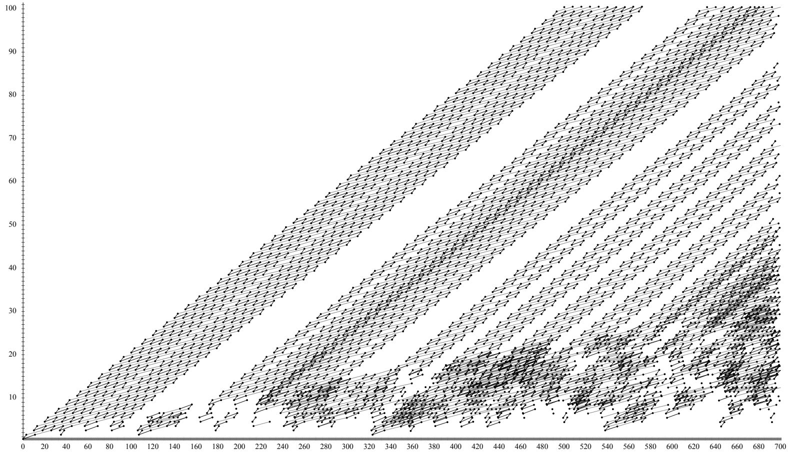
Some homological localization theorems

Haynes Miller

UIUC, July 17, 2017

The challenge: explain this picture



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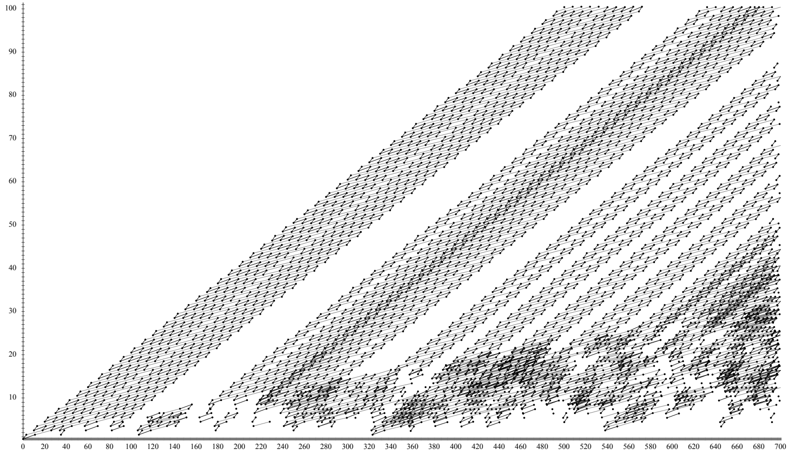

Conway's Game of Life?

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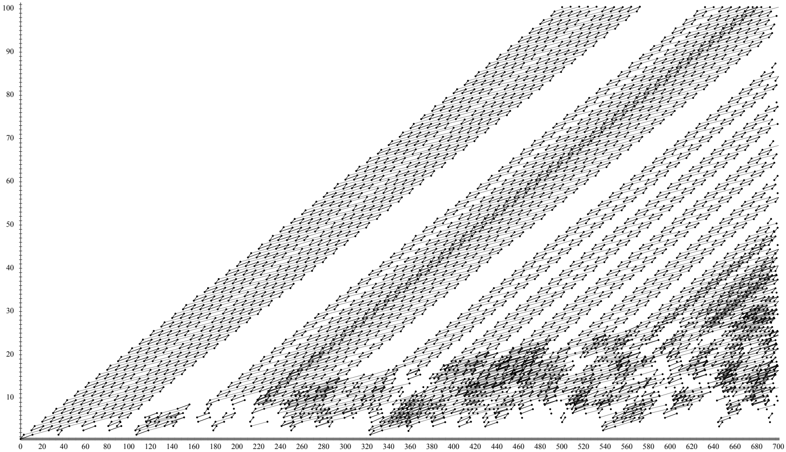
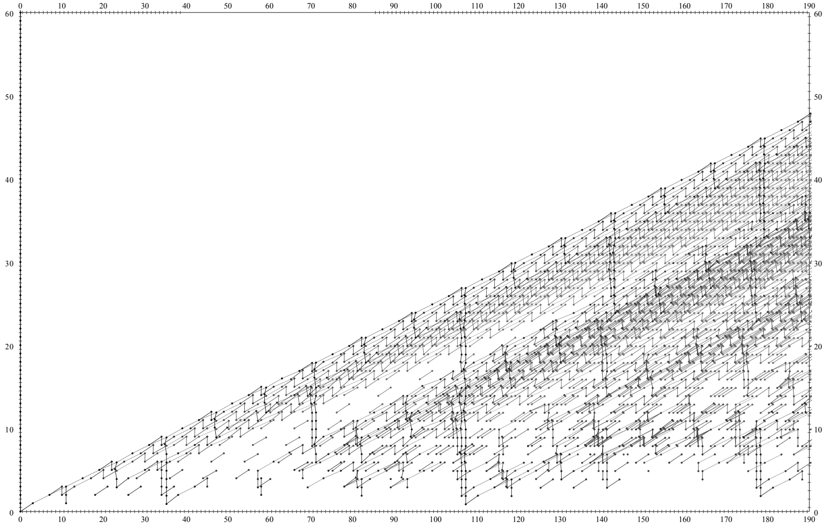


Conway's Game of Life? The rings of Saturn?

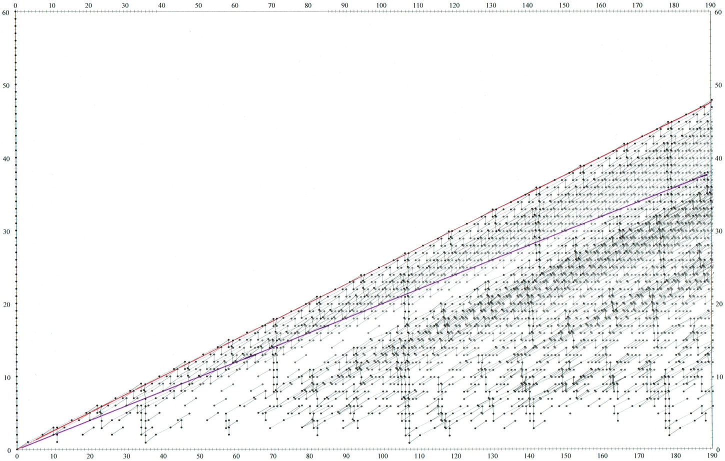
Slope-by-slope computation of Ext

The Adams E_2 term at $p = 3$:



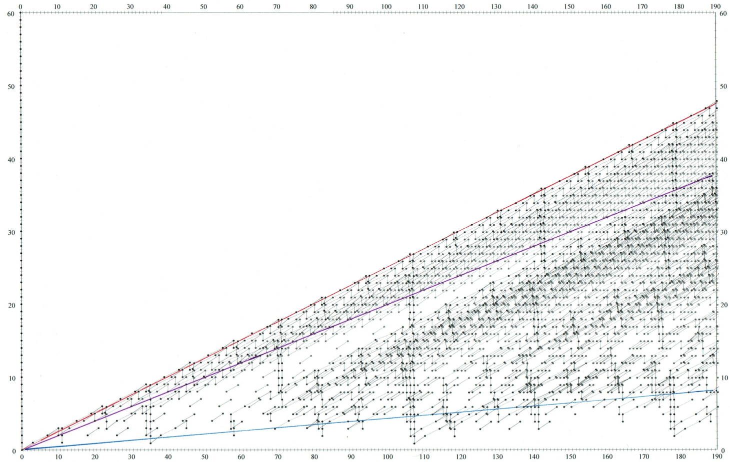
Slope-by-slope computation of Ext

Slopes 1/4 (old), 1/5 (quite new)



Slope-by-slope computation of Ext

Slope 1/23 (next up)



The Adams Spectral Sequence (1958–1969)

A “unit” map $S \to R$ in spectra determines a diagram

$$\begin{tikzcd}
S \arrow[dr] & & \overline{R} \arrow[ll, dotted] \arrow[dr] & & \overline{R} \wedge \overline{R} \arrow[ll, dotted] & \cdots \\
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Apply $\pi_*(- \wedge X)$ to get an exact couple and a spectral sequence with

$$E_1^s = R_*(\overline{R}^{\wedge s} \wedge X) \Longrightarrow \pi_*(X)$$

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If R is a ring-spectrum such that R_*R is flat over R_*, then R_*R is a Hopf algebroid and

$$E_1^* = C^*(R_*R; R_*X)$$

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$$E_2^* = H^*(R_*R; R_*X)$$

and is determined by R_*X as a comodule over R_*R.

The Adams Spectral Sequence

Example: $R = Hk$, $k = \mathbb{F}_p$. Then $R_*R = A$, the dual Steenrod algebra. Plot filtration degree s vertically and $t - s =$ topological dimension horizontally.

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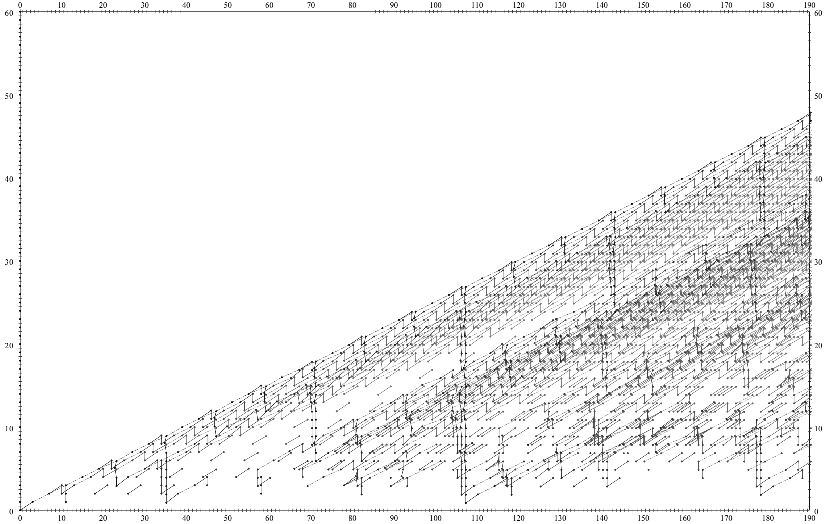

v_0-localization: algebra (1964)

At least we know that there's a vertical vanishing line: if $M_n = 0$ for $n < 0$ then $H^{s,t}(A; M) = 0$ for $t - s < 0$.

$H^{s,s}(A) = \langle v_0^s \rangle$, where v_0 represents $p\iota \in \pi_0(S)$. This acts on $H^*(A; M)$ for any M, and we may localize by inverting v_0.

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Theorem.

$$H^*(A; M) \to v_0^{-1} H^*(A; M)$$

is iso for $s > c + \dfrac{t-s}{2p-2}$, and

$$v_0^{-1} H^*(A; M) = k[v_0^{\pm 1}] \otimes H(M; \beta)\,.$$

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In particular, $v_0^{-1} H^*(A; M)$ depends only on the action of β on M. Using $A \to E[\tau_0]$, this can be written as

$$v_0^{-1} H^*(A; M) = v_0^{-1} H^*(E[\tau_0]; M)\,.$$

v_0-localization: topology (1981)

Theorem. Above a line of slope $1/(2p-2)$, the Adams spectral sequence coincides with the mod p homology Bockstein spectral sequence.

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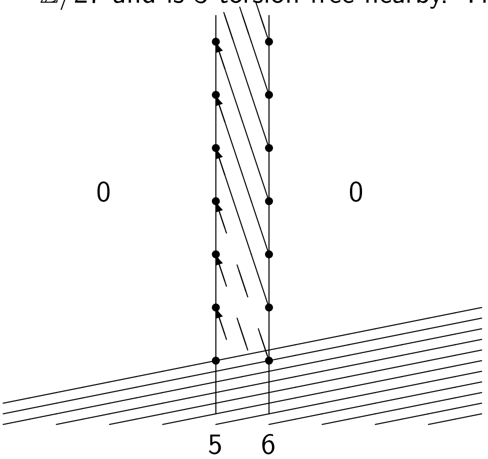
E.g. $H_5(X) = \mathbb{Z}/27$ and is 3-torsion free nearby. Then:



v_1-localization (1981)

Corollary. If $H(M;\beta) = 0$, we get a vanishing line of slope $1/(2p-2)$.

Example: (p odd) $M = A\square_{A/\tau_0} N$ is Bockstein-acyclic.

For example, if $N = H_*(X)$, this is $H_*(V(0) \wedge X)$, where $V(0) = S \cup_p e^1$.

Then

$$H^*(A; M) = H^*(A/\tau_0; N)$$

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Then

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Now $\tau_1 \in A/\tau_0$ is primitive, and produces $v_1 \in H^1(A/\tau_0)$ acting on $H^*(A/\tau_0; N)$.

Theorem. The localization map

$$H^*(A/\tau_0; N) \to v_1^{-1} H^*(A/\tau_0; N)$$

is an isomorphism above a line of slope $1/(p^2 - p - 1)$ (e.g. $1/5$ if $p = 3$).

v_1-localization (1981)

(Still p odd) There's a *formula* for

$$v_1^{-1} H^*(A/\tau_0; N)\,.$$

To state it, note the split Hopf algebra extension

$$E[\tau_1] \to A/\tau_0 \to A/(\tau_0, \tau_1)$$

A comodule over $E[\tau_1]$ is a graded vector space with a differential ∂ of degree $-|\tau_1| = 1 - 2p$. This makes $A/(\tau_0, \tau_1)$ into a differential Hopf algebra, and an A/τ_0-comodule is the same thing as a differential $A/(\tau_0, \tau_1)$-comodule.

v_1-localization (1981)

Theorem.

$$v_1^{-1}H^*(A/\tau_0; N) = k[v_1^{\pm 1}] \otimes \mathbb{H}^*(A/(\tau_0, \tau_1); N)\,.$$

v_1-localization (1981)

Theorem.

$$v_1^{-1} H^*(A/\tau_0; N) = k[v_1^{\pm 1}] \otimes \mathbb{H}^*(A/(\tau_0, \tau_1); N) .$$

There's a spectral sequence converging to this hypercohomology:

$$E_2 = H^*(H(A/(\tau_0, \tau_1)); H(N)) \Longrightarrow \mathbb{H}^*(A/(\tau_0, \tau_1); N)$$

$$H(A/(\tau_0, \tau_1)) = k[\xi_1, \xi_2, \ldots]/(\xi_1^p, \xi_2^p, \ldots)$$

With $N = k$, this spectral sequence collapses and we find that

$$v_1^{-1} E_2^*(V(0)) = k[v_1^{\pm 1}] \otimes E[h_{1,0}, h_{2,0}, \ldots] \otimes k[b_{1,0}, b_{2,0}, \ldots]$$

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$$v_1^{-1} E_2^*(V(0)) = k[v_1^{\pm 1}] \otimes E[h_{1,0}, h_{2,0}, \ldots] \otimes k[b_{1,0}, b_{2,0}, \ldots]$$

In the localized Adams spectral sequence,

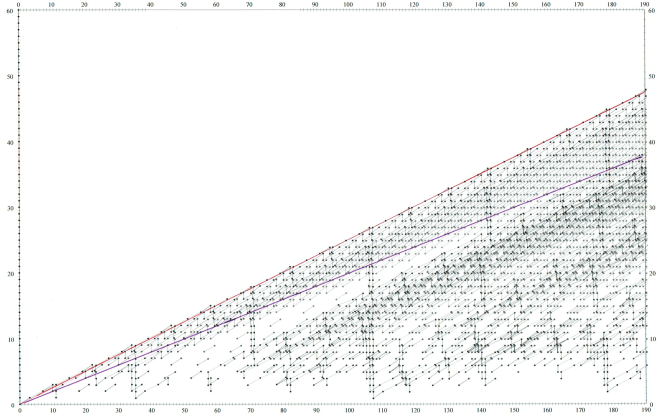
$$d_2 h_{n,0} = v_1 b_{n-1,0} + \cdots$$

resulting in

$$v_1^{-1} \pi_*(V(0)) = k[v_1^{\pm 1}] \otimes E[h_{1,0}] \, .$$

The v_1 wedge (2015)

There's a Bockstein spectral sequence relating $E_2(S \cup_p e^1)$ to $E_2(S)$, and Michael Andrews has worked it out – explaining this picture above a line of slope $1/(p^2 - p - 1)$, or $1/5$ when $p = 3$.

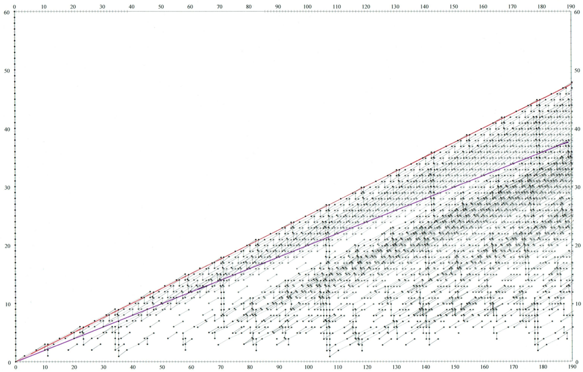


The v_1 wedge (2015)

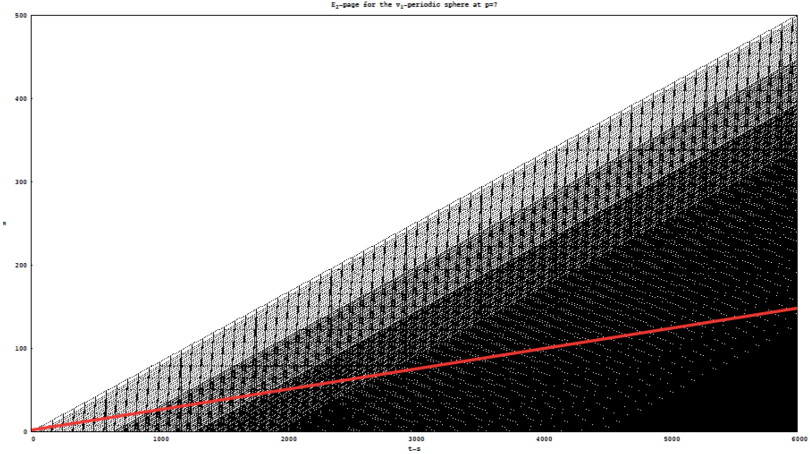
Here are Andrews's Bockstein differentials: Let $p^{[n]} = \dfrac{p^n - 1}{p - 1}$.

$$d_{p^{[n]}} v_1^{p^{n-1}} = v_1^{-p^{[n-1]}} h_{n,0}$$

$$d_{p^n - 1}(v_1^{-p^{[n]}} h_{n,0}) = v_1^{-p \cdot p^{[n]}} b_{n,0}$$



“v_1-periodic” E_2 term, $p = 7$: slope $1/41$



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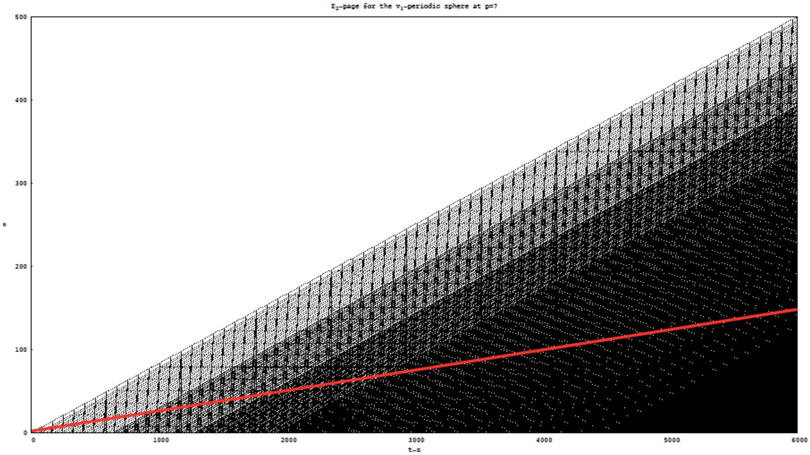


From this calculation Andrews deduces Adams differentials above the $1/(p^2 - p - 1)$ line, differentials accounting for the order of Im J. To my knowledge these are the first examples with $d_r \neq 0$ for arbitrarily large r in the Adams spectral sequence for the sphere.

Hidden periodicity: Novikov weight (1967)

Novikov observed that when $p > 2$, the dual Steenrod algebra admits a second grading, giving τ_n "weight" 1. The result is that the extension spectral sequence for

$$P \to A \to E[\tau_0, \tau_1, \ldots]$$

collapses to an isomorphism

$$H^*(A) = H^*(P; Q)$$

with

$$Q = H^*(E[\tau_0, \tau_1, \ldots]) = k[v_0, v_1, \ldots]$$

$E_2(S)$ splits into a sum

$$H^*(A) = \bigoplus_n H^*(P; Q^n)$$

Reduced powers vanishing line (1981)

For M bounded below, $H^*(P; M)$ exhibits a vanishing line of slope

$$1/(p^2 - p - 1)\,.$$

With $p = 3$ this is $2/10 = 1/5$.

The primitive element $\xi_1 \in P$ produces

$$h \in H^{1,2(p-1)}(P)$$

and its “transpotence” class

$$b \in H^{2,2p(p-1)}(P)$$

b is non-nilpotent. It acts along the vanishing edge, and

$$H^*(P; M) \to b^{-1} H^*(P; M)$$

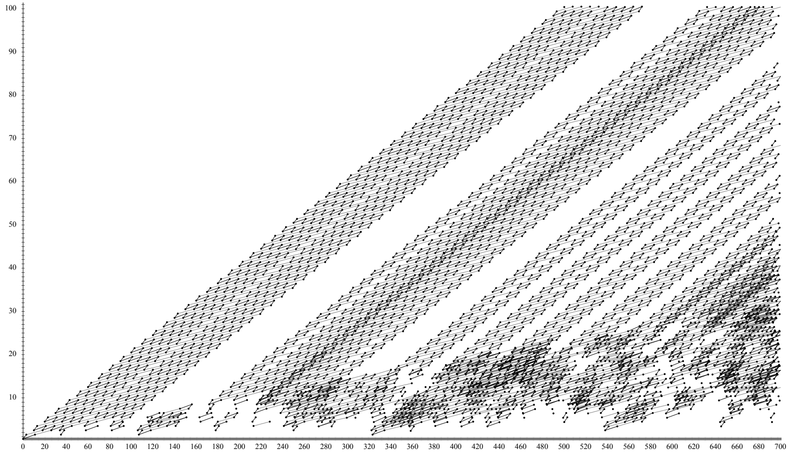
is an iso above a line of slope $1/(p^3 - p - 1)$, e.g. $1/23$ for $p = 3$.

$b^{-1}H^*(P; M)$

So if we can understand $b^{-1}H^*(P; M)$, at least for $M = Q^n$, we will understand the Adams E_2 term above a line of slope $1/(p^3 - p - 1)$, or $1/23$ for $p = 3$: a big improvement.

This is just the odd-primary analogue of understanding $v_0^{-1}H^*(A)$ at $p = 2$!

$H^*(P)$ for $p = 3$:



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Joint with Eva Belmont (2017)

Harvey Margolis (1983) and John Palmieri (2001) set up a stable homotopy category of chain complexes of comodules over a Hopf algebra P.

Analogies:

Spectra	Comodules
S^0	k
Hk	P
$\wedge$	$\otimes^{\Delta}$
$\pi_*(X)$	$\mathbb{H}^*(P; M)$
$R_*(X)$	$\mathbb{H}^*(P; R \otimes M)$

Margolis-Palmieri Adams spectral sequence

Suppose R is a ring-spectrum; for example a P-comodule algebra. We can form

$$\begin{tikzcd}
k \arrow[dr] & & \overline{R} \arrow[ll, dotted] \arrow[dr] & & \overline{R} \otimes \overline{R} \arrow[ll, dotted] & & \cdots \\
& R \arrow[ur] & & R \otimes \overline{R} \arrow[ur] & & &
\end{tikzcd}$$

Apply $\pi_*(- \otimes M)$ to get an exact couple and a spectral sequence with

$$E_1^s = \mathbb{H}^*(P; R \otimes \overline{R}^{\otimes s} \otimes M) = R_*(\overline{R}^{\otimes s} \otimes M) \Longrightarrow H^*(P; M)$$

This replaces the Cartan-Eilenberg spectral sequence

$$H^*(R; H^*(D; M)) \Longrightarrow H^*(P; M)$$

which makes sense only when R is the Hopf kernel of a normal map $P \to D$.

MPASS: Flatness

If R is a ring-spectrum such that

$$R_*R = H^*(P; R \otimes R)$$

is flat over

$$R_* = H^*(P; R)$$

then $H^*(P; R \otimes R)$ is a Hopf algebroid and

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and is determined by R_*X as a comodule over R_*R.

P, D, and R

Try this with the dual reduced powers

$$P = k[\bar{\xi}_1, \bar{\xi}_2, \ldots]$$

and the the P-comodule algebra

$$R = k[\bar{\xi}_1^p, \bar{\xi}_2, \ldots] .$$

That is,

$$R = P \Box_D k$$

where

$$D = k[\xi_1]/\xi_1^p$$

(R is the analogue of $H_*(H\mathbb{Z})$ as an A-comodule when $p = 2$).

Then

$$R_* M = H^*(P; R \otimes M) = H^*(D; M)$$

$$R_* = H^*(P; R) = H^*(D) = E[h] \otimes k[b] .$$

$b^{-1}R$

$R_*M = H^*(D; M)$ is rarely flat over R_*, certainly not if $M = R$. But we're interested in $b^{-1}H^*(P)$, so let's invert b on R. We can invert b on the level of "spectra": replace R by a fibrant object, represent b by a map $\Sigma^2 R \to R$, and take the colimit to form a new "2-periodic" ring spectrum $b^{-1}R$ with "homotopy"

$$\mathbb{H}^*(P; b^{-1}R) = b^{-1}H^*(D) = E[h] \otimes k[b^{\pm 1}]$$

Its self-homology is

$$b^{-1}R_*R = b^{-1}H^*(D; P)$$

This is *still* not flat over $b^{-1}R_* = b^{-1}H^*(D)$ ...

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This is *still* not flat over $b^{-1}R_* = b^{-1}H^*(D)$ ... **unless** $p = 3$.

$b^{-1}H^*(P)$

For this reason (and others) we'll take $p = 3$ now.

So $|h| = (1, 4)$ and $|b| = (2, 12)$.

Then the self-homology

$$b^{-1}H^*(D; P)$$

is a Hopf algebroid over

$$b^{-1}H^*(D) = E[h] \otimes k[b^{\pm 1}]\,.$$

$b^{-1}H^*(P)$

Here's a wonderful surprise (still for $p = 3$):

Theorem (Belmont) There are primitives

$$e_n \in H^{1,2(3^n+1)}(D; P)$$

such that

$$b^{-1}H^*(D; P) = b^{-1}H^*(D) \otimes E[e_2, e_3, \ldots]$$

as Hopf algebras.

MPASS E_2

Consequently in the localized Margolis-Palmieri Adams spectral sequence

$$E_2 = b^{-1}H^*(D) \otimes k[w_2, w_3, \ldots] \Longrightarrow b^{-1}H^*(P)\,.$$

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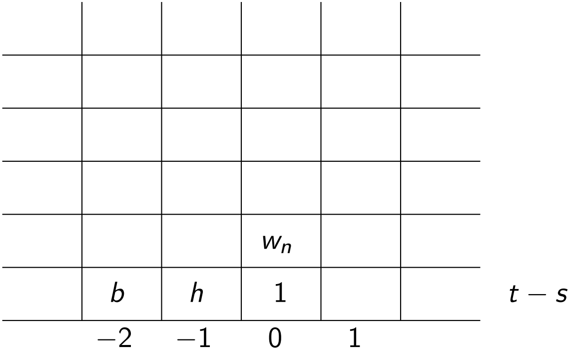
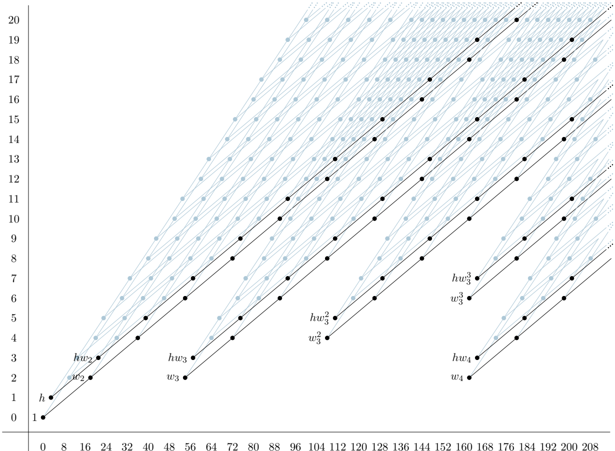
If we draw the MPASS in the standard Adams way, E_2 looks like this; $s - t$ is total cohomological degree.



Comparison with data

The class w_n contributes a b-tower in $H^*(P)$ starting in degree $(0, 2(3^n - 5))$: $(0, 8), (0, 44), (0, 154), \ldots$. Here's the polynomial subalgebra generated by bw_n's (marked as w_n).



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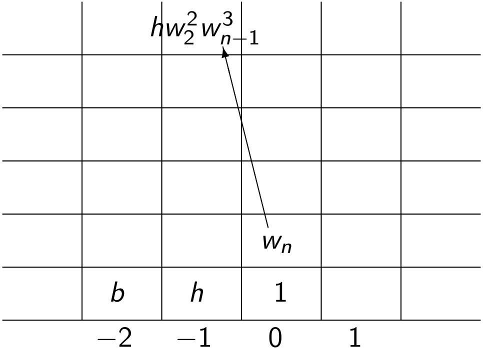
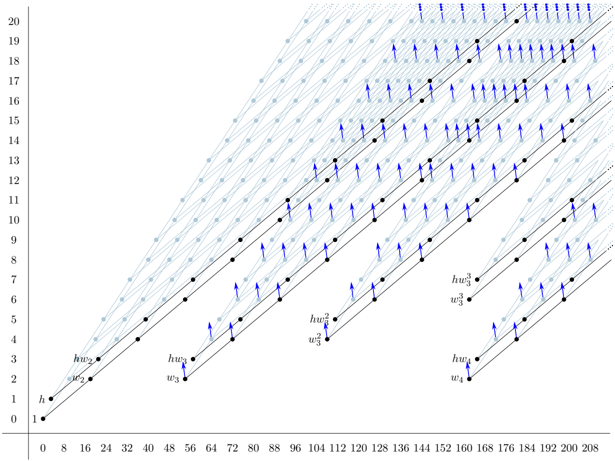
Conjecture. Only d_4 and d_8 are nontrivial, and

$$d_4 w_n = h w_2^2 w_{n-1}^3$$



E_4



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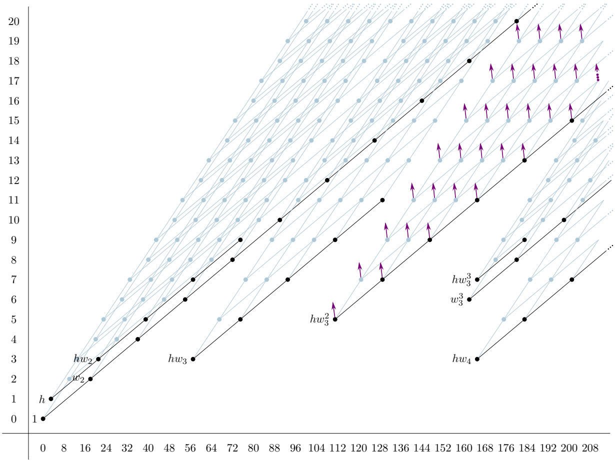

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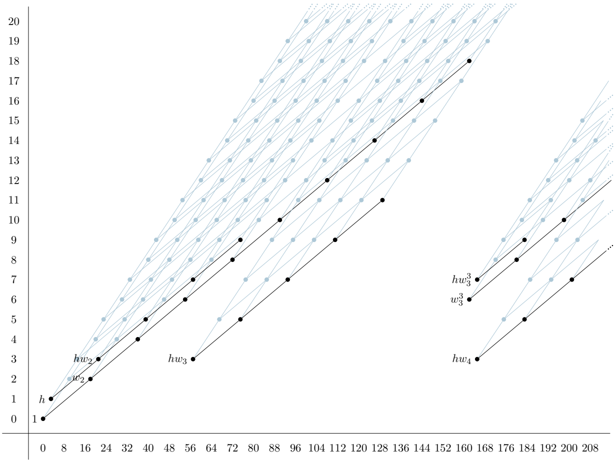
E_∞



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D-comodules

A D-comodule structure is a graded vector space M with an operator $\partial : M \to M$ of degree -4 such that $\partial^3 = 0$.

Define

$$W = k[w_2, w_3, \ldots]\,, \quad |w_n| = 2(3^n - 5)\,,$$

with D-comodule-algebra structure determined by

$$\partial w_n = w_2^2 w_{n-1}^3$$

extended as a derivation.

Conjecture

$$b^{-1}H^*(P) = b^{-1}H^*(D; W)\,.$$

D-comodules

This fits the data. For example, it implies that $b^{-1}H^*(P)$ is free over the exterior algebra $E[h]$. We have a sketch of an argument.

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Moreover, it seems that the MPASS coincides under this isomorphism with the spectral sequence associated with the weight filtration on W, putting each w_n in degree 1. Then

$$d_4 x = h \partial x \,.$$

The only remaining nonzero differential is d_8, and

$$d_8(hx) = b \partial^2 x \,.$$

Acknowledgements

Thanks to Christian Nassau for his charts,

`www.nullhomotopie.de`

and Hood Chatham for his spectral sequence package,

`www.ctan.org/pkg/spectralsequences`

And two announcements

Tunisian Journal of Mathematics

an international publication organized by the Tunisian Mathematical Society

with editorial board including

Benoit Fresse, Sadok Kallel, Haynes Miller, Said Zarati

is open for business, using EditFlow.

and

Happy birthday, Paul!