Rack modules

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A *rack* is a set X together with a binary operation (written xy) such that

$$x(yz) = (xy)(xz) \quad \text{and} \quad x\cdot : X \to X \quad \text{is bijective}\,.$$

An X module is a collection of abelian groups A_x together with maps

$$A_y \xrightarrow{\alpha_x} A_{xy} \xleftarrow{\beta_y} A_x$$

such that each α_x is an isomorphism and the following diagrams commute.

$$\begin{array}{ccc} A_z & \xrightarrow{\alpha_y} & A_{yz} \\ {\scriptstyle\alpha_x}\downarrow & & \downarrow{\scriptstyle\alpha_x} \\ A_{xz} & \xrightarrow{\alpha_{xy}} & A_{x(yz)} \end{array}$$

$$\begin{array}{ccc} A_y & \xrightarrow{\beta_z} & A_{yz} \\ {\scriptstyle\alpha_x}\downarrow & & \downarrow{\scriptstyle\alpha_x} \\ A_{xy} & \xrightarrow{\beta_{xz}} & A_{x(yz)} \end{array}$$

$$\begin{array}{ccc} A_x & \xrightarrow{\beta_z} & A_{xz} \\ {\scriptstyle\beta_y}\downarrow & \overset{\beta_{yz}}{\searrow} & \downarrow{\scriptstyle\alpha_{xy}} \\ A_{xy} & \xrightarrow{\beta_{xz}} & A_{x(yz)} \end{array}$$

where the last diagram signifies that the diagonal is the sum of the two edges. That is,

$$\alpha_x\alpha_y = \alpha_{xy}\alpha_x\,, \quad \alpha_x\beta_z = \beta_{xz}\alpha_x\,,$$

and

$$\beta_{yz} = \alpha_{xy}\beta_z + \beta_{xz}\beta_y\,.$$

A *derivation* of a rack X with values in an X-module A is a choice of $\sigma_x \in A_x$ for each $x \in X$ satisfying

$$\sigma_{xy} = \alpha_x\sigma_y + \beta_y\sigma_x\,.$$

The abelian group of derivations from X with values in A forms an abelian group $\mathrm{Der}(X, A)$.

There is a canonical X-module, the "Kähler differentials" Ω_X. It is the abelianization of X as a rack over itself, characterized by

$$\mathrm{Hom}_{X-mod}(\Omega_X, A) = \mathrm{Der}(X, A)\,.$$

It comes equipped with the "universal derivation" $\sigma : X \to \Omega_X$.

Example. The singleton set $*$ admits a unique rack structure, and a module for it is simply a module over the ring

$$R = \mathbb{Z}[\alpha^{\pm 1}, \beta]/\beta\beta'\,, \quad \beta + \beta' = 1 - \alpha\,.$$

The group of derivations from $*$ into an R-module M is

$$\mathrm{Der}(*, M) = \ker\,(\beta'|M)\,,$$

so the R-module of Kähler differentials is

$$\Omega_* = R/(\beta')$$

and the universal derivation sends $*$ to $1 \in R/(\beta')$.

An X-module is *automorphic* if $\beta_y = 0 : A_x \to A_y$ for every x, y. For example an automorphic $*$-module is an abelian group equipped with an automorphism α.

More generally, suppose that we are given an abelian group A together with a map $\alpha : X^2 \to \mathrm{Aut}(A)$ such that

$$\alpha_{x,yz}\alpha_{y,z} = \alpha_{xy,xz}\alpha_{x.z}\,.$$

Then we can take $A_x = A$ for every x, and $\alpha_x : A_y \to A_{xy}$ given by $\alpha_{x,y}$. This gives us an automorphic X-module. In particular, if $\alpha_{x,y}$ is independent of y, this is equivalent to an action on A of the group G_X associated to X:

$$G_X = \langle X : x \cdot y = xyx^{-1} \rangle$$

where we have written $\cdot$ for the rack operation and juxtaposition for the group operation.

A differential into an automorhic X-module is an assignment $\sigma_x \in A$ for each $x \in X$ such that

$$\sigma_{xy} = \alpha_x \sigma_y\,.$$

An X-module is *differential* if there is an abelian group A and isomorphisms $A \simeq A_x$ for every x under which $\alpha_x : A_y \to A_x$ corresponds to the identity map on A for every x, y. The relations then imply that the structure of an X-module is completed by giving an endomorphism β_x of A for every $x \in X$, with the property that for all x, y, z,

$$\beta_{x(yz)}(1 - \beta_{xy}) = \beta_{xz} \, .$$

For example, a differential $*$-module is an abelian group equipped with a differential β.

A derivation into a differential X-module A is a function $\sigma : X \to A$ such that

$$\sigma_{xy} = \sigma_y + \beta_y \sigma_x \, .$$

Given a morphism of racks $f : X \to X'$, there are functors

$$f_* : X - mod \leftrightarrows X' - mod : f^*$$

The pull-back functor f^* is easy to describe. Given an X'-module (A, α, β),

$$(f^* A)_x = A_{f(x)} \quad \text{for all } x \in X$$

and for all $x, y \in X$

$$\begin{array}{ccccc} (f^*A)_y & \xrightarrow{\alpha_x} & (f^*A)_{xy} & \xleftarrow{\beta_y} & (f^*A)_x \\ \| & & \| & & \| \\ A_{f(y)} & \xrightarrow{\alpha_{f(x)}} & A_{f(x)f(y)} & \xleftarrow{\beta_{f(y)}} & A_{f(x)} \end{array}$$

The functor f_* is the left adjoint to f^*, and is harder to describe. But sometimes we can.

An X-module is *constant* if it is pulled back from a module over the trivial rack. So if M is an R-module, the corresponding constant X module M_X has

$$(M_X)_x = M \quad \text{for all } x \in X$$

and

$$\alpha_x = \alpha \, , \quad \beta_y = \beta \quad \text{for all } x, y \in X \, .$$

A derivation from X to a constant module M is a function $\sigma : X \to M$ such that

$$\sigma_{xy} = \alpha \sigma_y + \beta \sigma_x \quad \text{for all } x, y \in X \, .$$

The adjunction shows that if $\pi : X \to *$ then

$$\pi_* \Omega_X = R\langle X \rangle / (xy = \alpha y + \beta x)\,.$$

Now let's try to construct a "Hochschild complex." For a start, what should "free" mean? Here's the proposal. There is a functor u from X-mod to sets over X, simply forgetting all the structure. This functor has a left adjoint, which I will write F_X. It is characterized by

$$\mathrm{Hom}_X(F_X S, A) = \prod_{x \in X} \mathrm{Map}(S_x, A_x)\,.$$

For example, if $X = *$, F_X is the free R-module functor. Write

$$\iota : S \to uFS$$

for the unit of the adjunction; it is a map over X.

The universal derivation $\sigma : X \to \Omega_X$ is a map over X and hence extends to an X-module surjection

$$\Omega_X \xleftarrow{\overline{\sigma}} F_X X$$

where X is regarded as a set over itself by the identity map.

How do we reflect the identity satisfied by the universal (and hence any) derivation? Regard X^2 as a set over X via the rack multiplication. Then there are three maps $X^2 \to uF_X X$ in $\mathbf{Set}/X$, given by

$$(x, y) \quad \mapsto \quad \alpha_x \iota_y \quad , \quad \iota_{xy} \quad , \quad \beta_y \iota_x\,,$$

and the identity specifies that their alternating sum is zero in Ω_X. So we have a presentation

$$0 \longleftarrow \Omega_X \xleftarrow{\overline{\sigma}} F_X X \xleftarrow{\partial} F_X X^2\,.$$

We want to extend this presentation to an exact complex, and based on the example provided by associative algebras we hope that the next term will be $F_X X^3$. It seems that the appropriate thing to do is to specify that X^n is to be regarded as a set over X by means of the map

$$\pi_n : (x_1, \ldots, x_n) \mapsto x_1(x_2 \cdots (x_{n-1} x_n) \cdots)\,.$$

Note that it is not possible to give meaning to X^0; there is no way to present Ω_X as a kernel. Andruskiewitch and Graña insist on unnaturally building something by choosing a basepoint at this point.

There are five natural maps $X^3 \to F_X X^2$ over X. Fix $w \in X$ and restrict to

$$\pi_3^{-1} w = \{(x, y, z) : w = x(yz) = (xy)(xz)\} .$$

The maps send (x, y, z) to

$$\alpha_x \iota_{y,z} \quad , \quad \alpha_{xy} \iota_{x,z} \quad , \quad \beta_{xz} \iota_{x,y} \quad , \quad \iota_{x,yz} \quad , \quad \iota_{xy,xz} .$$

Apply $\partial : FX^2 \to FX$ to these five elements:

$$\begin{aligned}
\alpha_x \iota_{y,z} &\mapsto \alpha_x(\alpha_y \iota_z - \iota_{yz} + \beta_z \iota_y) \\
\alpha_{xy} \iota_{x,z} &\mapsto \alpha_{xy}(\alpha_x \iota_z - \iota_{xz} + \beta_z \iota_x) \\
\beta_{xz} \iota_{x,y} &\mapsto \beta_{xz}(\alpha_x \iota_y - \iota_{xy} + \beta_y \iota_x) \\
\iota_{x,yz} &\mapsto \alpha_x \iota_{yz} - \iota_{x(yz)} + \beta_{yz} \iota_x \\
\iota_{xy,xz} &\mapsto \alpha_{xy} \iota_{xz} - \iota_{(xy)(xz)} + \beta_{xz} \iota_{xy} .
\end{aligned}$$

Applying the three identities satisfied by the α's and the β's, we see that choosing signs $+, -, -, +, -$ produces a signed sum of zero.

So define $\partial : FX^3 \to F_X X^2$ by

$$\iota_{x,y,z} \mapsto \alpha_x \iota_{y,z} - \alpha_{xy} \iota_{x,z} - \beta_{x,z} \iota_{x,y} + \iota_{x,yz} - \iota_{xy,xz} .$$

Then $\partial^2 = 0$.

Perhaps contracting homotopies can be built using the fact that $x\cdot$ is bijective. So there's an operator $x \mapsto x'$ on X characterized by $xx' = x$. Notice then that $X \to X^2$ by $x \mapsto (x, x')$ is a map over X.

A&G effectively describe the rest of this complex.

We use a variant of the useful notation introduced by Andruskiewitch and Graña:

$$x_1 \cdots x_n = x_1(x_2 \cdots x_n)$$

They observe that for $i < n$,

$$(x_1 \cdots x_i)(x_1 \cdots x_{i-1} x_{i+1} \cdots x_n) = (x_1 \cdots x_n) .$$

To see this note that it's the definition for $i = 1$. Then use self-distributivity:

$$(x_1 \cdots x_i)(x_1 \cdots x_{i-1} x_{i+1} \cdots x_n) = (x_1(x_2 \cdots x_i) x_2 \cdots x_{i-1} x_{i+1} \cdots x_n)$$

which, by induction on i, is $x_1 \cdots x_n$.

Define maps $d_i : X^n \to F_X X^{n-1}$ over X:

$$\begin{aligned} d_1(x_1, \dots, x_n) &= (x_1 x_2, \dots, x_1 x_n) \\ d_2(x_1, \dots, x_n) &= (x_1, x_2 x_3, \dots, x_2 x_n) \\ &\cdots \\ d_{n-1}(x_1, \dots, x_n) &= (x_1, \dots, x_{n-2}, x_{n-1} x_n) \end{aligned}$$

Also

$$\begin{aligned} a_1(x_1, \dots, x_n) &= \alpha_{x_1}(x_2, \dots, x_n) \\ a_2(x_1, \dots, x_n) &= \alpha_{x_1 x_2}(x_1, x_3, \dots, x_n) \\ &\cdots \\ a_{n-1}(x_1, \dots, x_n) &= \alpha_{x_1 \cdots x_{n-1}}(x_1, \dots, x_{n-2}, x_n) \end{aligned}$$

and

$$b(x_1, \dots, x_n) = \beta_{x_1 \cdots x_{n-2} x_n}(x_1, \dots, x_{n-1}) .$$

Reminder: The operators α and β here are the ones in the free X-module construction F_X.

For example, with $n = 2$ we have

$$d_1(x_1, x_2) = x_1 x_2 , \quad a_1(x_1, x_2) = \alpha_1 x_2 , \quad b(x_1, x_2) = \beta_2 x_1 .$$

With $n = 3$,

$$a_1(x_1, x_2, x_3) = \alpha_{x_1}(x_2, x_3) , \quad a_2(x_1, x_2, x_3) = \alpha_{x_1 x_2}(x_1, x_3) ,$$

$$d_1(x_1, x_2, x_3) = (x_1 x_2, x_1 x_3) , \quad d_2(x_1, x_2, x_3) = (x_1, x_2 x_3) ,$$

and

$$b(x_1, x_2, x_3) = \beta_{x_1 x_3}(x_1, x_2) .$$

The rack axioms and the identities involving α and β imply relations among these operators. For example as maps $X^3 \to F_X X$,

$$a_1 d_1 = d_1 a_2 , \quad a_1 d_2 = d_1 a_2 , \quad b d_1 = d_1 b$$

are just true;

$$d_1 d_2 = d_1 d_1$$

follows from the rack axiom; and

$$a_1 a_1 = a_1 a_2\,, \quad a_1 b = b a_1\,, \quad b d_2 = b a_2 + bb$$

follow from the three relations.

The last relation indicates that we are not trying to describe an indexing category; rather an indexing ringoid, and then we study additive functors from it into, for example, X-modules.

Next step is to write down the general relations among these operators. A&G's proof that $\partial^2 = 0$ should carry over. What about maps going in the other direction, "degeneracies"? For example, while we don't have a unit element, we do have a function $' : X \to X$ characterized by $xx' = x$. (Sometimes $x' = x$.) So I can define $X \to F_X X^2$ by sending x to (x, x').

And there should be an operator $FX^{n-1} \to FX^n$ that is only a map of objects over X but hopefully at least additive, providing us with a contracting homotopy.

Here's the "face" structure of the ringoid $\mathcal{R}$ over which $F_X X^\bullet$ is a module. The objects form the set $\{1, 2, \ldots\}$. For all $n > 1$ there are operators

$$d_i, a_i, b \in \mathcal{R}(n, n-1)\,, \quad 1 \le i < n\,.$$

They satisfy the following relations in $\mathcal{R}(n+1, n-1)$.

$$d_i d_j = d_{j-1} d_i \quad \text{for} \quad i < j \tag{1}$$

$$a_i a_j = a_{j-1} a_i \quad \text{for} \quad i < j \tag{2}$$

$$d_i a_j = \begin{cases} a_{j-1} d_i & \text{for} \quad i < j \\ a_j d_{i+1} & \text{for} \quad i \ge j \end{cases} \tag{3}$$

$$b d_i = d_i b \quad \text{and} \quad b a_i = a_i b \quad \text{for} \quad 1 \le i < n \tag{4}$$

$$b d_n = b a_n + bb \tag{5}$$

These are painfully derived using the following information. (1) uses the rack axioms. (2) uses the $\alpha\alpha$ identity. (3) is the most painful one. It uses the rack identity on subscripts when $i < j - 1$. (4) uses the rack identity on subscripts in the d_1 case, and the $\alpha\beta$ identity for the ba_i case. (5) of course uses the last identity.

Maybe it's nicer to write these relations like this: for $i < j$,

$$d_i d_j = d_{j-1} d_i\,, \quad a_i a_j = a_{j-1} a_i\,, \quad a_i d_j = d_{j-1} a_i\,, \quad d_i a_j = a_{j-1} d_i\,.$$

Also, for $i < n$,

$$d_i b = b d_i \quad \text{and} \quad a_i b = b a_i \,,$$

while

$$b(d_n - a_n) = bb \,.$$

The symmetry between a and d in these relations is amazing.

There is an additive functor from this pre-additive category into the additive category of functors from racks to abelian groups, taking n to the functor $X \mapsto F_X X^n$. It's augmented to the functor $X \mapsto \Omega_X$.

An $\mathcal{R}$-module A determines a chain complex as follows. Define operators $A_{n+1} \to A_n$ by

$$\partial' = d_n - d_{n-1} + \cdots + (-1)^n d_1 \quad , \quad \partial' = a_n - a_{n-1} + \cdots + (-1)^n a_1 \,.$$

Then the usual calculation shows that

$$\partial'\partial' = \partial'\partial'' = \partial''\partial' = \partial''\partial'' = 0 \,,$$

while

$$b\partial' = b d_n \quad , \quad b\partial'' = b a_n \,.$$

So

$$\partial = \partial' - \partial'' - b : A_{n+1} \to A_n$$

defines a differential.

For example, with $X = *$, this is the complex of modules over the ring $R = \mathbb{Z}[\alpha^{\pm 1}, \beta]/(\beta(1 - \alpha - \beta))$ given by

$$R \xleftarrow{1-\alpha-\beta} R \xleftarrow{-\beta} R \xleftarrow{1-\alpha-\beta} R \xleftarrow{-\beta} \cdots \,.$$

In this case at least, the complex is exact.

The chain complex associated to an $\mathcal{R}$-module has a *maximal augmentation*

$$M(A_\bullet) = A_1/\partial A_2 \,.$$

In the case of the free resolution, $M(F(X^\bullet)) = \Omega_X$.

FGG prove the following important result. Suppose we have a map $X \to G$ from a set to a group. The set $X \times G$ becomes a rack under the operation

$$(x, f)(y, g) = (y, g f^{-1} x f)$$

(where we leave the map $X \to G$ undenoted). Proof: $(x,f)(z,h) = (z, hf^{-1}xf)$, so

$$((x,f)(y,g))(x,f)(z,h)) = (z, (hf^{-1}xf)(f^{-1}x^{-1}fg^{-1})y(gf^{-1}xf)) = (z, hg^{-1}ygf^{-1}xf)$$

while

$$(x,f)((y,g)(z,h)) = (x,f)(z, hg^{-1}yg) = (z, (hg^{-1}yg)f^{-1}xf)\,.$$

Theorem. (FGG) The free rack on a set X is given in terms of the inclusion of X into the free group GX by

$$FX = X \times GX\,.$$

For example, the free rack on a singleton is $\mathbb{Z}$ with rack structure given by $xy = y + 1$ for all $x, y \in \mathbb{Z}$ and inclusion sending $*$ to 1.

The free X-module functor F_X may be expressed in terms of the free rack functor F: Given a rack X, a set $S \downarrow X$ over X, and a rack $Y \downarrow X$ over X, consider

$$\begin{array}{ccc}
\mathrm{Map}_{\mathbf{Set}/X}(S,Y) & & \mathrm{Map}_{\mathbf{Rack}/X}(FS,Y) \\
\downarrow & & \downarrow \\
\mathrm{Map}(S,Y) & = & \mathrm{Map}_{\mathbf{Rack}}(FS,Y) \\
\Downarrow & & \Downarrow \\
\mathrm{Map}(S,X) & = & \mathrm{Map}_{\mathbf{Rack}}(FS,X)
\end{array}$$

Both columns are equalizers, so the top sets are canonically isomorphic. If Y is in fact an abelian object in $\mathbf{Rack}/X$, then this isomorphism leads to

$$\mathrm{Map}_{\mathbf{Set}/X}(S,Y) = \mathrm{Hom}_{X-\mathrm{mod}}(Ab_X(FS), Y)$$

so

$$F_X S = Ab_X(FS)\,.$$