Toda's realization theorem

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$\mathcal{A}$ is the Steenrod algebra. Let M be an $\mathcal{A}$ module which is bounded below and of finite type. I want to know whether there is a spectrum with this as its cohomology.

Let

$$M \leftarrow F_0 \xleftarrow{d_0} F_1 \xleftarrow{d_1} \cdots$$

be a free resolution. We may assume that F_s is trivial below dimension s more than the connectivity of M. If M is Bockstein-acyclic, the connectivity of F_{s+1} can be chosen to be $2(p-1)$ larger than the connectivity of F_s.

Let K^s be the GEM with $H^*(K^s) = \Sigma^{1-s} F_s$. We have a diagram

$$K_0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} K^2 \xrightarrow{d^2} \cdots$$

where each d has degree -1, which induces the resolution in cohomology. We wish to embed it into a diagram

$$\begin{array}{ccccccccccccc}
* = Y^0 & & \xleftarrow{j^0} & & Y^1 & & \xleftarrow{j^1} & & Y^2 & & \xleftarrow{j^2} & & Y^3 & & \cdots \\
& {\scriptstyle k^0}\searrow & & \nearrow{\scriptstyle i^0} & & {\scriptstyle k^1}\searrow & & \nearrow{\scriptstyle i^1} & & {\scriptstyle k^2}\searrow & & \nearrow{\scriptstyle i^2} & & {\scriptstyle k^3}\searrow & \\
& & K^0 & & \xrightarrow{d^0} & & K^1 & & \xrightarrow{d^1} & & K^2 & & \xrightarrow{d^2} & & K^3
\end{array}$$

where the arrows labelled i and d have degree -1. In cohomology, the maps j will fit into a commutative diagram

$$\begin{array}{ccccccccccc}
& & M & & \xrightarrow{=} & & M & & \xrightarrow{=} & & M \\
& {\scriptstyle s_2}\nearrow & & \searrow{\scriptstyle p_2} & & {\scriptstyle s_3}\nearrow & & \searrow{\scriptstyle p_3} & & {\scriptstyle s_4}\nearrow & \\
H^*(Y^1) = F_0 & & \xrightarrow{j_1} & & H^*(Y^2) & & \xrightarrow{j_2} & & H^*(Y^3) & & \cdots
\end{array}$$

$Y^1 = \Sigma^{-1} K^0$, and the map $k^1 : Y^1 \to K^1$ is d^0. Let Y^2 be the fiber of k^1.

We want to factor $d^1 : K^1 \to \Sigma K^2$ through $i^1 : K^1 \to \Sigma Y^2$. This can be done since $k^1 : Y^1 \to K^1$ is just $d^0 : \Sigma^{-1} K^0 \to K^1$, and $d^1 d^0 = 0$. The map $k^2 : Y^2 \to K^2$ can be varied by adding a map of the form $Y^2 \xrightarrow{j^1} Y^1 \longrightarrow K^2$.

Let $Y^3 \to Y^2$ be the fiber of k^2 and let $i^2 : K^2 \to \Sigma Y^3$ be the boundary homomorphism.

Next we want to factor $d^2 : K^2 \to \Sigma K^3$ through the map $i^2 : K^2 \to \Sigma Y^3$. So I want to know that k^2 can be chosen so that $Y^2 \xrightarrow{k^2} K^2 \xrightarrow{d^2} \Sigma K^3$ is null. Since the target is a GEM, it is equivalent to ask that this map be zero in cohomology.

Since $\operatorname{coker}(F_1 \to F_0) = M$, the long exact sequence for the cofibration sequence $Y^2 \to Y^1 \to K^1$ gives exactness of the top row in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{p} & H^*(Y^2) & \longrightarrow & \Sigma^{-1}\ker d_0 & \longrightarrow & 0 \\
 & & & & \uparrow{\scriptstyle k_2} & \nearrow & \downarrow & & \\
\Sigma^{-1}F_4 & \xrightarrow{d_3} & \Sigma^{-1}F_3 & \xrightarrow{d_2} & H^*(K^2) = \Sigma^{-1}F_2 & \xrightarrow{d_1} & \Sigma^{-1}F_1 & &
\end{array}$$

The composite $d_1 d_2$ is zero, and the right vertical is a monomorphism, so the composite $k_2 d_2$ factors through the inclusion $p : M \to H^*(Y^2)$ by a map $c : \Sigma^{-1}F_3 \to M$. Since $pcd_3 = k_3 d_2 d_3 = 0$, the map c is a cocycle representing a class in

$$\operatorname{Ext}^{3,1}_{\mathcal{A}}(M, M)$$

If we assume that this group is zero, then c is a coboundary, which is to say that it factors through $d_2 : \Sigma^{-1}F_3 \to H^*(K^2) = \Sigma^{-1}F_2$ by a map $b : H^*(K^2) \to M$. The map $pb : \Sigma^{-1}F_2 \to H^*(Y^2)$ is the effect in cohomology of exactly the sort of map by which we are allowed to alter k^2; and $pbd_2 = pc = k_2 d_2$, so if we replace k_2 by $k_2 - pb$, then $k_2 d_2 = 0$, as desired.

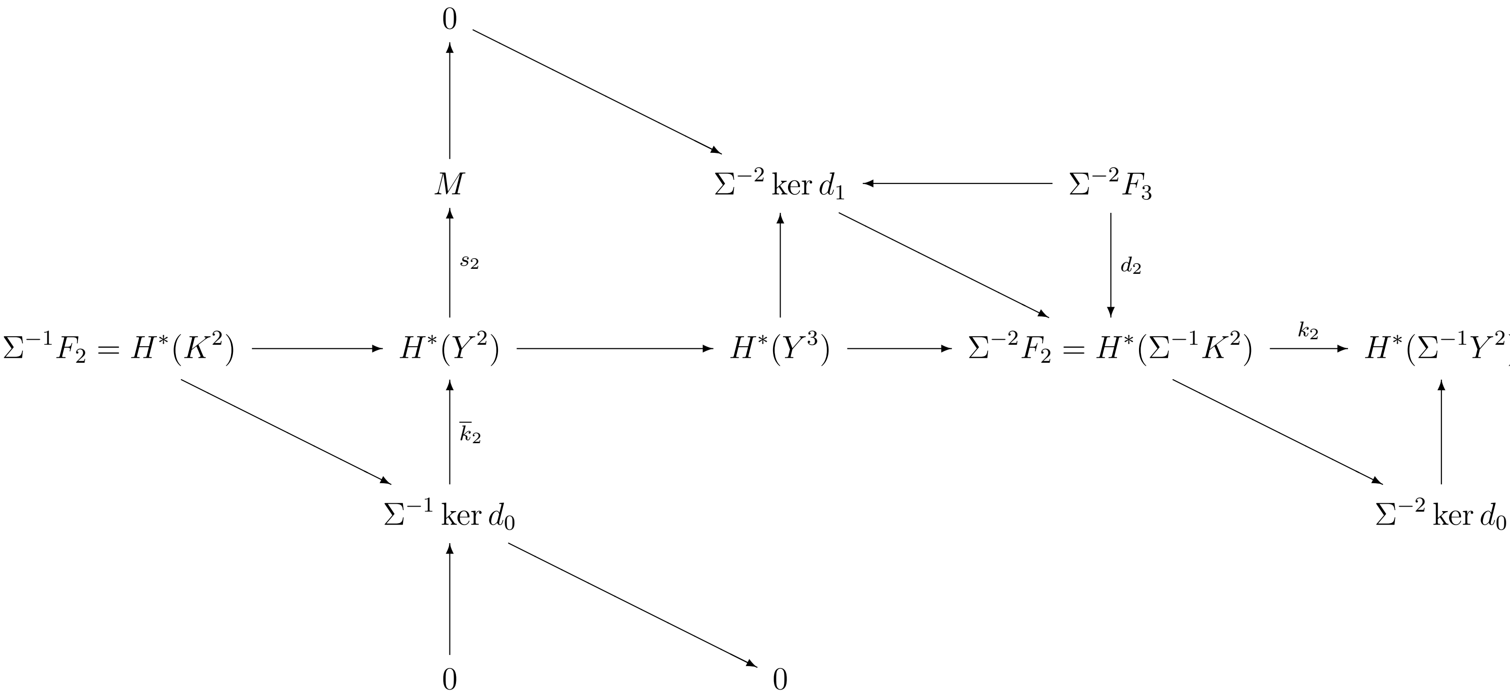
Notice that this choice of k_2 then factors through the surjection $\Sigma^{-1}F_2 \to \Sigma^{-1}\ker d_0$, and thus splits the top sequence in the diagram. Let $s_2 : H^*(Y^2) \to M$ be the corresponding splitting of p.

So d^2 factors as $d^2 = k^3 i^2$. The map $k^3 : Y^3 \to K^3$ can be varied by any map of the form $Y^3 \xrightarrow{j^2} Y^2 \longrightarrow K^3$.

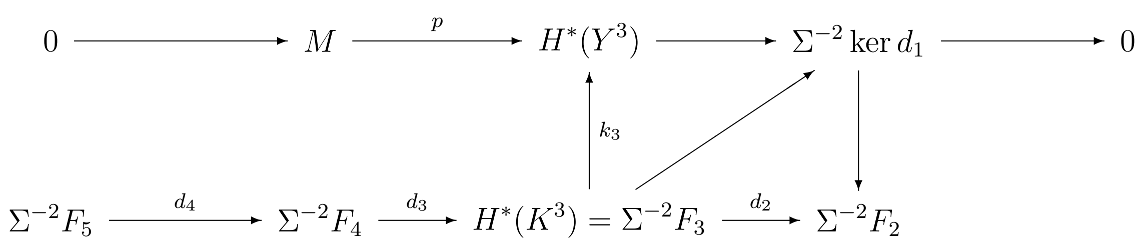
Let $j^3 : Y^4 \to Y^3$ be the fiber of $k^3 : Y^3 \longrightarrow K^3$.

Next we want to factor $d^3 : K^3 \longrightarrow \Sigma K^4$ through $i^3 : K^3 \longrightarrow \Sigma Y^4$; that is, we want to know that k^3 can be chosen so that $Y^3 \xrightarrow{k^3} K^3 \xrightarrow{d^3} \Sigma K^4$ is null.

For this we need to analyze the cohomology of Y^3. We have a diagram



in which the straight lines are exact. Chasing it around, we find that the top row in the following sequence is exact.



We are at the inductive step; the composite k_2d_3 factors as $\Sigma^{-1}F_4 \xrightarrow{c} M \xrightarrow{p} H^*(Y^3)$, and c is a cocyle, determining an element of

$$\mathrm{Ext}_{\mathcal{A}}^{4,2}(M, M)$$

If this cohomology class vanishes, the map k_3 can be altered so that $k_3d_3 = 0$, and the process continues.