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VANISHING LINES FOR MODULES OVER THE STEENROD ALGEBRA

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Introduction

The purpose of this paper is to establish criteria for freeness and for the existence of a vanishing line for modules over sub-Hopf algebras of the mod p Steenrod algebra. Here, a left module M over a connected K-algebra A is said to have a *vanishing line over A of slope d* provided that there exists an intercept $-c$ such that $\mathrm{Tor}^A_{s,t}(K,M)=0$ for all (s,t) satisfying $t<ds-c$. The criteria for freeness considered here have been dealt with by Adams and Margolis [3] in the case $p=2$, and by Moore and Peterson [13] in case p is odd. For $p=2$, the existence of these vanishing lines was proved independently by Adams, Anderson, and Mahowald, in special cases, and by Anderson and Davis [5] in general. For p odd, special cases were obtained by Liulevicius [9] and May [unpublished]. The general case has remained open and represents the main new result presented here.

To state the result, give the dual Steenrod algebra the basis of monomials in Milnor's generators [12]. Let P^s_t be dual to $\xi_t^{p^s}$, and (if $p\neq 2$) let Q_t be dual to τ_t. Then $(Q_t)^2=0$, and $(P^s_t)^p=0$ for $s<t$, so we can define for a module M the groups

$$H(M;Q_t)=\frac{\ker(Q_t\,|\,M)}{\mathrm{im}(Q_t\,|\,M)},\qquad H(M;P^s_t)=\frac{\ker(P^s_t\,|\,M)}{\mathrm{im}((P^s_t)^{p-1}\,|\,M)}.$$

Main Theorem. *Let A be a sub-Hopf algebra of the* mod p *Steenrod algebra, and let M be a $(k-1)$-connected A-module.*

(i) [3, 13]. *If $H(M;P^s_t)=0$ for all $P^s_t\in A$ with $s<t$, and (if $p\neq 2$) $H(M;Q_t)=0$ for all $Q_t\in A$, then M is A-free.*

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(ii) ([5] *for* $p=2$). *If* $H(M; P_t^s)=0$ *for all* $P_t^s \in A$ *such that* $s<t$ *and* $p|P_t^s|<2d$ (*where* $|\ |$ *denotes dimension*), *and* (*if* $p\neq 2$) $H(M; Q_t)=0$ *for all* $Q_t \in A$ *such that* $|Q_t|<d$, *then* M *has a vanishing line of slope* d, *with intercept* $-c$ *depending only on* d *and* k. *Furthermore, if* M *is of finite type, then* $\mathrm{Ext}_A^{s,t}(M, \mathbb{F}_p)=0$ *for* $t<ds-c$ *as well.*

Our proof closely follows the Anderson–Davis strategy [5]. In our extension to p possibly odd, however, we have overcome two significant technical obstacles. The Anderson–Davis method requires a proof of the nilpotence of certain cohomology classes. In [5], this is accomplished by direct computation in the bar construction. This is unpleasant for $p=2$, and seems intractible for p odd. We show that for all p, the required nilpotence follows from a simple calculation with Steenrod operations in the cohomology of a cocommutative Hopf algebra (see Proposition 4.1 below). Second, certain arguments for which long exact sequences sufficed in [5] must be replaced for p odd by manipulations with the spectral sequence associated to an extension of algebras in which the quotient is a polynomial algebra truncated at height p. We have been inspired here by the paper [14] of Quillen and Venkov.

Our use of Steenrod operations allows us to avoid the considerable amount of computation in the Steenrod algebra or its bar construction present in [3, 5, 13], and in this respect represents a simplification even in the proof of the freeness result and even at $p=2$. We have tried to present a complete account of the proof of the Theorem, relying whenever possible on general properties of Hopf algebras and of the (co)homology groups of algebras. In Section 1 we recall cap and cup products and the spectral sequence of an extension, and in Section 2 we study the (co)homology of modules over certain elementary algebras. Section 3 is the heart of the paper; there we establish three conditions under which a vanishing line over a sub-algebra $B \subset A$ extends to a vanishing line over A. The hypothesis that A is a sub-Hopf algebra of the Steenrod algebra is fitted together with these extension results in an inductive argument in Section 4.

Our decision to express the results and proofs in this paper in terms of $\mathrm{Tor}_*^A(N,M)$ represents a compromise. On the one hand we could have dealt with coalgebras, comodules, and Cotor; these notions are less familiar but for many purposes more convenient. On the other hand, we could have stuck with algebras but used $\mathrm{Ext}_A^*(M,K)$; this certainly has a higher level of recognition, but is only of interest for M of finite type, and furthermore has awkward variance properties. The disadvantage of our choice is that the familiar notion of a nilpotent action by a polynomial generator must be replaced by the dual concept of a conilpotent action (Definition 2.3) appropriate to the cap product pairing between cohomology and homology.

1. Preliminaries

We begin by establishing notational conventions concerning algebras, modules over algebras, and the homology and cohomology of an algebra with coefficients in a module.

We will always work within the category of $\mathbb{Z}$-graded modules over a fixed ground field K. If $x \in M^n$, we write $|x| = n$. The k-fold suspension of a module M is the module $\Sigma^k M$ such that $(\Sigma^k M)^n = M^{n-k}$.

All our *algebras* will be assumed connected, associative and defined over K. The augmentation $\varepsilon : A \to K$ of an algebra gives K its unique A-module structure.

A module M is $(k-1)$-*connected* provided $M^i = 0$ for $i < k$, and is *connective* (*bounded below* in the terminology of [3]) provided it is $(k-1)$-connected for some k. An algebra is $(k-1)$-*connected* provided its augmentation ideal is $(k-1)$-connected as a module.

Given a left A-module M, we shall write

$$H^A_*(M) = \mathrm{Tor}^A_*(K, M)$$

and for a right A-module N, we write

$$H^*_A(N) = \mathrm{Ext}^*_A(K, N).$$

Thus, if $P_* : 0 \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$ is a projective resolution of M by left A-modules, with augmentation $\varepsilon : P_0 \to M$, and Q_* is a projective resolution of K by right A-modules, then

$$H^A_*(M) = H(K \otimes_A P_*) = H(Q_* \otimes_A M), \qquad H^*_A(N) = H(\mathrm{Hom}_A(Q_*, N)).$$

These modules are of course bigraded, but we suppress the second grading – internal degree – except when required for clarity. Given a left A-module M, we can form the dual right A-module $M^* = \mathrm{Hom}_K(M, K)$, where the action is given by $\{f \circ a\}(m) = \{f\}(am)$ for $f \in \mathrm{Hom}_K(M, K)$, $a \in A$, $m \in M$. There are natural duality homomorphisms defined:

Lemma 1.1 [6, VI.5]. *Let M be a left A-module.*

(i) *There is a natural duality isomorphism*

$$\varrho : \mathrm{Ext}^*_A(K, \mathrm{Hom}_K(M, K)) \to \mathrm{Hom}_K(\mathrm{Tor}^A_*(K, M), K),$$

i.e. $\varrho : H^*_A(M^*) \to \mathrm{Hom}_K(H^A_*(M), K)$.

(ii) *If M and A have finite type over K, and M is connective, then the natural map* $\mathrm{Ext}^*_A(M, K) \to \mathrm{Ext}^*_A(K, M^*) = H^*_A(M^*)$ *is an isomorphism and the dual of ϱ reduces to*

$$\varrho^* : \mathrm{Tor}^A_*(K, M) \to \mathrm{Hom}_K(\mathrm{Ext}^*_A(M, K), K).$$

Lemma 1.2. *Let M be a connective A-module.*

(i) *M is A-free if and only if $H^A_1(M) = 0$.*

(ii) *Given* $s \geq 0$, *suppose* $H^A_{s,t}(M) = \mathrm{Tor}^A_{s,t}(K,M) = 0$ *for all* $t < T$. *If* N *is any* $(k-1)$-*connected right* A-*module, then* $\mathrm{Tor}^A_{s,t}(N,M) = 0$ *for all* $t < T + k$.

Proof. Let $\bar{\varepsilon}: K \otimes_A M \to M$ split the natural surjection, and extend $\bar{\varepsilon}$ to an A-module map $\varepsilon: A \otimes_K (K \otimes M) \to M$. Then $K \otimes \varepsilon$ is an isomorphism; and since a connective A-module M' is 0 if and only if $K \otimes_A M' = 0$, we find that ε is surjective. This observation allows the construction of a minimal resolution P_* of M, for which $H^A_s(M) = K \otimes_A P_s$. Thus if $H^A_1(M) = 0$, the augmentation ε in this resolution is an isomorphism, and M is free. For (ii) just use a minimal resolution of M to compute $\mathrm{Tor}^A_*(N,M)$. □

We shall write simply H^A_* and H^*_A for $H^A_*(K)$ and $H^*_A(K)$. H^*_A is a (bigraded) algebra, and it acts naturally on $H^*_A(N)$ and on $H^A_*(M)$ by ∪- and ∩-products. Since the ∩-product may be unfamiliar, we review these constructions.

Given projective resolutions P_* and Q_* of K by right A-modules, let $f: P_s \to \Sigma^m K$ and $g: Q_t \to \Sigma^n N$ be cocycles representing $\alpha \in H^{s,m}_A$ and $\beta \in H^{t,n}_A(N)$. The map f lifts to an A-module map $f_0: P_s \to \Sigma^m Q_0$ and hence to a chain map with components $f_r: P_{s+r} \to \Sigma^m Q_r$. The composite $\Sigma^m g \circ f_t$ is then a cocycle representing $\beta \cup \alpha \in H^{s+t,n+m}_A(N)$.

Next, given a cycle $x \in (P_{s+t} \otimes_A M)_{m+n}$ representing $\gamma \in H^A_{s+t,m+n}(M)$, the element $(f_t \otimes_A 1)(x) \in (P_t \otimes_A M)_n$ represents $\alpha \cap \gamma \in H^A_{t,n}(M)$.

One can show that these actions are compatible under the Kronecker pairing

$$\langle\ ,\ \rangle : H^{s,m}_A(M^*) \otimes H^A_{s,m}(M) \to K$$

induced by Lemma 1.1:

$$\langle \beta \cup \alpha, \gamma \rangle = \langle \beta, \alpha \cap \gamma \rangle .$$

A diagram

$$B \xrightarrow{\ \iota\ } A \xrightarrow{\ \pi\ } C$$

of algebras is called an *extension* provided $\pi\iota$ factors through K, $C \cong K \otimes_B A$ under the natural map, and A is free over B. In particular, if A is a connected co-commutative coassociative Hopf algebra, and B is a normal sub-Hopf algebra, $B \to A \to A//B$ is an extension of algebras [11, 4.9].

There is then a spectral sequence [6, p. 349]

$$E^2_{s,t} = H^C_s(H^B_t(M)) \Rightarrow H^A_{s+t}(M).$$

If we let H^*_C act on $H^A_*(M)$ via the map $\pi^*: H^*_C \to H^*_A$, this becomes a spectral sequence of H^*_C-modules.

Finally, we recall the change of rings isomorphism:

Lemma 1.3 [6, p. 117]. *Let* B *be a subalgebra of* A *such that* A *is* B-*free. If* M *is a left* A-*module and* N *a right* B-*module, then*

$$\mathrm{Tor}^B_*(N,M) \cong \mathrm{Tor}^A_*(N \otimes_B A, M).$$

2. Modules over 'elementary' algebras

From now on K will have characteristic $p \neq 0$. In Section 3 we shall study Hopf algebras and their (co)homology by building them up from Hopf algebras having one of the following simple algebra structures:

Type E: $\quad E = E[x] = K[x]/x^2, \quad |x|$ odd or $p = 2$.
Type D: $\quad D = D[x] = K[x]/x^p, \quad |x|$ even and $p \neq 2$.

We study the (co)homology of finite tensor products of these algebras in this section.

Lemma 2.1. (i) $H_E^* = K[v]$ *for* $v \in H_E^{1,|x|}$.
(ii) $H_D^* = E[h] \otimes K[b]$ *for* $h \in H_D^{1,|x|}$, $b \in H_D^{2,p|x|}$.

Proof. Standard; see for example [8]. □

Lemma 2.2. (i) *For any E-module M,* $v : H_{s+1}^E(M) \to H_s^E(M)$ *is a monomorphism for* $s \geq 0$ *and an isomorphism for* $s > 0$.
(ii) *For any D-module M,* $b : H_{s+2}^D(M) \to H_s^D(M)$ *is a monomorphism for* $s \geq 0$ *and an isomorphism for* $s > 0$.

Proof. Let C be either E or D. The result is true by Lemmas 1.1 and 2.1 if C acts on M via $\varepsilon : C \to K$. For M finite-dimensional, the proof is by induction on $\dim_K M$. Consider the short exact sequence

$$0 \to N \to M \xrightarrow{\pi} K \otimes_C M \to 0$$

of C-modules, where C acts on $K \otimes_C M$ via ε and π is the natural map. The five-lemma and the inductive hypothesis applied to N then give the result for M. The general case follows since homology commutes with direct limits. □

Definition 2.3. An element a of an algebra A acts *conilpotently* on an A-module M provided that

$$\bigcap_n a^n M = 0.$$

Corollary 2.4. (i) *If v acts conilpotently on* $H_*^E(M)$, *and M is connective, then M is E-free.*
(ii) *If b acts conilpotently on* $H_*^D(M)$, *and M is connective, then M is D-free.*

Proof. Combine Lemmas 1.2 and 2.2. □

If M is an $E=E[x]$-module, define

$$H_t(M;x)=\frac{\ker(x|M^t)}{\operatorname{im}(x)\cap M^t}.$$

If M is a $D=D[x]$-module, define

$$H_t(M;x)=\frac{\ker(x|M^t)}{\operatorname{im}(x^{p-1})\cap M^t},\qquad H_t(M;x^{p-1})=\frac{\ker(x^{p-1}|M^t)}{\operatorname{im}(x)\cap M^t}.$$

Lemma 2.5. (i) *For* $s>0$,

$$H^E_{s,t}(M)=H_{t-s|x|}(M;x).$$

(ii) [13, Prop. 3.4]. *For* $s>0$,

$$H^D_{2s,t}(M)=H_{t-ps|x|}(M;x^{p-1}),\qquad H^D_{2s+1,t}(M)=H_{t-(ps+1)|x|}(M;x).$$

Proof. These are easy computations with a minimal resolution, which we leave to the reader. □

We remark that 'chain complexes' with $d^p=0$ appear to have first been considered by Meyer [16].

Corollary 2.6. (i) *If M is a connective E- or D-module such that $H(M;x)=0$, then M is free.*

(ii) *If M is a $(k-1)$-connected E-module then $H^E_{s,t}(M)=0$ for $t<s|x|+k$.*

(iii) *If M is a $(k-1)$-connected D-module, then $H^D_{2s,t}(M)=0$ for $t<ps|x|+k$ and $H^D_{2s+1,t}(M)=0$ for $t<(ps+1)|x|+k$.*

Proof. Combine Lemmas 1.2 and 2.5. □

Corollary 2.6 forms the foundation for the next proposition. For precursors, see [3, Thm. 2.1] and [13, Lemmas 5.2 and 5.3].

Proposition 2.7. *Let A be a tensor product of algebras of types E and D, generated by elements $x_1,\dots,x_r$. Assume that the sequence sending i to $2|x_i|$ if $|x_i|$ is odd, and to $p|x_i|$ if $|x_i|$ is even, has no repetitions. Then a connective A-module M is A-free if and only if $H(M;x_i)=0$ for all i.*

Proof. By Lemmas 2.5 and 1.2, the proposition is true for $r=1$. We proceed by induction on r. If B is the subalgebra generated by x_r, then the extension

$$B\to A\to C$$

gives rise to a spectral sequence

$$E^2_{s,t}=H^C_s(H^B_t(M))\Rightarrow H^A_{s+t}(M).$$

Since M is B-free by assumption, the spectral sequence collapses to the equality

$$H_s^C(K\otimes_B M)=H_s^A(M).$$

We claim that $K\otimes_B M=M/x_rM$ is C-free; then $H_s^A(M)=0$ for $s>0$, so M is A free by Lemma 1.2.

It remains to check the claim. By the inductive hypothesis, it suffices to show that

$$H(M/x_rM;x_j)=0 \quad \text{for } 1\le j<r.$$

We suppose that $|x_r|$ and $|x_j|$ are both even, and leave the other cases to the reader. The short exact sequences

$$0\to\Sigma^{|x_r|}M/x_r^{p-1}M\xrightarrow{x_r}M\to M/x_rM\to 0,$$
$$0\to\Sigma^{(p-1)|x_r|}M/x_rM\xrightarrow{x_r^{p-1}}M\to M/x_r^{p-1}M\to 0$$

give rise to boundary homomorphisms [13, Prop. 3.2]

$$H_t(M/x_rM;x_j)\to H_{t+|x_j|-|x_r|}(M/x_r^{p-1}M;x_j^{p-1})$$
$$\to H_{t+p(|x_j|-|x_r|)}(M/x_rM;x_j).$$

Now $H(M;x_j)=0$ by assumption, so $H(M;x_j^{p-1})=0$ by Lemma 1.2, and it follows that both boundary maps are isomorphisms. Since $|x_j|\neq|x_r|$ and M is connective, we conclude that $H(M/x_rM;x_j)=0$ as desired. □

3. Extension of vanishing lines

We establish three key propositions guaranteeing a vanishing line for M over A given a vanishing line for M over a subalgebra B of A. The proofs are postponed until all three statements have been given.

Proposition 3.0. *Suppose that $B\to A\to C$ is an extension of algebras with $C=E[x]$ (respectively $C=D[x]$). Let M be a connective A-module, and assume that $v\in H_E^{1,|x|}$ (respectively $b\in H_D^{2,p|x|}$) acts conilpotently on $H_*^A(M)$.*

(i) *If M is B-free, then it is A-free.*

(ii) *If d and c are integers such that $d\ge|x|$ (respectively $2d\ge p|x|$) and $H_{s,t}^B(M)=0$ for $t<ds-c$, then $H_{s,t}^A(M)=0$ for $t<ds-c'$, where $c'=c+|x|$ (respectively $c'=c+(p-1)|x|$).*

Part (ii) of Proposition 3.0 in case $C=D[x]$ is an immediate consequence of the following sharper result, stated in terms of vanishing *functions*.

Proposition 3.1. *Suppose $B\to A\to D$ is an extension of algebras with $D=D[x]$. Let M be a connective A-module, and assume that $b\in H_D^{2,p|x|}$ acts conilpotently on $H_*^A(M)$. Let T be a function such that for all $s\ge 0$, $H_{s,t}^B(M)=0$ for $t<T(s)$, and*

suppose that for all $s \geq 0$, $T(s+2) \geq T(s) + p|x|$. *Then* $H^A_{s,t}(M) = 0$ *for all* $t < T'(s)$, *where*

$$T'(s) = \min\{T(s) - (p-1)|x|, T(s+1) - p|x|\}.$$

Proposition 3.2. *Let* B *be a subalgebra of* A *such that* $B^i = A^i$ *for* $i < d$ *and* A *is* B-*free. Let* M *be an* A-*module. If* $H^B_{s,t}(M) = 0$ *for* $t < ds - c$, *then also* $H^A_{s,t}(M) = 0$ *for* $t < ds - c$.

Our third proposition is a special feature of Hopf algebras. For us, a *Hopf algebra* is always connected, with associative and commutative diagonal.

Proposition 3.3. *Let* A *be a finite-dimensional Hopf algebra, and let* $B \subsetneq A$ *be a sub-Hopf algebra. Let* $C = K \otimes_B A$, *and assume that* $C^i = 0$ *for all* i *satisfying either* $0 < pi < 2d$ *and* i *even or* $0 < i < d$ *and* i *odd. If an* A-*module* M *has a vanishing line over* B *of slope* d, *then it has a vanishing line over* A *of slope* d *as well; and the change in the intercept depends only on* C.

Proof of Proposition 3.0 in case $C = E$. We follow [5, Theorem 3.2]: the short exact sequence of A-modules

$$0 \to \Sigma^{|x|} M \to A \otimes_B M \xrightarrow{\varphi} M \to 0,$$

in which φ is multiplication, gives rise to a long exact sequence in $H^A_*(\text{-})$:

$$\cdots \to H^A_{s+1,t+|x|}(M) \xrightarrow{v} H^A_{s,t}(M) \to H^B_{s,t+|x|}(M) \to \cdots.$$

Here we have rewritten $H^A_*(A \otimes_B M)$ using Lemma 1.3. See [1, 2.6.1] for the identification of the map marked v as v-multiplication.

(i) Since M is B-free, $H^B_s(M) = 0$ for $s > 0$, so the long exact sequence shows that $v \mid H^A_{s+1}(M)$ is epimorphic for $s > 0$. Since v acts conilpotently, we conclude that $H^A_s(M) = 0$ for $s > 0$. Thus M is A-free by Lemma 1.2.

(ii) The long exact sequence shows that $v \mid H^A_{s+1,t+|x|}(M)$ is epimorphic for $t + |x| < ds - c$. Iterating, using $d \geq |x|$, we find that for $t < ds - (c + |x|)$,

$$H^A_{s,t}(M) \subseteq \bigcap_{n \geq 0} v^n H^A_{s+n,t+n|x|}(M).$$

By the conilpotence of v we conclude that $H^A_{s,t}(M) = 0$. $\square$

Proof of Proposition 3.0(i) in case $C = D$. We no longer have such a long exact sequence, and must instead study the spectral sequence of the extension:

$$E^2_{s,t} = H^D_s(H^B_t(M) \Rightarrow H^A_{s+t}(M). \tag{$*$}$$

Since M is B-free, this collapses to the equality

$$H^D_s(K \otimes_B M) = H^A_s(M).$$

Since b acts conilpotently, $H_s^D(K\otimes_B M)=0$ for $s>0$ by Corollary 2.4; and hence, by Lemma 1.2, M is A-free. □

Proof of Proposition 3.1. Again, we study the spectral sequence (∗). The point of the proof is that the Quillen–Venkov induction argument [14; 15, (5.2)] shows that any change in b-multiplication being epimorphic in the spectral sequence occurs because of a differential hitting $E^r_{s,t}$ with $s=0$ or 1; and the connectivities of these modules, for $r=2$ and hence for $r\geq 2$, are controlled by Corollary 2.6.

It suffices to show that

$$b\,|\,H^A_{s+2,t+p|x|}(M) \text{ is onto for } t<T'(s). \qquad (**)$$

For, let $u\in H^A_{s,t}(M)$ with $t<T'(s)$. Then there exists $u_1\in H^A_{s+2,t+p|x|}(M)$ such that $bu_1=u$. Since $T(s+2)\geq T(s)+p|x|$ implies that $T'(s+2)\geq T'(s)+p|x|$, we can iterate the procedure to find u_n for all n such that $b^nu_n=u$. Thus $u=0$ by conilpotence.

We shall show by induction on r that in the spectral sequence (∗),

$(1)^r$ $\quad b\,|\,E^r_{s+2,t,n+p|x|}$ is one-to-one for all $s\geq 0$;

$(2)^r_s$ $\quad b\,|\,E^r_{s+2,t,n+p|x|}$ is onto for: $s=0$: $n<T(t)$,
$1\leq s\leq r-3$: $n<T'(s+t)$,
$s=r-2$: $n<T(s+t)-(p-1)|x|$,
$s\geq r-1$: n arbitrary.

Since $T'(q)\leq T(q)$, we conclude that $b\,|\bigoplus_{s+t=q}E^\infty_{s+2,t,n+p|x|}$ is onto for $n<T'(q)$, and (∗∗) follows by induction over the filtration on $H^A_{q+2,n+p|x|}(M)$.

For $r=2$, $(1)^2$ and $(2)^2_*$ hold by Lemma 2.2, so we proceed to the inductive step.

Proof of $(1)^r$. Let $[u]\in E^r_{s+2,t,n+p|x|}$ have $b[u]=0$. By $(1)^{r-1}$, either $u=0$ or there exists $y\in E^{r-1}_{s+r-1,t-r+2,n}$ such that $d^{r-1}y=bu$. By $(2)^{r-1}_{s+r-1}$, there exists z such that $bz=y$. Thus $b(d^{r-1}z-u)=0$; but, again by $(1)^{r-1}$, this implies $d^{r-1}z=u$, so $[u]=0$. This proves $(1)^r$.

Proof of $(2)^r_s$ *for* $s<r-3$. These cases follow from $(2)^{r-1}_s$ and the triviality of $d^{r-1}\,|\,E^{r-1}_{s,t}$ for $s<r-1$.

Proof of $(2)^r_{r-3}$. Given $[u]\in E^r_{s,t,n}$ with $s=r-3$ and $n<T(s+t)-(p-1)|x|$, $(2)^{r-1}_s$ implies that there exists $y\in E^{r-1}_{s+2,t,n+p|x|}$ such that $by=u$. Now $d^{r-1}y\in E^{r-1}_{0,s+t+1,n+p|x|}$, and this group is 0 for $n+p|x|<T(s+t+1)$, so y is then a d^{r-1}-cycle. Thus, in the presence of both conditions, $b\,|\,E^r_{s+2,t,n+p|x|}$ is onto as desired.

Proof of $(2)^r_{r-2}$. Given $[u]\in E^r_{s,t,n}$, with $s=r-2$, $(2)^{r-1}_s$ implies that there exists $y\in E^{r-1}_{s+2,t,n+p|x|}$ such that $by=u$. Now $d^{r-1}y\in E^{r-1}_{1,s+t,n+p|x|}$, and this group is 0 for $n+p|x|<|x|+T(s+t)$, so y is then a d^{r-1}-cycle, and this case follows.

Proof of $(2)_s^r$ *for* $s \geq r-1$. Given $[u] \in E_{s,t}^{r-1}$, $(2)_s^{r-1}$ implies that there exists $y \in E_{s+2,t}^{r-1}$ such that $by = u$. Thus $bd^{r-1}y = d^{r-1}u = 0$. By $(1)^{r-1}$, this implies that y is a d^{r-1}-cycle, so $b[y] = [u]$, and $b \mid E_{s+2,t}^r$ is epic as desired.

Proof of Proposition 3.2. The short exact sequence

$$0 \to I \to K \otimes_B A \to K \to 0$$

of right A-modules yields a long exact sequence in $\mathrm{Tor}^A(\text{-}, M)$:

$$\cdots \to H_s^B(M) \to H_s^A(M) \to \mathrm{Tor}_{s-1}^A(I, M) \to \cdots, \tag{$*$}$$

where we have used the change of rings isomorphism (Lemma 1.3) to rewrite $\mathrm{Tor}_s^A(K \otimes_B A, M)$. The proposition is clear for $s = 0$, and we proceed by induction on s. Since I is $(d-1)$-connected, Lemma 1.2 and the induction hypothesis imply that $\mathrm{Tor}_{s-1,t}^A(I, M) = 0$ for $t < d(s-1) - c + d = ds - c$. By assumption, $H_{s,t}^B(M) = 0$ for $t < ds - c$, so the middle term in $(*)$ is 0 as desired. □

The proof of Proposition 3.3 requires several lemmas, which we state now and prove at the end of the section.

Lemma 3.4. *Every finite-dimensional Hopf algebra has a central sub-Hopf algebra* C *of type E or D.*

Lemma 3.5. *If* B *is a proper sub-Hopf algebra of the finite-dimensional Hopf algebra* A, *then there is a sub-Hopf algebra* N *of* A *properly containing* B *as a normal sub-Hopf algebra.*

Lemma 3.6. *Let* C *be a finite-dimensional Hopf algebra such that* $C^i = 0$ *for all* i *satisfying either* $0 < pi < 2d$ *and* i *even or* $0 < i < d$ *and* i *odd. There exists an integer* c *such that for all* $(k-1)$*-connected* C*-modules* M, $H_{s,t}^C(M) = 0$ *for* $t < ds - c + k$.

Proof of Proposition 3.3. Suppose first that B is normal in A, with quotient Hopf algebra C, and consider the associated spectral sequence

$$E_{s,t}^2 = H_s^C(H_t^B(M)) \Rightarrow H_{s+t}^A(M).$$

Since $H_t^B(M)$ is $(dt-c-1)$-connected, and C satisfies the conditions of Lemma 3.6, there exists an integer c' such that $E_{s,t,n}^2 = 0$ for $n < ds - c' + dt - c = d(s+t) - (c'+c)$. The conclusion now follows from the convergence of the spectral sequence. If B is not normal in A, use induction over the chain of normal extensions guaranteed by Lemma 3.5. □

Proof of Lemma 3.4 [15]. Since A is finite-dimensional, there exists a primitive x of maximal dimension. Let C be the subalgebra generated by x; it is a sub-Hopf algebra since x is primitive. Furthermore, $x^p = 0$ by maximality of $|x|$, since x^p is primitive. Thus C is isomorphic to either $E[x]$ or $D[x]$. If C were not central, there would exist $y \neq 0$ in A of minimal dimension such that $[x,y] \neq 0$. A computation of $\Delta[x,y]$ shows that minimality of $|y|$ forces y to be primitive; but then $[x,y]=0$ by maximality of $|x|$. □

Proof of Lemma 3.5. We proceed by induction on $\dim_K A$. Let C be a central sub-Hopf algebra of A of type E or D. Then either $C \subsetneq B$ or $C \cap B = K$. If $C \subsetneq B$, the inductive hypothesis guarantees a proper normal extension $\bar{N}$ of $B//C$ in $A//C$, and the pull-back of $\bar{N}$ to A is the desired subalgebra N. If $C \cap B = K$, let N be the sub-algebra of A generated by B and C. Since B and C are sub-Hopf algebras, N is a sub-Hopf algebra, and since C is central, B is normal in N. □

Proof of Lemma 3.6. By Lemma 3.5 there is a filtration $K = F_0 \subset F_1 \subset \cdots \subset F_r = A$ by sub-Hopf algebras such that F_{i-1} is normal in F_i with quotient $E[x_i]$ or $D[x_i]$. We will show that the intercept c can be taken to be

$$c_A = (\tfrac{1}{2}p - 1) \sum_{|x_i| \text{ even}} |x_i|,$$

a number easily seen to be independent of the filtration.

We prove Lemma 3.6 by induction on $\dim_K A$. For $A = K$, the result is trivial. For $\dim_K A > 1$, let B be a central sub-Hopf algebra of type E or D as in Lemma 3.4, with quotient C, and consider the resulting spectral sequence with

$$E^2_{s,t,n} = H^C_{s,n}(H^B_{t,*}(M)).$$

If $B = E[x]$, then the inductive hypothesis and Lemma 2.5 imply that $E^2_{s,t,n} = 0$ for $n < ds - c_C + (t|x| + k)$, and hence for $n < d(s+t) - c_C + k$ since $|x| \geq d$. If $B = D[x]$, then again the inductive hypothesis and Lemma 2.5 imply that $E^2_{s,t,n} = 0$ for $n < ds - c_C + ((\tfrac{1}{2}p(t-1)+1)|x| + k)$, and hence for $n < d(s+t) - (c_C + (\tfrac{1}{2}p - 1)|x|) + k$ since $p|x| \geq 2d$. This completes the induction, by convergence of the spectral sequence. □

4. Proof of the Main Theorem

We now put the general considerations of Sections 2 and 3 together with the special structure of the sub-Hopf algebras of the Steenrod algebra $\mathscr{A}^*$ in order to prove the Main Theorem stated in the introduction. The prime p continues to be arbitrary, but K is henceforth the prime field $\mathbb{F}_p$.

Recall [12] that the dual of the Steenrod algebra is

$$\mathscr{A}_* = K[\xi_1, \xi_2, \ldots] \otimes E[\tau_0, \tau_1, \ldots]$$

as an algebra (omitting the τ's if $p=2$). Give $\mathscr{A}_*$ the basis of monomials in these generators, and let P^s_t be dual to $\xi_t^{p^s}$, and (if $p \neq 2$) let Q_t be dual to τ_t. Then $|P^s_t| = 2^s(2^t - 1)$ if $p=2$ and $|Q_t| = 2p^t - 1$.

According to [4], any sub-Hopf algebra of the Steenrod algebra is generated by the P_t^s's and Q_t's contained in it. Dually, any quotient Hopf algebra of $\mathscr{A}_*$ is of the form

$$\mathscr{A}_*/(\xi_1^{p^{e(1)}}, \xi_2^{p^{e(2)}}, \ldots, \tau_0^{2^{k(0)}}, \tau_1^{2^{k(1)}}, \ldots) \tag{$*$}$$

where $0 \le e(i) \le \infty$ and $0 \le k(i) \le 1$. The sequences e and k must satisfy certain obvious conditions imposed by the Milnor diagonal.

It is easy to check, for example, that $K[\xi_t]/(\xi_t^{p^t})$ is a quotient Hopf algebra of $\mathscr{A}_*$ (by sending all other generators to 0). Thus the elements P_t^s for $s<t$ lie in a divided polynomial sub-Hopf algebra of $\mathscr{A}^*$, and hence all have height p.

The following proposition, whose proof will be given at the end of the section, plays a central role in the proof of the main theorem.

Proposition 4.1 ([5] for $p=2$). *Let A be a sub-Hopf algebra of the Steenrod algebra such that the description $(*)$ of its dual has*

$$e(i)=0 \quad \textit{for } i<t.$$

Then there is a map $A \to D[P_t^s]=D$ for $s=e(t)-1$ ($A \to E[P_t^s]=E$ for $p=2$). Write $x=P_s^t$. If $s \ge t$, the pullback of the generator

$$b_s \in H_D^{2,p|x|} \qquad (h_s \in H_E^{1,|x|} \textit{ if } p=2)$$

to A is nilpotent.

A Steenrod operation proof of this proposition appears in Lin [7] for $p=2$.

Proof of the Main Theorem. Any skeleton of the Steenrod algebra is contained in a finite-dimensional sub-Hopf algebra: for example, the duals of

$$\mathscr{A}_*/(\xi_1^{p^n}, \xi_2^{p^{n-1}}, \ldots, \xi_{n+1}, \ldots; \tau_{n+1}, \tau_{n+2}, \ldots)$$

for $n \ge 0$ exhaust $\mathscr{A}^*$. The same is therefore true of the sub-Hopf algebra A of $\mathscr{A}^*$. To prove part (i), it suffices by Lemma 1.2 to show that $H_{s,t}^A(M)=0$ for all $s>0$ and all t. Since this group agrees with the homology over a sufficiently large sub-Hopf algebra of A, we may assume that A is finite. For part (ii), Proposition 3.2 similarly lets us suppose that A is finite-dimensional. The proof is then by induction on $\dim_K A$. For clarity, we treat only part (ii), and let the reader make the minor modifications necessary to produce a proof of part (i).

Write A in the form $(*)$, and let

$$t=\min\{j : e(j)>0 \text{ or } k(j-1)>0\}.$$

There are three cases to consider.

Case 1: $e(t)=0$. Then there is an extension of Hopf algebras

$$B \to A \to E$$

where $E = E[Q_{t-1}]$. If $|Q_{t-1}| \geq d$ we apply Proposition 3.3 to the inclusion $B \subsetneq A$ to see that the assumed vanishing line over B of slope d extends to one over A. If $|Q_{t-1}| < d$, we note that the extension splits (uniquely), and apply Proposition 3.3 to the resulting inclusion $E \subsetneq A$, using the fact that M is E-free, since we assume $H(M, Q_{t-1}) = 0$. This gives a vanishing line over A of slope greater than $|Q_{t-1}|$, so the generator $v_{t-1} \in H_E^{1,|Q_{t-1}|}$ acts conilpotently on $H_*^A(M)$. Hence, by Proposition 3.0, the vanishing line of slope d over B is passed on to A.

If $e(t) > 0$, we have an extension of Hopf algebras

$$B \to A \to D$$

in which $D = D[P_t^s]$, with $s = e(t) - 1$. If $p|P_t^s| \geq 2d$, we apply Proposition 3.3 to the inclusion $B \subsetneq A$ to see that the vanishing line over B extends to one over A. Henceforth, therefore, we suppose that $p|P_t^s| < 2d$.

Case 2: $s < t$. By assumption, $H(M; P_t^s) = 0$. Also, $H(M; P_j^i) = 0$ for all (i, j) such that $P_j^i \in A$ and $0 \leq i < j$, $t < j$, and $i + j \leq s + t - 1$, and (if $p \neq 2$) $H(M; Q_j) = 0$ for all j such that $Q_j \in A$ and $t \leq j \leq s + t$: for each of these elements has dimension less than $|P_t^s|$. Together with P_t^s, they generate a subalgebra C of A isomorphic as an algebra to a tensor product of Hopf algebras of types E and D. The dimensions are such that we can conclude that M is C-free by Proposition 2.7. We now apply Proposition 3.3 to the inclusion $C \subsetneq A$ to obtain a vanishing line over A of slope greater than $\frac{1}{2}p|P_t^s|$. Thus the generator $b_s \in H_D^{2,p|P_t^s|}$ acts conilpotently on $H_A^*(M)$, and hence, by Proposition 3.0, the vanishing line of slope d over B is passed on to A.

Case 3: $s \geq t$. Under the Hopf algebra map $A \to D$, the generator $b_s \in H_D^{2,p|P_t^s|}$ pulls back to a nilpotent element, by Proposition 4.1. Hence b_s acts conilpotently on $H_*^A(M)$, so Proposition 3.0 again finishes the proof.

The assertion about cohomology vanishing lines follows by Lemma 1.1. □

Proof of Proposition 4.1. Let C be the dual of $K[\xi_t]/(\xi_t^{p^{s+1}})$, and consider the obvious Hopf algebra map $A \to C$. Suppose first that p is odd. As an algebra,

$$C \cong \bigotimes_{i=0}^{s} D[P_t^i],$$

so that

$$H_C^* \cong E[h_0, \ldots, h_s] \otimes K[b_0, \ldots, b_s].$$

The Steenrod operations [7, 8, 10, 15] act in the cohomology of the Hopf algebra C by

$$\tilde{P}^0 h_i = h_{i+1}, \qquad \beta\tilde{P}^0 h_i = b_i,$$

for $0 \leq i \leq s$, with $h_{s+1} = 0$. Furthermore, the action is unstable with respect to homological degree ($\tilde{P}^r x = x^p$ if $x \in H^{2r}$, and $\tilde{P}^r x = 0$ if $x \in H^s$ with $2s < r$), and satisfies the usual Cartan formula. Thus, writing $\tilde{P} = \sum_{r=0}^{\infty} \tilde{P}^r$ for the total Steenrod operation, $\tilde{P} b_i = b_{i+1} + b_i^p$.

Since h_i is the class of $[\xi_t^{p^i}]$ in the cobar construction [1, 8], the reduced diagonal

$$\bar{\Delta}\xi_{2t}^{p^{s-t}} = \xi_t^{p^s} \otimes \xi_t^{p^{s-t}}$$

in the dual of A yields the relation

$$h_s h_{s-t} = 0$$

in H_A^*. Applying $\widetilde{\beta P}^1 \widetilde{\beta P}^0$, we obtain a relation among the b's:

$$b_s^p b_{s-t+1} = 0.$$

Now apply $\tilde{P}^{p^{t-1}} \cdots \tilde{P}^{p^2} \tilde{P}^p$ to find

$$b_s^{p^t} b_s = 0,$$

so b_s is nilpotent as claimed.

For $p=2$, we have

$$C \cong \bigotimes_{i=0}^{s} E[P_t^i]$$

as algebras, so $H_C^* = K[h_0, \dots, h_s]$, with Steenrod operations given by $\widetilde{Sq}^0 h_i = h_{i+1}$ for $0 \le i \le s$, with $h_{s+1} = 0$. As before, $h_s h_{s-t} = 0$ in H_A^*, and the Steenrod operation $\widetilde{Sq}^{2^{t+1}} \cdots \widetilde{Sq}^1$ applied to this relation yields

$$h_s^{2^t} h_s = 0,$$

so h_s is nilpotent as claimed. □

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