The Riemann-Roch formula

We briefly recall the topological Riemann-Roch formula. A reference for this is the book *Cohomology Theories* of Eldon Dyer.

Let $\pi : E \to B$ be a smooth fiber bundle with compact total space, and fiber-dimension d, and let τ be the bundle (over E) of tangents along the fiber. We can embed E into a trivial vector bundle $\mathbb{R}^n_B = B \times \mathbb{R}^n$ over B. If ν denotes the normal bundle of this embedding then canonically

$$\nu \oplus \tau = E^n_{\mathbb{R}} \,.$$

Let h be a multiplicative cohomology theory, and suppose u_h is a Thom class for the vector bundle ν. The *Gysin map* associated to π and u_h is the composite π^h_* in

$$\begin{array}{ccc} h^*(E) & \xrightarrow{\pi^h_*} & h^{*-d}(B) \\ \downarrow \cong & & \downarrow \cong \\ \overline{h}^{*+n-d}(E^\nu) & \xrightarrow{c^*} & \overline{h}^{*+n-d}(\Sigma^n B_+) \end{array}$$

where

$$c : \Sigma^n B_+ \to E^\nu$$

is the collapse map and the vertical isomorphisms are the Thom isomorphisms.

Now let k and h be two multiplicative cohomology theories, and $\theta : k \to h$ a multiplicative natural transformation. Suppose ξ is a vector bundle over X with fiber-dimension d, and that u_k and u_h are Thom classes for this bundle with respect to the two cohomology theories. Then there is a unique class $\rho(\xi) \in h^d(X)$ such that $\epsilon(\rho(\xi)) = 1$ and $\rho(\xi) \cup \theta u_k = u_h$.

If u is a Thom class for ξ and v is a Thom class for η, then the cross-product uv is a Thom class for the Whitney sum $\xi \oplus \eta$. It follows that the characteristic class ρ is "exponential":

$$\rho(\xi \oplus \eta) = \rho(\xi)\rho(\eta) \,.$$

Let $i : X \to X^\xi$ be the inclusion of the zero section. The Thom class pulls back to the Euler class under this map:

$$e_h(\xi) = i^* u_h \in \overline{h}^d(X) \,.$$

If $e_h(\xi)$ happens to be a non-zero-divisor in $h^*(X)$, then so is $\theta e_k(\xi)$ (which is just another h-theoretic Euler class for ξ), and we may write

$$\rho(\xi) = \frac{e_h(\xi)}{\theta e_k(\xi)} .$$

If u_k is a k-Thom class and u_h an h-Thom class for the normal bundle ν as above, then we may compute:

$$\theta\pi_*^k(a) = \theta c^*(a \cup u_k) = c^*(\theta a \cup \theta u_k) = c^*(\theta(a)\rho(\nu)^{-1} \cup u_h) = \pi_*^h(\theta(a)\rho(\nu)^{-1}) .$$

Since the characteristic class ρ is exponential,

$$\rho(\nu)^{-1} = \rho(\tau) ,$$

and we may express this as

$$\theta\pi_*^k(a) = \pi_*^h(\theta(a)\rho(\tau)) .$$

Example 1. The eponymous example takes unitary K-theory for k, rational cohomology for h, and the Chern character for θ. For a line bundle λ the standard choice of a Thom class gives

$$e_K(\lambda) = 1 - \lambda$$

(ignoring the Bott class). If we write this as $\Lambda_{-1}\lambda$, we have at our disposal a description of the Euler class for a general complex vector bundle arising from this:

$$e_K(\xi) = \Lambda_{-1}(\xi) .$$

The Chern character of $1 - \lambda$ is $1 - e^{-x}$, where $x = e_H(\lambda) = -c_1(\lambda)$. In the universal case, the Euler classes are non-zero-divisors, and we can write

$$\rho(\lambda) = \frac{x}{1 - e^{-x}} .$$

The Todd class is the exponential characteristic class determined by this value on line bundles, so the Riemann-Roch formula reads

$$\mathrm{ch}\,\pi_*^K(a) = \pi_*^H(\mathrm{ch}(a)\,\mathrm{Td}(\tau)) .$$

Example 2. Now take k and h both to be unitary K-theory, and let θ be the stable operation associated to ψ^{-1}. On vector bundles, this Adams operation forms the Hermitian dual complex bundle. For line bundles this is the inverse.

Use the standard Thom classes, so that (dividing by -1 to make the operation stable)

$$\psi^{-1} e_K(\lambda) = \frac{1-\lambda^{-1}}{-1} = \lambda^{-1} - 1$$

and

$$\rho(\lambda) = \frac{1-\lambda}{\lambda^{-1}-1} = \lambda$$

Since ρ is exponential,

$$\rho(\lambda_1 \oplus \cdots \oplus \lambda_n) = \lambda_1 \cdots \lambda_n = \det(\lambda_1 \oplus \cdots \oplus \lambda_n)$$

so by the splitting principle

$$\rho(\xi) = \det(\xi)\,.$$

The Riemann-Roch formula is then

$$\psi^{-1}\pi_*^K a = \pi_*^K(\psi^{-1}(a)\det(\tau))\,.$$

This is "topological Serre duality."