_0 \longleftarrow P_1 \longleftarrow \dots$ is a projective resolution of I as a B-module. □

Now, if $i : A \longrightarrow B$ is perfect, we have an isomorphism $F_A : A^F \otimes_A (A \oplus I) \longrightarrow A \oplus I$, and this gives an isomorphism $F_A : A^F \otimes_A I \longrightarrow I$ of non-unital algebras.

Now suppose that M is an A-module, and regard M as a B-module via ϵ. Then I acts as zero on M, and we use that to prove the following:

Proposition 11.1.3. *For $i : A \longrightarrow B$ perfect and M any A-module viewed as a B-module via the augmentation $\epsilon : B \longrightarrow A$ we have*

$$Tor_B^i(I, M) = 0 \tag{11.1.10}$$

and

$$Ext_B^i(I, M) = 0 \tag{11.1.11}$$

for all i.

Proof. We show that the maps

$$(A^F \otimes_A I) \otimes_B M \xrightarrow{F_A \otimes 1} I \otimes_B M \tag{11.1.12}$$

and

$$Hom_B(I, M) \xrightarrow{F_A^*} Hom_B(A^F \otimes_A I, M) \tag{11.1.13}$$

are both isomorphisms and zero. They are isomorphism because $i : A \longrightarrow B$ is perfect. They are zero because, for example, given any map $f : I \longrightarrow M$ of B-modules, we find that $F_A^* f$ is given by $F_A^* f(a \otimes b) = f(ab^p) = b^{p-1} f(ab) = 0$, so $F_A^* f$ is zero. The same argument applies to a projective resolution of I to show that $Ext_B^i(I, M) = 0$ for $i > 0$. The argument for Tor is similar. $\square$

Combining the above two results we get the following:

Theorem 11.1.4. *Suppose that $i : A \longrightarrow B$ is perfect, and let M be any A-module regarded as a B-module via the augmentation $\epsilon : B \longrightarrow A$. Then we get*

$$Tor_i^B(A, M) = 0 \tag{11.1.14}$$

and

$$Ext_B^i(A, M) = 0 \tag{11.1.15}$$

for $i > 0$, while $A \otimes_B M \cong M$ and $Hom_B(A, M) \cong M$.

11.2   Formal groups

Most of what we need to know about formal groups can be found in [54]. Recall that given two Morava E-theories E and F of the same height, the maximal ideals in $E_0 F$ coming from m_E and m_F coincide. Furthermore, $E_0 F/m$ represents isomorphisms of formal group laws. Let $W(\Gamma_1, \Gamma_2) = k_1 \otimes_L W \otimes_L k_2$, where L is the Lazard ring (isomorphic to MU_*, or MUP_0, where MUP is the 2-periodic complex cobordism spectrum) and $W = L[t_0^{\pm 1}, t_1, \ldots]$.

Proposition 11.2.1. *([54, remark 17.4]) If E and F are the Morava E-theories associated to two formal groups Γ_1 and Γ_2 of height n, then*

$$(\pi_0 E \wedge_S F)/m \cong W(\Gamma_1, \Gamma_2). \tag{11.2.16}$$

Proposition 11.2.2. *([54, corollary 21.6]) The ring $W(\Gamma_1, \Gamma_2)$ is a perfect k_1-algebra.*

Proposition 11.2.3. *Given any multiplication on K, we have*

$$\pi_0(K \wedge_S K^{op}) \cong \big(\pi_0(E \wedge_S E)\big)/m \otimes \Lambda(\alpha_0, \ldots, \alpha_{n-1}). \tag{11.2.17}$$

additively, and each α_i squares to something that acts trivially on K_*.

Proof. This is clear additively, and the claim about the multiplicative structure follows as in the proof of proposition 8.2.4. □

Now we are in a position to prove the following theorem:

Theorem 11.2.4. *Let E be either E_n or $\widehat{E(n)}$. If $E = E_n$ let $K = K_n$ and if $E = \widehat{E(n)}$ let $K = K(n)$. Then the canonical maps*

$$THH^S(K) \longrightarrow THH^E(K) \tag{11.2.18}$$

and

$$THH_E(K) \longrightarrow THH_S(K) \tag{11.2.19}$$

are weak equivalences.

Proof. We have spectral sequences calculating π_* of both sides, where the E_2-terms are $Tor_{\pi_*(K \wedge_E K^{op})}(K_*, K_*)$ and $Tor_{\pi_*(K \wedge_S K^{op})}(K_*, K_*)$ in the first case and corresponding Ext groups in the second case. For $(E, K) = (E_n, K_n)$, proposition 11.2.3 shows that the E_2-terms are isomorphic, and since the isomorphisms are induced by the obvious maps this proves the theorem.

The case $(E, K) = (\widehat{E(n)}, K(n))$ is similar, using $L[t_1, t_2, \ldots]$ instead of W. □

One interesting consequence of this theorem is the following:

Corollary 11.2.5. *Let E be either E_n or $\widehat{E(n)}$. If $E = E_n$ let $K = K_n$ and if $E = \widehat{E(n)}$ let $K = K(n)$. Then the spaces of A_∞ E-algebra structures on K and A_∞ S-algebra structures on K are equivalent.*

78

Bibliography

[1] V. Angeltveit and J. Rognes. Hopf algebra structure on topological hochschild homology. *Algebraic and Geometric Topology*, 5(49):1223–1290, 2005.

[2] Christian Ausoni and John Rognes. Algebraic k-theory of the fraction field of topological k-theory. *In preparation*.

[3] Scott Axelrod and I. M. Singer. Chern-Simons perturbation theory. II. *J. Differential Geom.*, 39(1):173–213, 1994.

[4] Nils Andreas Baas and Ib Madsen. On the realization of certain modules over the Steenrod algebra. *Math. Scand.*, 31:220–224, 1972.

[5] A. Baker and A Jeanneret. Brave new Bockstein operations. *Preprint*. Available at http://www.maths.gla.ac.uk/~ajb/dvi-ps.html.

[6] A. Baker and A. Lazarev. Topological Hochschild cohomology and generalized Morita equivalence. *Algebraic and Geometric Topology*, 4(29):623–645, 2004.

[7] Andrew Baker and Andrej Lazarev. On the Adams spectral sequence for R-modules. *Algebr. Geom. Topol.*, 1:173–199 (electronic), 2001.

[8] M. Basterra and M. A. Mandell. Multiplicative structures on topological Hochschild homology. *To appear*.

[9] Clemens Berger and Ieke Moerdijk. Axiomatic homotopy theory for operads. *Comment. Math. Helv.*, 78(4):805–831, 2003.

[10] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Springer-Verlag, Berlin, 1973. Lecture Notes in Mathematics, Vol. 347.

[11] J. M. Boardman and R. M. Vogt. Tensor products of theories, application to infinite loop spaces. *J. Pure Appl. Algebra*, 14(2):117–129, 1979.

[12] J. Michael Boardman. Conditionally convergent spectral sequences. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 49–84. Amer. Math. Soc., Providence, RI, 1999.

[13] M. Bökstedt, W. C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic K-theory of spaces. *Invent. Math.*, 111(3):465–539, 1993.

[14] Marcel Bökstedt. Topological hochschild homology. *Unpublished.*

[15] Marcel Bökstedt. The topological hochschild homology of $\mathbb{Z}$ and $\mathbb{Z}/p$. *Unpublished.*

[16] Raoul Bott and Clifford Taubes. On the self-linking of knots. *J. Math. Phys.*, 35(10):5247–5287, 1994. Topology and physics.

[17] A. K. Bousfield. Homotopy spectral sequences and obstructions. *Israel J. Math.*, 66(1-3):54–104, 1989.

[18] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger. H_∞ *ring spectra and their applications*, volume 1176 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[19] Michael Ching. A note on the composition product of symmetric sequences in a symmetric monoidal category. *In preparation.* Available at http://www.math.jhu.edu/~mching/.

[20] Michael Ching. Bar constructions for topological operads and the Goodwillie derivatives of the identity. *Geom. Topol.*, 9:833–933 (electronic), 2005.

[21] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology*, 43(1):1–47, 2004.

[22] Vladimir Drinfeld. On the notion of geometric realization. *Mosc. Math. J.*, 4(3):619–626, 782, 2004.

[23] Gerald Dunn. Tensor product of operads and iterated loop spaces. *J. Pure Appl. Algebra*, 50(3):237–258, 1988.

[24] W. G. Dwyer and J. P. C. Greenlees. Complete modules and torsion modules. *Amer. J. Math.*, 124(1):199–220, 2002.

[25] W. G. Dwyer, J. P. C. Greenlees, and S. Iyengar. Duality in algebra and topology. *Preprint.*

[26] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[27] Zbigniew Fiedorowicz and Rainer Vogt. Topological Hochschild homology of E_n-ring spectra. *Preprint, math.AT/0410367.*

[28] William Fulton and Robert MacPherson. A compactification of configuration spaces. *Ann. of Math. (2)*, 139(1):183–225, 1994.

[29] Paul Goerss. Associative MU-algebras. *Preprint.* Available at http://www.math.northwestern.edu/~pgoerss/.

[30] Paul G. Goerss and Michael J. Hopkins. André-Quillen (co)-homology for simplicial algebras over simplicial operads. In *Une dégustation topologique [Topological morsels]: homotopy theory in the Swiss Alps (Arolla, 1999)*, volume 265 of *Contemp. Math.*, pages 41–85. Amer. Math. Soc., Providence, RI, 2000.

[31] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[32] M. J. Hopkins and H. R. Miller. Lubin-Tate deformations in algebraic topology. *Preprint*.

[33] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[34] Thomas J. Hunter. On the homology spectral sequence for topological Hochschild homology. *Trans. Amer. Math. Soc.*, 348(10):3941–3953, 1996.

[35] G. M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, (10):vi+137 pp. (electronic), 2005. Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714].

[36] Stanley O. Kochman. Symmetric Massey products and a Hirsch formula in homology. *Trans. Amer. Math. Soc.*, 163:245–260, 1972.

[37] A. Lazarev. Hoschschild cohomology and moduli spaces of strongly homotopy associative algebras. *Homology Homotopy Appl.*, 5(1):73–100 (electronic), 2003.

[38] A. Lazarev. Towers of MU-algebras and the generalized Hopkins-Miller theorem. *Proc. London Math. Soc. (3)*, 87(2):498–522, 2003.

[39] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[40] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1998. Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.

[41] Martin Markl. A compactification of the real configuration space as an operadic completion. *Preprint, hep-th/9608067*, 1996.

[42] Martin Markl. Simplex, associahedron, and cyclohedron. In *Higher homotopy structures in topology and mathematical physics (Poughkeepsie, NY, 1996)*, volume 227 of *Contemp. Math.*, pages 235–265. Amer. Math. Soc., Providence, RI, 1999.

[43] Martin Markl, Steve Shnider, and Jim Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[44] J. P. May and R. Thomason. The uniqueness of infinite loop space machines. *Topology*, 17(3):205–224, 1978.

[45] J. Peter May. *The geometry of iterated loop spaces*. Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 271.

[46] John McCleary. *A user's guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[47] J. E. McClure and R. E. Staffeldt. On the topological Hochschild homology of bu. I. *Amer. J. Math.*, 115(1):1–45, 1993.

[48] James E. McClure and Jeffrey H. Smith. A solution of Deligne's Hochschild cohomology conjecture. In *Recent progress in homotopy theory (Baltimore, MD, 2000)*, volume 293 of *Contemp. Math.*, pages 153–193. Amer. Math. Soc., Providence, RI, 2002.

[49] James E. McClure and Jeffrey H. Smith. Operads and cosimplicial objects: an introduction. In *Axiomatic, enriched and motivic homotopy theory*, volume 131 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 133–171. Kluwer Acad. Publ., Dordrecht, 2004.

[50] Shichirô Oka. Multiplications on the Moore spectrum. *Mem. Fac. Sci. Kyushu Univ. Ser. A*, 38(2):257–276, 1984.

[51] T. Pirashvili and B. Richter. Hochschild and cyclic homology via functor homology. *K-Theory*, 25(1):39–49, 2002.

[52] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*. AMS Chelsea Pub., Providence, R.I., 2nd edition, 2004.

[53] C. Rezk. The units in a ring spectrum and the logarithm. *Preprint*, 2004.

[54] Charles Rezk. Notes on the Hopkins-Miller theorem. In *Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997)*, volume 220 of *Contemp. Math.*, pages 313–366. Amer. Math. Soc., Providence, RI, 1998.

[55] Alan Robinson. Obstruction theory and the strict associativity of Morava K-theories. In *Advances in homotopy theory (Cortona, 1988)*, volume 139 of *London Math. Soc. Lecture Note Ser.*, pages 143–152. Cambridge Univ. Press, Cambridge, 1989.

[56] Alan Robinson. Classical obstructions and S-algebras. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 133–149. Cambridge Univ. Press, Cambridge, 2004.

[57] Alan Robinson and Sarah Whitehouse. Operads and Γ-homology of commutative rings. *Math. Proc. Cambridge Philos. Soc.*, 132(2):197–234, 2002.

[58] J. Rognes. Galois extensions of structured ring spectra. *Available on the Hopf archive.*

[59] R. Schwänzl and R. M. Vogt. E_∞-spaces and injective Γ-spaces. *Manuscripta Math.*, 61(2):203–214, 1988.

[60] Stefan Schwede. The stable homotopy category is rigid. *Preprint.* Available at http://www.math.uni-bonn.de/people/schwede/.

[61] James Dillon Stasheff. Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid.*, 108:293–312, 1963.

[62] Jim Stasheff. From operads to "physically" inspired theories. In *Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995)*, volume 202 of *Contemp. Math.*, pages 53–81, Providence, RI, 1997. Amer. Math. Soc.

[63] N. P. Strickland. Products on MU-modules. *Trans. Amer. Math. Soc.*, 351(7):2569–2606, 1999.

[64] R. W. Thomason. Uniqueness of delooping machines. *Duke Math. J.*, 46(2):217–252, 1979.

[65] Charles A. Weibel and Susan C. Geller. Étale descent for Hochschild and cyclic homology. *Comment. Math. Helv.*, 66(3):368–388, 1991.