Nishida relations and Singer construction

Steenrod began the analysis of the cohomology of an extended square, by means of the diagram

$$\begin{array}{ccccc} X^2 & \xrightarrow{i} & E\pi \times_\pi X^2 & \xleftarrow{\delta} & B\pi \times X \\ & & \big\downarrow{\scriptstyle p} & & \\ & & B\pi & & \end{array}$$

He showed that the map

$$H^*(E\pi \times_\pi X^2) \xrightarrow{(i^*,\delta^*)} H^*(X^2) \oplus [H^*(B\pi) \otimes H^*(X)] \tag{1}$$

is injective. He also described an operation

$$P : H^p(X) \to H^{2p}(E\pi \times_\pi X^2)$$

characterized by the equation

$$i^* P x = x \times x \in H^{2p}(X^2)$$

One way to obtain this is to consider the universal example, $X = K_p$, and contemplate the Serre spectral sequence for the Borel construction of the pair $(K_p \times K_p, K_p \vee K_p)$. Analogously, one may construct an operation

$$[-,-] : H^p(X) \otimes H^q(X) \to H^{p+q}(E\pi \times_\pi X^2)$$

characterized by requiring $[x, y]$ to restrict to $x \times y + y \times x \in H^{p+q}(X^2)$. To construct it one again considers the universal example, now $K_p \times K_q$. Now you should work modulo a certain equivariant subspace of $(K_p \times K_q)^2$. To describe it, indicate the first pair of Eilenberg Mac Lane spaces with primes and the second pair with double primes. Then the relevant subspace is

$$(K'_p \times K'_q \vee K''_p \times K''_q) \cup_{K'_p \vee K'_q \vee K''_p \vee K''_q} (K'_p \times K''_p \vee K'_q \times K''_q)$$

Then $\iota'_p \times \iota''_q + \iota'_q \times \iota''_p$ is in the bottom dimension and so survives, to a unique class written $[\iota_p, \iota_q]$. It is killed by the diagonal map δ.

Using the projection map $p : E\pi \times_\pi X^2 \to B\pi$, $H^*(E\pi \times_\pi X^2)$ becomes an $H^*(B\pi)$-module over A. Write t for the generator of $H^1(B\pi)$. In terms of this action, Steenrod operations are defined by

$$\delta^* P x = \sum_i t^{|x|-i} \mathrm{Sq}^i x \tag{2}$$

Claim. The action of the Steenrod algebra on the class $[x,y]$ is given by

$$\mathrm{Sq}^n[x,y] = \sum_{i+j=n} [\mathrm{Sq}^i x, \mathrm{Sq}^j y]$$

and on Px by

$$\mathrm{Sq}^n(Px) = \sum_j \binom{|x|-j}{n-2j} t^{n-2j} P(\mathrm{Sq}^j x) + \sum_{2i<n} [\mathrm{Sq}^i x, \mathrm{Sq}^{n-i} x] \tag{3}$$

We can use (1) to check these claims. The first claim is immediate. For the second, note that restricting to X^2 kills t, so the sum is zero if n is odd and has only the term $j = \frac{n}{2}$ if n is even. The restriction is then checked using the Cartan formula.

If we apply δ^* to (3) we get

$$\delta^*(\mathrm{Sq}^n Px) = \sum_{i,j} \binom{|x|-i}{n-2i} t^{|x|+n-i-j} \mathrm{Sq}^j \mathrm{Sq}^i x$$

while

$$\mathrm{Sq}^n(\delta^* Px) = \sum_{i,j} \binom{|x|-i}{n-j} t^{|x|+n-i-j} \mathrm{Sq}^j \mathrm{Sq}^i x$$

So we hope that

$$\sum_{i,j} \binom{|x|-i}{n-2i} \mathrm{Sq}^j \mathrm{Sq}^i = \sum_{i,j} \binom{|x|-i}{n-j} \mathrm{Sq}^j \mathrm{Sq}^i$$

This is precisely the identity Nick Kuhn showed me how to prove in Lemma 1.3 of [1].

More generally, modulo brackets we have

$$\mathrm{Sq}^n(t^j Px) \equiv \sum_i \binom{j+|x|-i}{n-2i} t^{j+n-2i} P(\mathrm{Sq}^i x) \tag{4}$$

To see this use the Cartan formula and $\mathrm{Sq}^k t^j = \binom{j}{k} t^{j+k}$ to write

$$\mathrm{Sq}^n(t^j Px) \equiv \sum_{k,i} \binom{j}{n-k} \binom{|x|-i}{k-2i} t^{j+n-2i} P(\mathrm{Sq}^i x)$$

Then use $(1+t)^j(1+t)^{|x|-i} = (1+t)^{j+|x|-i}$ to see that

$$\sum_k \binom{j}{n-k} \binom{|x|-i}{k-2i} = \binom{j+|x|-i}{n-2i}$$

and (4) follows.

Nishida [2] works in homology, not cohomology. He works with elements in $H_*(E\pi \times_\pi X^2)$ of the form $e_k \otimes Py$ where $y \in H_*(X)$ and $k \geq 0$, of dimension $k + 2|y|$, with the property that

$$\langle [w, x]\,, e_k \otimes Py\rangle = 0 \quad , \quad \langle t^j Px\,, e_k \otimes Py\rangle = \delta_k^j \langle x, y\rangle$$

He asserts that the right action of the Steenrod algebra is given by

$$(e_k \otimes Py)\mathrm{Sq}^n = \sum_i \binom{|y| + k - n}{n - 2i} e_{k-n+2i} \otimes P(y\mathrm{Sq}^i)$$

To check this, we pair the equation against $t^j Px$. The terms in the resulting sum are zero except possibly when $j = k - n + 2i$, when we get

$$\binom{|y| + k - n}{n - 2i} \langle x\,, y\,\mathrm{Sq}^i\rangle$$

On the other hand,

$$\langle \mathrm{Sq}^n(t^j Px)\,, e_k \otimes Py\rangle = \sum_i \binom{|x| + j - i}{n - 2i} \langle t^{j+n-2i} P(\mathrm{Sq}^i x)\,, e_k \otimes Py\rangle$$

The terms in this sum are again zero except when $k = j + n - 2i$, when we get

$$\binom{|x| + j - i}{n - 2i} \langle \mathrm{Sq}^i x\,, y\rangle$$

These terms are equal, since $|y| + k - n = |x| + j - i$.

The *Singer construction* is the functor R from unstable A-modules to unstable A-modules with a compatible action of $H^*(B\pi)$, given by

$$RM = H^*(B\pi) \otimes \Phi M$$

with Steenrod action described by (4). If we write $\mathrm{Sq}^{|x|+j+1} \otimes x$ in place of $t^j Px$, (4) is (1.5) of [3]. (The sum in the top case in (1.5) is automatically zero if $a > 2i$ by instability.)

The considerations above show that the map

$$\delta^* : RM \to H^*(B\pi) \otimes M$$

defined by (2) determines an $H^*(B\pi)$-module map over A. This map is injective. To see this, filter M by declaring $F^d M = \{x \in M : |x| \geq d\}$, and filter ΦM by declaring $F^d \Phi M = \Phi F^d M$. Extend this to filtrations of $H^*(B\pi)$ modules. Then δ^* is filtration preserving, and $\delta^*(t^j x) \equiv t^{|x|+j} x$ modulo smaller filtration.

This shows that RM is an A-module and is unstable as such.

The unstable condition can also be verified directly. We write $t^j Px$ for a decomposable tensor; then we define an A-action by equation (3).

Suppose there is a nonzero term in the sum; say the ith term. Then $i \leq |x|$ since M is unstable. Thus in order for the binomial coefficient to be nonzero we must have $n-2i \leq |x|-i$. Adding these two inequalities gives $n \leq 2|x|$, so RM is again unstable.

In any case, we now have the pullback square

$$\begin{array}{ccc} H^*(E\pi \times_\pi X^2) & \longrightarrow & R(H^*(X)) \\ \downarrow & & \downarrow \\ (H^*(X)^{\otimes 2})^\pi & \longrightarrow & \Phi H^*(X) \end{array}$$

of algebras over the Steenrod algebra.

References

[1] H. Miller, An algebraic analogue of a conjecture of G. Whitehead, Proc. Amer. Math. Soc. 84 (1982) 131–137.

[2] G. Nishida, Cohomology operations in iterated loop spaces, Proc. Japan Acad. 44 (1968) 104–109.

[3] W. M. Singer, The construction of certain algebras over the Steenrod algebra, Journal of Pure and Applied Algebra 11 (1977) 53–59.