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The Sullivan conjecture on maps from classifying spaces

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Dedicated to John C. Moore
on the occasion of his sixtieth birthday

Introduction

In this paper we shall prove the following theorem, resolving in the affirmative a conjecture of D. Sullivan [40: p. 5.118].

THEOREM A. *Let G be a discrete group which is locally finite (i.e. every finitely generated subgroup is finite), and let X be a connected finite dimensional* CW *complex. Then the space of pointed maps to X from the classifying space of G has the weak homotopy type of a point*:

$$\pi_*\mathrm{map}_*(BG, X) = 0.$$

This theorem presents a curious feature of loop spaces of finite dimensional complexes X: For any $n \geq 0$ and any locally finite group G, every pointed map $BG \to \Omega^n X$ is null-homotopic through pointed maps. Thus for example no essential map from $\mathbf{R}P^m$ to $\Omega^n X$ can be extended over $\mathbf{R}P^{m+s}$ for all s.

Alex Zabrodsky has pointed out that the following extension is a corollary.

THEOREM A′. *Let W be a connected* CW *complex such that each homotopy group $\pi_i(W)$ is locally finite and such that $\pi_i(W)$ is nonzero for only finitely many i. Let X be a connected finite dimensional* CW *complex. Then*

$$\pi_*\mathrm{map}_*(W, X) = 0.$$

We give the simple deduction of this from Theorem A in Section 9. Zabrodsky [44] has carried this further, to obtain information even when $\pi_*(W)$ is not torsion. We remark also that C. McGibbon and J. Neisendorfer [28] have used Theorem A (or rather, Theorem C below) to prove the conjecture of J-P.

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Serre [38]: Let X be a simply connected space such that $H_*(X; F_p)$ is nonzero for only finitely many degrees. Then either $\pi_*(X)$ contains no elements of order p at all, or it contains them in arbitrarily large dimensions.

One may view Theorem A as an unstable analogue of the Burnside ring conjecture of G. B. Segal [2]. A proof of this conjecture has recently been completed by G. Carlsson. For a finite group G, it computes the stable cohomotopy of BG, and asserts, among other things, that

$$\{BG, S^q\} = \left[BG, \varinjlim_n \Omega^n S^{n+q}\right]$$

is trivial for $q > 0$. On the other hand if $q \leq 0$ it predicts that $\{BG, S^q\}$ will be nonzero—indeed, when $q = 0$, it will generally be uncountable. This is in contrast to the consequence of Theorem A: For any $q \in \mathbf{Z}$,

$$\varinjlim_n [BG, \Omega^n S^{n+q}] = 0.$$

The moral one draws is that none of the essential maps to $\varinjlim \Omega^n S^{n+q}$ can be compressed to $\Omega^n S^{n+q}$, for any n. This may be regarded as a dramatic instance of J. F. Adams' dictum [1], "Cells now, maps later."

We will now outline the proof of Theorem A. There are three subsidiary results. Two use the notion of a nilpotent space [11: II, 4.3, p. 59]. This is a path-connected space such that π_1 is nilpotent and acts nilpotently on π_n for all $n > 1$. Also, we say that a graded abelian group is *bounded* provided that it is nonzero in only finitely many degrees. Finally, all classifying spaces will be assumed to be CW complexes, and all base points will be vertices. Z_p denotes the group of order p.

THEOREM B. *Let X be a finite dimensional* CW *complex and G a torsion group. Then any pointed map $BG \to X$ induces the trivial map of fundamental groups.*

THEOREM C. *Let X be a nilpotent space such that $\overline{H}_*(X; F_p)$ is bounded. Then* $\mathrm{map}_*(BZ_p, X)$ *is weakly contractible.*

THEOREM D. *Let X be a nilpotent space and G a locally finite group. Assume that* $\mathrm{map}_*(BZ_p, X)$ *is weakly contractible for every prime p occurring as the order of an element of G. Then* $\mathrm{map}_*(BG, X)$ *is weakly contractible.*

Note that Theorems C and D provide a strengthening of Theorem A in case X is nilpotent: all that is required for $\mathrm{map}_*(BG, X)$ to be weakly contractible is that $H_*(X; F_p)$ be bounded for every prime p occurring as the order of an element of G. This strengthening extends to Theorem A′ as well.

To prove Theorem A, take an element of $\pi_n \mathrm{map}_*(BG, X)$ and form its adjoint, $f\colon \Sigma^n BG \to X$. (It is now apparent that we intend the compactly generated topology [42: I, § 4] associated to the usual compact open topology on all mapping spaces.) The map f lifts to the universal cover of X: if $n > 0$ this is obvious, while in the other case we appeal to Theorem B. We may thus assume that X is simply connected, and so in particular nilpotent; and Theorems C and D then finish the proof.

Most of the paper is written in the setting of simplicial sets. The standard comparison theorems (see [11: VIII, § 4], for example) easily provide a translation.

We conclude the introduction by sketching the proofs of Theorems B, C, and D, and explaining the organization of the paper.

Theorem B is an exercise in covering spaces and K-theory; it is carried out in Section 10. Theorem D also is not a long story, though more technical; the essential idea is to find a replacement for $B(G/H)$ when H is not a normal subgroup of G, and use it in an induction. This is done, following an approach due to M. J. Hopkins, in Section 9.

Theorem C is the heart of the matter. In Section 1, we recall work of Bousfield, Dror, Dwyer, and Kan, showing that it would suffice to prove that certain Ext sets

$$(*) \qquad \mathrm{Ext}^s_{\mathbf{CA}}\big(\overline{H}_*(\Sigma^n BZ_p), \overline{H}_*(X)\big)$$

are zero for all $n \geq s$. These are defined cosimplicially, and depend on the structure of $\overline{H}_*(X)$ both as a coalgebra and as a module over the Steenrod algebra A. When $n = s = 0$ we have only a pointed set, but its triviality in this case is easy (1.15). When $n > 0$, $\overline{H}_*(\Sigma^n BZ_p)$ is a cogroup object, and (*) forms a group—indeed, an F_p-vector space. There is then a standard homological technique for tweezing apart the effects of the two structures, recalled in Section 2. This allows us to proceed in two steps: (1) show that the relevant homological construction on coalgebras (namely, formation of the derived functors of primitives) preserves boundedness (Theorem 2.6); and (2) show that the relevant homological construction on unstable A-modules (namely, formation of the derived functors of $\mathrm{Hom}_A(\overline{H}_*(\Sigma^n BZ_p), -)$) yields zero given bounded input (Theorem 2.7).

Step (1) is carried out in Sections 3, 4, and 5. In these sections we choose to work with algebras and simplicial objects, rather than with coalgebras and cosimplicial objects, because of their greater familiarity and in order to make reference to other work easier. As explained at the end of Section 5, the dualization causes no trouble.

In Section 3 we review in our context D. G. Quillen's "homotopical algebra" setting for studying derived functors from nonadditive categories. This is useful as an orientation, and is moreover essential in the proof. The derived functors of indecomposables are defined and generalized to the "Quillen homology" of a simplicial supplemented commutative graded F_p-algebra. A new element here is the introduction of a canonical resolution. This resolution is naturally filtered, and the resulting spectral sequence is studied in Section 4. When combined with a small amount of computation, it leads to a boundedness result (Theorem 4.2) for Quillen homology. Finally, in Section 5, the Eilenberg-MacLane $\overline{W}$ construction is studied, and shown to serve as a suspension in this theory: its application merely shifts the degrees of the Quillen homology. The boundedness result of the preceding section then easily yields the required result for derived functors of indecomposables (Theorem 5.1).

Theorem 5.1, incidentally, results in a logarithmic "lower vanishing curve" for E_2 of the Bousfield-Kan unstable Adams spectral sequence [10] for a space with bounded mod p homology. This is given as Theorem 8.9 below, and may have some independent interest.

Step (2) is carried out in Sections 6 and 8. The vanishing result for $\mathrm{Ext}^s(\overline{H}_*(\Sigma^n BZ_p), -)$ in the category of unstable right A-modules follows easily from the surprising fact that $\overline{H}_*(BZ_p)$ is a summand over A of a direct limit of projective objects. This splitting result is due to G. Carlsson [14] when $p = 2$, and provided the initial motivation for the entire project. Carlsson's proof must be modified somewhat to deal with odd primes. The projective objects turn out to be familiar: each is the homology of the Spanier-Whitehead dual of a Brown-Gitler spectrum [12], [16]. This observation leads to a novel characterization of Brown-Gitler spectra, presented in Section 7.

In summary, the proof proceeds by constructing a long chain of spectral sequences and then showing that, miraculously, the initial term of the initial spectral sequence is trivial. It may be used somewhat more generally to address the following question. Suppose that $\overline{H}_*(W; \mathbf{Z})$ is altogether p-torsion. Is $\mathrm{map}_*(W, X)$ weakly contractible for every nilpotent space X with bounded mod p homology? The work gives a sufficient condition entirely in terms of the A-module structure of $\overline{H}_*(W; F_p)$; see Theorem 8.8.

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Notation. We will use the following symbols. For categories:

$\mathbf{S}$ = category of sets;
$\mathbf{S}_*$ = category of pointed sets;
$\mathbf{R}$ = category of modules over a ring R.

If $\mathbf{C}$ is any category,

$n\mathbf{C}$ = category of nonnegatively graded objects over $\mathbf{C}$;
$n_+\mathbf{C}$ = category of positively graded objects over $\mathbf{C}$;
$s\mathbf{C}$ = category of simplicial objects over $\mathbf{C}$;
$s_+\mathbf{C}$ = category of simplicial objects over $\mathbf{C}$ which are *connected*, i.e., such that the coequalizer of $X_1 \rightrightarrows X_0$ exists and is a terminal object in $\mathbf{C}$;
$s^\circ\mathbf{C}$ = category of cosimplicial objects over $\mathbf{C}$.

Also, if p is a prime number,

Z_p = the group with p elements;
F_p = the field with p elements.

Finally, $H_*(-)$ will always denote homology with coefficients in F_p for some prime p evident from the context.

1. The Bousfield-Kan construction

In this section we shall review the work of Bousfield and Kan [10], [11], as supplemented by Dror, Dwyer, and Kan [17], insofar as it is relevant to our approach to the Sullivan conjecture. Their work results in a homological criterion, applicable under suitable conditions, for the weak contractibility of a mapping space: Theorem 1.13 below.

We remind the reader that "space" means "simplicial set."

Bousfield and Kan begin in [11: I] by constructing, for any ring R, a cosimplicial "resolution" of a pointed space X. To explain this, suppose first that S is a pointed set: $S \in \mathbf{S}_*$. Let $\bar{R}S$ be the free R-module generated by S, modulo the relation $[*] = 0$, where $*$ is the basepoint of S. There are natural transformations η: $S \to \overline{R}S$ and μ: $\overline{R}\overline{R}S \to \overline{R}S$, by $\eta(s) = [s]$ and $\mu(r[r'[s]]) = rr'[s]$, and $(\overline{R}, \mu, \eta)$ forms a *triple* [5].

Any triple (T, μ, η) on a category $\mathbf{C}$ determines a functor $T^\bullet$: $\mathbf{C} \to s^\circ\mathbf{C}$ to the category of cosimplicial objects [11: X.2.1, p. 267] over $\mathbf{C}$:

$$(T^\bullet C)^n = T^{n+1}C,$$
$$d^i = T^i\eta T^{n-i}C\colon T^{n-1}C \to T^nC, \tag{1.1}$$
$$s^i = T^i\mu T^{n-i}C\colon T^{n+1}C \to T^nC.$$

Moreover, η: $C \to (T^\bullet C)^0$ determines a "coaugmentation," a map from the constant cosimplicial object $\underline{C}$ with C in each codegree.

Thus we have, for $X \in s\mathbf{S}_*$, a map $\underline{X} \to \overline{R}^\bullet X$ of cosimplicial pointed spaces. $\overline{R}^\bullet X$ is naturally *group-like* [11: X.4.8, p. 275]: $(\overline{R}^\bullet X)^n$ has a natural group structure for each $n \geq 0$, and each codegeneracy map s^i is a homomorphism.

Bousfield and Kan next show how to "collapse" a cosimplicial space X to obtain a space $\operatorname{tot} X$. For this, notice that there is a "tautologous" cosimplicial space Δ, with the simplicial n-simplex Δ^n in codegree n and with the evident cosimplicial operators. This object enables one to define, for any $W, X \in s^\circ s\mathbf{S}$, a mapping space map $(W, X) \in s\mathbf{S}$ with

$$\operatorname{map}(W, X)_n = s^\circ s\mathbf{S}(\Delta^n \times W, X).$$

Here $\Delta^n \times W$ denotes the evident product in $s^\circ s\mathbf{S}$.

The most naive thing one could do now is take $W = *$. The resulting space may be described using "cohomotopy": for $C \in s^\circ\mathbf{S}$, let $\pi^0(C)$ denote the equalizer in

$$\pi^0(C) \to C^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} C^1. \tag{1.2}$$

For $X \in s^\circ s\mathbf{S}$, then, $\pi^0(X)$ is a space, and

$$\operatorname{map}(*, X) = \pi^0(X).$$

This is clearly not an adequate invariant of X, and one phrasing of the reason is that $*$ is not "cofibrant" in the model category structure on $s^\circ s\mathbf{S}$ [11: X § 5, p. 277]. It is weakly equivalent in that structure to Δ, however, which, as it turns out, is cofibrant; so define

$$\operatorname{tot} X = \operatorname{map}(\Delta, X). \tag{1.3}$$

An immediate consequence of the definition is that if $X \in s\mathbf{S}$ then naturally

$$\operatorname{tot} \underline{X} = X.$$

Also:

LEMMA 1.4. *If $W \in s\mathbf{S}_*$ and $X \in s^\circ s\mathbf{S}_*$, then naturally*

$$\operatorname{map}_*(W, \operatorname{tot} X) = \operatorname{tot} \operatorname{map}_*(W, X). \qquad \square$$

If R is a ring and $X \in s\mathbf{S}_*$, then the map of pointed spaces

$$X = \operatorname{tot} \underline{X} \xrightarrow{\eta} \operatorname{tot} \overline{R}^{\bullet} X \stackrel{\text{def}}{=} R_\infty X$$

is the Bousfield-Kan *R-completion* of X. The first question to ask is: How close is η to being a weak equivalence? Since we are interested in mapping spaces, the following theorem is suitable for our purposes. We thank A. K. Bousfield for the statement and proof in the present generality.

THEOREM 1.5. *Let W be a connected space such that $\overline{H}_*(W; \mathbf{Z}[1/p]) = 0$, and let X be a nilpotent fibrant space. Then η: $X \to F_{p\infty} X$ induces a weak equivalence*

$$\operatorname{map}_*(W, X) \to \operatorname{map}_*(W, F_{p\infty} X).$$

Proof. Recall from [17] that there is up to homotopy a fiber square

$$\begin{array}{ccc} X & \longrightarrow & \prod_l X_{Z_l} \\ \Big\downarrow & & \Big\downarrow \\ X_{\mathbf{Q}} & \longrightarrow & \left(\prod_l X_{Z_l}\right)_{\mathbf{Q}} \end{array}$$

where X_G denotes the Bousfield $H_*(-\,; G)$-localization of X [9]. Thus there is, up to homotopy, an analogous fiber square of mapping spaces with source space W. Now Proposition 12.2 of [9] easily implies that $\operatorname{map}_*(B, C)$ is contractible whenever B is h_*-acyclic and C is h_*-local. If $h_*(-) = H_*(-\,; \mathbf{Z}[1/p])$, it follows that $\operatorname{map}_*(W, Y_G) \cong *$, where Y is any space and $G = \mathbf{Q}$ or $G = Z_l$ with $l \neq p$. Thus the fiber square implies that the map

$$\operatorname{map}_*(W, X) \to \operatorname{map}_*\left(W, X_{Z_p}\right)$$

is an equivalence, and the proposition follows since $X_{Z_p} \simeq F_{p\infty} X$ by [9: § 4]. $\square$

Using Lemma 1.4 we may rewrite the weak equivalence resulting from Theorem 1.5 as

$$\operatorname{map}_*(W, X) \to \operatorname{tot} \operatorname{map}_*\left(W, \overline{F}_p^{\bullet} X\right). \tag{1.6}$$

The cosimplicial space $\operatorname{map}_*(W, \overline{F}_p^{\bullet} X)$ is group-like since $\overline{F}_p^{\bullet} X$ is. We now recall that if $X \in s^\circ s\mathbf{S}_*$ is group-like (or even merely "fibrant" [11: X.4.6, p. 275]), Bousfield and Kan provide a condition guaranteeing that tot X is k-connected. It is expressed in terms of "cohomotopy." If $C \in s^{\circ}\mathbf{S}_*$, then we defined the pointed set $\pi^0(C)$ in (1.2). If C is a cosimplicial abelian group we may define $\pi^n(C)$ for all $n \geq 0$ as the nth cohomology group of the cochain complex with C^n in degree n and boundary map $\Sigma(-1)^i d^i$.

PROPOSITION 1.7 [11: X.7.3, p. 285]. *Let $k \geq 0$ and let $X \in s^{\circ}s\mathbf{S}_*$ be group-like. If $\pi^s\pi_t(X) = 0$ for $0 \leq t - s \leq k$, then $\pi_r(\mathrm{tot}\, X) = 0$ for $r \leq k$.* □

Notice that when $t = 0$, so that $\pi_t(X)$ is merely a cosimplicial pointed set and only $\pi^0\pi_t(X)$ is defined, we have only to consider $s = 0$. When $t > 0$, $\pi_t(X)$ is a cosimplicial abelian group (since X is group-like) so that $\pi^s\pi_t(X)$ is defined and is an abelian group for all $s \geq 0$. Only the groups for which $s \leq t$ are relevant to us at present, however; the significance of the groups $\pi^s\pi_t(X)$ for $s > t$ is an interesting question.

The objects $\pi^s\pi_t\,\mathrm{map}_*(W, \bar{F}_p^{\bullet}X)$ were described in earlier work [10] of Bousfield and Kan, as follows. Let **CA** be the category of connected commutative graded F_p-coalgebras without unit, with a right action of the Steenrod algebra A which is

(1.8) unstable—

$$xP^n = 0 \quad \text{if } |x| \leq 2pn - 1,$$
$$x\beta P^n = 0 \quad \text{if } |x| \leq 2pn + 1$$

(here and throughout the paper, we adopt the convention that if $p = 2$ then $P^n = Sq^{2n}$ and $\beta = Sq^1$, so that $\beta P^n = Sq^{2n+1}$);

(1.9) compatible with the diagonal—Δ: $C \to C \otimes C$ is A-linear, when we allow A to act diagonally on the tensor product;

(1.10) related to the Verscheibung ξ: $C_{pk} \to C_k$ by

$$\begin{aligned} \xi x &= xSq^n \quad \text{if } p = 2 \text{ and } |x| = 2n, \\ &= xP^n \quad \text{if } p > 2 \text{ and } |x| = 2pn. \end{aligned}$$

There is an adjoint pair, with left adjoint the forgetful functor J:

$$n_+\mathbf{F}_p \underset{J}{\overset{G}{\rightleftarrows}} \mathbf{CA} \tag{1.11}$$

(and indeed, **CA** is the category of coalgebras over the resulting cotriple on $n_+\mathbf{F}_p$). Write G also for the resulting triple on **CA** [10].

Mod p homology of pointed connected spaces naturally takes values in **CA**, and the triple G is compatible with the triple $\bar{F}_p$ on $s_+\mathbf{S}_*$: there is a natural isomorphism θ such that the following diagram commutes.

$$\begin{tikzcd}
\bar{H}_*(X) \arrow[rr, "="] \arrow[d, "\eta"] & & \bar{H}_*(X) \arrow[d, "\eta"] \\
\bar{H}_*(\bar{F}_p X) \arrow[rr, "\theta"] & & G\bar{H}_*(X) \\
\bar{H}_*(\bar{F}_p^2 X) \arrow[r, "\theta"] \arrow[u, "\mu"] & G\bar{H}_*(\bar{F}_p X) \arrow[r, "\theta"] & G^2\bar{H}_*(X). \arrow[u, "\mu"]
\end{tikzcd}$$

Moreover, for any connected simplicial vector space $V \in s_+\mathbf{F}_p$ and any pointed connected simplicial set $W \in s_+\mathbf{S}_*$, the natural map

$$[W,V] \to \operatorname{Hom}_{\mathrm{CA}}(\bar{H}_*(W), \bar{H}_*(V))$$

is an isomorphism. We stress that here $\bar{H}_*(V)$ denotes the reduced mod p homology of V as a pointed simplicial set. It follows that

$$\pi_t \operatorname{map}_*(W, \bar{F}_p^{\bullet} X) \cong [\Sigma^t W, \bar{F}_p^{\bullet} X] \cong \operatorname{Hom}_{\mathrm{CA}}(\bar{H}_*(\Sigma^t W), G^{\bullet}\bar{H}_*(X)).$$

The right-hand side here is a cosimplicial vector space when $t > 0$, and its cohomotopy deserves to be called the derived functor of $\operatorname{Hom}_{\mathrm{CA}}(\bar{H}_*(\Sigma^t W), -)$ evaluated at $\bar{H}_*(X)$. Thus we may write (for $s = 0$ only, if $t = 0$)

$$(1.12) \qquad \pi^s \pi_t \operatorname{map}_*(W, \bar{F}_p^{\bullet} X) = \operatorname{Ext}^s_{\mathrm{CA}}(\bar{H}_*(\Sigma^t W), \bar{H}_*(X)).$$

Combining (1.5), (1.7), and (1.12), we have

THEOREM 1.13. *Let W be a connected space such that $\bar{H}_*(W; \mathbf{Z}[1/p]) = 0$, and let X be a nilpotent fibrant space. If*

$$\operatorname{Ext}^s_{\mathrm{CA}}(\bar{H}_*(\Sigma^t W), \bar{H}_*(X)) = 0$$

for all $s, t \geq 0$ with $0 \leq t - s \leq k$, then

$$\pi_r \operatorname{map}_*(W, X) = 0$$

for all $r \leq k$. □

Theorem C of the introduction results from this and the following theorem, whose proof will occupy six of the next seven sections.

THEOREM 1.14. *If $s \geq 0$ and $t > 0$, or if $s = t = 0$, then*

$$\operatorname{Ext}^s_{\mathrm{CA}}(\bar{H}_*(\Sigma^t BZ_p), C) = 0$$

provided that $C \in \mathbf{CA}$ is bounded.

We can make a start on proving this theorem without further ado. If $s = 0$, the result is implied by

LEMMA 1.15. *If M is a bounded A-module then*

$$\mathrm{Hom}_A\left(\overline{H}_*\left(\Sigma^t BZ_p\right), M\right) = 0.$$

Proof. This is a consequence of the fact that every element of $\overline{H}_*(\Sigma^t BZ_p)$ is hit by a positive-dimensional Steenrod operation. □

2. A Grothendieck spectral sequence for triple-derived functors

Our goal is to establish a spectral sequence which will enable us to study separately the effects on the groups $\mathrm{Ext}_{\mathbf{CA}}(\overline{H}_*(\Sigma^n BZ_p), C)$ of the diagonal in C and of the action of the Steenrod algebra on C.

We begin by establishing general notation for triple-derived functors [5]. So let S be a triple on a category $\mathbf{B}$ and let E: $\mathbf{B} \to \mathbf{A}$ be a functor to some abelian category. Then (1.1) displays a functor $S^\bullet$: $\mathbf{B} \to s^\circ\mathbf{B}$ to cosimplicial objects over $\mathbf{B}$. Apply E and evaluate the cohomotopy—i.e. the cohomology of the associated cochain complex—of the result:

$$R_S^s E(B) = \pi^s(ES^\bullet B). \tag{2.1}$$

If the triple is evident from the context it will be omitted from the notation. There is a natural map

$$\eta_B\colon E(B) \to R^0E(B) \tag{2.2}$$

and we will say that B is *R_S^*E-acyclic* if η_B is an isomorphism and $R_S^sE(B) = 0$ for $s > 0$. The functor E is *S-exact* when every object of $\mathbf{B}$ is R_S^*E-acyclic.

There is of course a categorically dual definition of the (left) derived functors of E with respect to a cotriple G on the source category $\mathbf{B}$; we write $L_*^G E$ for these. They will be used in Sections 3, 4, and 5.

The category $\mathbf{CA}$ of unstable A-coalgebras is closely related to two other categories.

(2.3) The category $\mathbf{U}$ of right A-modules satisfying the unstable condition (1.8). In particular, $M_i = 0$ for $i < 0$, since $P^0 = 1$. There is an evident adjoint pair

$$n\mathbf{F}_p \overset{F}{\rightleftarrows} \mathbf{U}$$

between $\mathbf{U}$ and the category of nonnegatively graded F_p vector spaces. The right adjoint F of the forgetful functor sends a vector space M to the subspace of all elements satisfying the unstable condition in the A-module $\mathrm{Hom}(A, M)$, with A acting by $(f\alpha)(\beta) = f(\alpha\beta)$, $\alpha, \beta \in A$. It is easy to see that $\mathbf{U}$ is abelian, that it has enough injectives, and that the injectives are exactly the F-injectives, i.e. summands of objects of the form $F(M)$. Given $M \in \mathbf{U}$, $\mathrm{Hom}_{\mathbf{U}}(M, -)$ is an additive functor to $\mathbf{F}_p$ (not to $n\mathbf{F}_p$), and we denote by $\mathrm{Ext}^s_{\mathbf{U}}(M, -)$ its sth derived functor.

(2.4) The category $\mathbf{C}$ of connected commutative graded F_p-coalgebras without unit. There is an evident adjoint pair

$$n_+\mathbf{F}_p \underset{S'}{\leftrightarrows} \mathbf{C}$$

between $\mathbf{C}$ and the category of positively graded F_p-vector spaces. We will occasionally consider the analogous category of coalgebras over a field k other than F_p: We then write $\mathbf{C}_k$. The right adjoint S' of the forgetful functor sends a positively graded vector space M to the largest commutative subcoalgebra of the tensor coalgebra without unit $T'M$. Denote the resulting triple on $\mathbf{C}$ by S'. The vector space of *primitives* in $C \in \mathbf{C}$ is $PC = \ker(\Delta\colon C \to C \otimes C)$. This gives a functor $P\colon \mathbf{C} \to n_+\mathbf{F}_p$, whose S'-derived functors $R^sP(-)$ will be of interest. It is easy to check that $\eta\colon P(C) \to R^0P(C)$ is an isomorphism.

We may regard $C \in \mathbf{CA}$ as a coalgebra and form its module of primitives. By (1.9) this is an A-module; by (1.8) it is unstable; and by (1.10) it is in fact a suspension in $\mathbf{U}$. Write

$$\Sigma^{-1}P\colon \mathbf{CA} \to \mathbf{U}$$

for the resulting functor. It has a left adjoint $\Sigma\colon \mathbf{U} \to \mathbf{CA}$ sending M in $\mathbf{U}$ to the A-module ΣM, with trivial diagonal. Thus we have a factorization

$$\mathrm{Hom}_{\mathbf{CA}}(\Sigma M, -) = \mathrm{Hom}_{\mathbf{U}}(M, -) \circ \Sigma^{-1}P(-).$$

THEOREM 2.5. (i) *There is a convergent cohomological spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathbf{U}}\big(M, R^t_G(\Sigma^{-1}P)(C)\big) \Rightarrow \mathrm{Ext}^{s+t}_{\mathbf{CA}}(\Sigma M, C)$$

natural in $M \in \mathbf{U}$ and $C \in \mathbf{CA}$.

(ii) *There is for each t an isomorphism of graded vector spaces*

$$R^t_G(\Sigma^{-1}P)(C) \cong \Sigma^{-1}R^t_{S'}P(C)$$

natural in $C \in \mathbf{CA}$.

Before proving this theorem, we note that together with Lemma 1.15 it allows us to obtain Theorem 1.14 from the next two results. Recall that a graded vector space is *bounded* if it is zero in almost all degrees.

THEOREM 2.6. *If $C \in \mathbf{C}$ is bounded, then $R^tP(C)$ is bounded for each $t \geq 0$.*

THEOREM 2.7. *If $N \in \mathbf{U}$ is bounded, then*

$$\operatorname{Ext}^s_{\mathbf{U}}\left(\overline{H}_*\left(\Sigma^n BZ_p\right), N\right) = 0$$

for all $s \geq 0$ and $n \geq 0$.

Sections 3, 4, and 5 are devoted to a proof of Theorem 2.6, and Sections 6 and 8 contain a proof of Theorem 2.7.

Remark 2.8. It is possible to prove certain cases of the Sullivan conjecture using only the following result of M. André [3: IX], rather than the sharper Theorem 2.6.

THEOREM 2.9. *Let $C \in \mathbf{C}_k$. If PC is bounded and of finite type, then so is $R^tP(C)$.*

Fix a locally finite group G. Theorem 2.9 suffices in the proof that $\operatorname{map}_*(BG, X)$ is weakly contractible provided that the target space X is a nilpotent CW complex whose mod p homology is bounded and of finite type for each prime p occurring as the order of an element of G, or is a finite dimensional CW complex admitting such a space as a covering space.

Actually, André derives functors with respect to the triple on $\mathbf{C}_k$ induced from the forgetful functor from $\mathbf{C}_k$ to the category $n_+\mathbf{S}$ of positively graded sets, rather than to the category $n_+\mathbf{F}_p$ as is done here. However, Proposition 2.11 may easily be used to show that the two derived functors coincide. This also follows from Theorem 3.4 (i) below. Moreover, André works in the setting of ungraded algebras; but neither the extension to graded objects nor the subsequent dualization poses a problem. Of course, it is immediate that if $C \in \mathbf{C}_k$ is of finite type then $R^sP(C)$ is of finite type, so that Theorem 2.9 is a corollary of Theorem 2.6.

We will establish a general "Grothendieck spectral sequence" for triple-derived functors. Such spectral sequences have been studied before: cf. [6] (where, however, a condition like (2.13) (b) was omitted). Our setting will be

(2.10)

$$\begin{tikzcd} \mathbf{C} \arrow[r, "F"] \arrow[loop below, "\mathbf{T}"] & \mathbf{B} \arrow[r, "E"] \arrow[loop below, "\mathbf{S}"] & \mathbf{A} \end{tikzcd}$$

where T is a triple on $\mathbf{C}$, S is a triple on $\mathbf{B}$, $\mathbf{A}$ is an abelian category, and F and E are functors.

PROPOSITION 2.11. *In the context of* (2.10), *assume that*

(a) *FTC is R_S^*E-acyclic for all $C \in \mathbf{C}$, and*

(b) *$ES^{s+1}F$ is T-exact for all $s \geq 0$.*

Then there is a natural isomorphism

$$R_S^*E(FC) \cong R_T^*(EF)(C).$$

Proof. Consider the augmentations

$$EFT^{\bullet}C \to ES^{\bullet}FT^{\bullet}C \leftarrow ES^{\bullet}FC$$

and the associated maps of double cochain complexes. Filter the left one by degree in $T^{\bullet}$; (a) implies that at E_1 the map is an isomorphism. Filter the right one by degree in $S^{\bullet}$; (b) implies that at E_1 the map is an isomorphism. So both maps of total complexes are homology isomorphisms. □

Remark 2.12. On (a): If $B \in \mathbf{B}$ is *S-injective*—that is, if η: $B \to SB$ is a split monomorphism—then it is easy to construct a contraction[(1)] for $B \to S^{\bullet}B$ [5: (2.1), p. 264]. So one may assure (a) by assuming that FTC is S-injective for all $C \in \mathbf{C}$.

On (b): Assume that T arises from an adjoint pair

$$\mathbf{D} \underset{\sigma}{\rightleftarrows} \mathbf{C}$$

and suppose that for each $s \geq 0$ the functor $ES^{s+1}F$ factors through σ. Now it is easy to construct a contraction for $\sigma C \to \sigma T^{\bullet}C$, and so $ES^{s+1}FC \to ES^{s+1}FT^{\bullet}C$ inherits one as well. Thus (b) holds in this situation.

PROPOSITION 2.13. *In the context of* (2.10), *assume further that* $\mathbf{B}$ *is abelian, and that*

(a) *FTC is R_S^*E-acyclic for all $C \in \mathbf{C}$, and*

(b) *ES^{s+1}: $\mathbf{B} \to \mathbf{A}$ is exact for all $s \geq 0$.*

Then there is a convergent cohomological spectral sequence

$$E_2^{s,t} = R_S^sE\big(R_T^tF(C)\big) \Rightarrow R_T^{s+t}(EF)(C).$$

---

(1) An *augmentation* of a simplicial object $X \in s\mathbf{C}$ is a map ε: $X \to \underline{A}$ to a constant simplicial object: that is, ε: $X_0 \to A$ such that $\varepsilon d_0 = \varepsilon d_1$. A *contraction* of an augmented simplicial object ε: $X \to \underline{A}$ is a map η: $A \to X_0$ together with maps h: $X_n \to X_{n+1}$ which behave like "s_{-1}":

$$hs_j = s_{j+1}h, \qquad j \geq 0,$$

$$d_0h = 1 = \varepsilon\eta,$$

$$d_ih = hd_{i-1}, \qquad i > 0.$$

If $\mathbf{C}$ is abelian, a contraction for ε defines a chain homotopy inverse for the map of associated chain complexes induced by ε, and ε thus induces an isomorphism in homotopy. Dual remarks apply in the cosimplicial situation under consideration here.

Proof. Form $ES^{\bullet}FT^{\bullet}C$ again, and look at the associated double cochain complex. Filter by degree in $T^{\bullet}$; (a) implies that

$$E_1^{s,t} = \begin{cases} EFT^{s+1}C & t = 0 \\ 0 & t > 0, \end{cases}$$

so the spectral sequence collapses to an isomorphism of the homology of the total complex with $R_T^*(EF)(C)$. The spectral sequence of the theorem is obtained by filtering by degree in $S^{\bullet}$; we have established its abutment. In it, (b) implies that

$$E_1^{s,t} = ES^{s+1}\big(R_T^t F(C)\big)$$

so that E_2 has the desired form. □

Remark 2.14. If FTC is always S-injective then of course (a) holds. If ES: $\mathbf{B} \to \mathbf{A}$ and S: $\mathbf{B} \to \mathbf{B}$ are both exact, then of course (b) holds.

Proof of Theorem 2.5. (i) For (2.10) take

$$\begin{tikzcd} \mathbf{CA} \arrow[r, "\Sigma^{-1}P"] \arrow[loop below, "G"] & \mathbf{U} \arrow[r, "\mathrm{Hom}_{\mathbf{U}}(M,\, -)"] \arrow[loop below, "F"] & \mathbf{F}_{\mathrm{p}}. \end{tikzcd}$$

The conditions of Remark 2.14 are easily checked, and a spectral sequence of the indicated form results.

(ii) For (2.10) take

$$\begin{tikzcd} \mathbf{CA} \arrow[r, "\Phi"] \arrow[loop below, "G"] & \mathbf{C} \arrow[r, "P"] \arrow[loop below, "S'"] & n\mathbf{F}_p \end{tikzcd}$$

where Φ is the forgetful functor. For any $M \in n_+\mathbf{F}_p$, GM is injective as a coalgebra, so by Remark 2.12, (2.11) (a) holds. The triple G arises from an adjoint pair

$$n_+\mathbf{F}_p \overset{G}{\rightleftarrows} \mathbf{CA},$$

and $PS'^{s+1}\Phi(C) = S'^{s}\Phi(C)$ depends only on the underlying vector space of C; so by Remark 2.12, (2.11) (b) holds too, and we have a natural isomorphism

$$R_{S'}^* P(\Phi C) \cong R_G^* P(C).$$

Since Σ is exact, the second clause of Theorem 2.5 follows. □

3. The homotopy theory of simplicial commutative algebras

Our purpose in this section is to set out the definition of the Quillen homology of a simplicial commutative algebra. We will tailor our work to suit the applications, and provide some proofs even when they are special cases of the general results of Quillen [36], [37] and André [3]. In particular, we make heavier use of cotriples than do those authors. This is because the spectral sequence needed in Section 4 arises from a canonical filtration on a cotriple-generated resolution. We add an internal gradation to the standard accounts. Only the cofibration/acyclic fibration side of the simplicial closed model category structure concerns us, so we restrict our attention to that aspect. Finally, we choose to work with supplemented algebras over a field k.

Let k be a field and let $\mathbf{A}_k$ denote the category of supplemented commutative graded k-algebras: henceforth, "algebras." We shall define two classes of morphisms in the category $s\mathbf{A}_k$ of simplicial algebras.

Definition 3.1. A map p: $Y \to B$ in $s\mathbf{A}_k$ is an *acyclic fibration* provided that it is surjective and induces an isomorphism in homotopy. A map i: $A \to X$ in $s\mathbf{A}_k$ is a *cofibration* provided that in any commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & Y \arrow[d, "p"] \\ X \arrow[r] & B \end{tikzcd} \tag{3.2}$$

in which p is an acyclic fibration, there is a map $X \to Y$ making both triangles commute. An object X is *cofibrant* provided that the canonical map $\underline{k} \to X$ is a cofibration.

It will be important for us to be able to recognize cofibrations, and for this purpose we have

Definition 3.3. A map i: $A \to X$ in $s\mathbf{A}_k$ is *almost free* provided that there is a sequence of subspaces $V_n \subseteq X_n$ such that (i) $s_i V_n \subseteq V_{n+1}$ for each i with $0 \leq i \leq n$, and (ii) the natural map

$$A_n \otimes SV_n \to X_n$$

is an isomorphism for all $n \geq 0$.

Here S: $n\mathbf{k} \to \mathbf{A}_k$ is the free algebra functor, left adjoint to the augmentation ideal functor.

THEOREM 3.4. (i) *Any almost free morphism is a cofibration.*

(ii) *Any map $A \to B$ admits a factorization $A \to X \to B$ in which $A \to X$ is almost free and $X \to B$ is an acyclic fibration.*

(iii) *Suppose that in* (3.2) $A \to X$ *is a cofibration and* $Y \to B$ *is an acyclic fibration. Then any two maps* $X \to Y$ *making the triangles commute are homotopic under* A *and over* B.

Here a homotopy between maps $f, g\colon X \to Y$ may be defined using A-algebra maps $h_i\colon X_n \to Y_{n+1}$, $0 \le i \le n$, satisfying the identities in [27: I.5.1, p. 12], or, equivalently, by means of a mapping object construction Y^{Δ^1} as indicated in [37: p. 67].

Notice that (i) and (ii) together imply:

COROLLARY 3.5. *A map* $A \to X$ *in* $s\mathbf{A}_k$ *is a cofibration if and only if it is a retract of an almost free map* $A \to Y$.

Definition 3.6. The *Quillen homology* of $X \in s\mathbf{A}_k$ is

$$H_*^Q(X) = \pi_*(QP)$$

where $P \to X$ is an acyclic fibration and P is cofibrant.

Theorem 3.4 guarantees that such a map exists and that the homotopy type of P is functorial in X. Thus $H_*^Q(X)$ is well-defined and functorial in $X \in s\mathbf{A}_k$.

If X is constant then $H_*^Q(X)$ is a sequence of derived functors defined by means of a cotriple in a fashion dual to (2.1). More precisely,

PROPOSITION 3.7. *If* $A \in \mathbf{A}_k$ *and* $\underline{A} \in s\mathbf{A}_k$ *is the associated constant simplicial algebra, then* $H_*^Q(\underline{A})$ *coincides with the cotriple derived functors* $L_*Q(A)$ *obtained using the cotriple* $\bar{S}$ *associated to the adjoint pair*

$$\mathbf{A}_k \underset{I}{\overset{S}{\leftrightarrows}} n\mathbf{k}.$$

Proof. We shall see in the proof of Theorem 3.4 that the cotriple complex $\bar{S}_\bullet A$ is cofibrant. □

Remark 3.8. The definition of the Quillen homology of a simplicial algebra is indeed a case of Quillen's general notion of homology [36], [37], namely, derived functors of abelianization. It is easy to see that an abelian object in $\mathbf{A}_k$ is an algebra with trivial multiplication, and that the projection $A \to k \oplus QA$ is abelianization. Thus, in the notation of [37],

$$L_nQ(A) = D_n(A/k).$$

From the point of view of commutative algebra it is more natural to consider $D_*(k/A)$; but the "transitivity sequence" [37: Theorem 5.1, p. 74] or [3: V, p. 61] shows that $D_n(k/A) = D_{n-1}(A/k)$ for all n.

Proof of Theorem 3.4. (i) We modify an argument of Kan [24]. Some terminology will be useful. Let Δ be the category whose objects are the ordered sets $[n] = \{0, 1, \ldots n\}$, $n \ge 0$, and whose morphisms are order-preserving maps.

A simplicial object X over a category $\mathbf{C}$ is then precisely a contravariant functor $X: \Delta \to \mathbf{C}$ [27: p. 4]. If $\phi: [n] \twoheadrightarrow [m]$ is surjective then $X_\phi = \phi^*$ is called a *degeneracy*. The operator s_i, for example, is induced from the surjection σ_i: $[m+1] \twoheadrightarrow [m]$ which repeats the value i. If $\phi: [n] \twoheadrightarrow [m]$ is a surjection and $n > m$, we let $i(\phi)$ be the largest integer such that $\phi(j) = j$ for all $j \leq i(\phi)$. Then ϕ factors as $\phi = \sigma_{i(\phi)}\phi'$. Finally, we will need the following consequence of the Dold-Kan Theorem [27: Theorem 22.4, p. 96].

Fact 3.9. If $\mathbf{C}$ is an abelian category, then for each $n \geq m$

$$\Sigma\phi^*: \bigoplus_{\phi: [n] \twoheadrightarrow [m]} X_m \to X_n$$

is a monomorphism.

Given any injection $A \to X$ in $s\mathbf{A}_k$, define the "relative skeleton" filtration of X by letting $(F_m X)_n$ be the A_n-subalgebra of X_n generated by degeneracies of elements of X_j for $j \leq m$ if $m \leq n$, and $(F_m X)_n = X_n$ if $m \geq n$.

Claim 3.10. $F_m X$ is a subobject of X in $s\mathbf{A}_k$.

We leave the proof of this claim, and of those below, to the reader.

Now let $A \to X$ be almost free, with generating vector spaces $V_n \subseteq X_n$. Let $F_m V_n$ be the subspace of V_n generated by degeneracies of elements of V_j for $j \leq m$ if $m \leq n$, and let $F_m V_n = V_n$ if $m \geq n$. For fixed m, $\{F_m V_n\}$ is closed under degeneracies.

Claim 3.11. $F_m X_n = A_n \otimes SF_m V_n$.

Therefore, $A \to F_m X$ is almost free for each $m \geq 0$, with generating vector spaces $F_m V_n$. We next claim that in fact $F_{m-1}X \to F_m X$ is almost free for each $m \geq 0$. Let W_m be any subspace of V_m complementary to $F_{m-1}V_m$, and let

$$W_{m,n} = \sum_{\phi: [n] \twoheadrightarrow [m]} \phi^* W_m.$$

Claim 3.12. $F_m V_n = W_{m,n} \oplus F_{m-1} V_n$.

The proof of this claim uses Fact 3.9. It follows that $F_{m-1}X \to F_m X$ is almost free, with generating vector spaces $\{W_{m,n}\}$.

To prove (3.4) (i), it will clearly suffice to show that each $F_{m-1}X \to F_m X$ is a cofibration, so we now assume that $A = F_{m-1}X$ and $F_m X = X$. Then for all $n > m$,

$$V_n = \bigoplus_{\phi: [n] \twoheadrightarrow [m]} \phi^* V_m.$$

The Moore normalization theorem [27: Theorem 22.2, p. 94] shows

Claim 3.13. We may assume that $d_i V_m = 0$ for all $i > 0$.

Now suppose given a commutative diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & Y \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

in which p is an acyclic fibration. The dotted lifting will clearly be determined by its value on V_m. For $W \in sn\mathbf{k}$, let $N(W)$ denote the normalized chain complex and $Z(W)$ the subcomplex of cycles. We have a commutative diagram:

$$\begin{tikzcd} V_m \arrow[d, "d_0"] \arrow[dr, dashed] \arrow[drr, "f"] & & \\ Z_{m-1}A \arrow[dr, "g"] & N_mY \arrow[r] \arrow[d, "d_0"] & N_mB \arrow[d, "d_0"] \\ & Z_{m-1}Y \arrow[r] & Z_{m-1}B. \end{tikzcd}$$

Since V_m is projective, the dotted fill-in exists provided that N_mY surjects to the evident pull-back. Now $N_m(p)$: $N_mY \to N_mB$ is surjective, since (by the normalization theorem) N_sW is naturally a quotient of W_s. Thus the following chain-level fact suffices. $\mathbf{C}$ denotes an arbitrary abelian category.

Claim 3.14. Let f: $B \to C$ be a homology isomorphism of nonnegatively graded chain complexes over $\mathbf{C}$. If f is an epimorphism then for each m, the map from B_m to the pull-back $P_m(f)$ in the diagram

$$\begin{tikzcd} B_m \arrow[dr, dashed] \arrow[drr, "f"] \arrow[ddr, "d"'] & & \\ & P_m(f) \arrow[r] \arrow[d] & C_m \arrow[d, "d"] \\ & Z_{m-1}B \arrow[r, "f"] & Z_{m-1}C \end{tikzcd}$$

is an epimorphism.

The proof of (3.4) (i) is completed by checking

Claim 3.15. The resulting map h is simplicial.

(ii) This is proved in [36: II.4.5], and a related proof occurs in [3: IX]. We give another proof, which yields a *canonical* factorization. For $A \in \mathbf{A}_k$, let $A/\mathbf{A}_k$ denote the category of morphisms in $\mathbf{A}_k$ from A. There is an adjoint pair

$$A/\mathbf{A}_k \underset{S^A}{\overset{I}{\rightleftarrows}} n\mathbf{k}$$

in which I sends $A \to X$ to $IX = \ker(X \to k)$, and $S^A M = A \otimes SM$. Write $\bar{S}^A$ for the associated cotriple on $A/\mathbf{A}_k$, with functor $S^A I$. We have the corresponding functor $\bar{S}^A_\bullet$ to simplicial objects over $A/\mathbf{A}_k$, with a description dual to (1.1), and an augmentation $\bar{S}^A_\bullet X \to X$. This is natural in the map $A \to X$.

Now suppose that A is a simplicial algebra. Then application of $\bar{S}^{A_n}$ in degree n gives a cotriple $\bar{S}^A$ on $A/s\mathbf{A}_k$. Given $A \to X$ in $s\mathbf{A}_k$, form the associated simplicial object

$$\bar{S}^A_\bullet X \in s(A/s\mathbf{A}_k).$$

There is an augmentation $\bar{S}^A_\bullet X \to \underline{X}$, and so upon taking diagonal simplicial objects, a factorization

$$A \xrightarrow{i} \operatorname{diag} \bar{S}^A_\bullet X \xrightarrow{p} X.$$

We claim that p is an acyclic fibration and that i is a cofibration.

The map p is clearly surjective. To see that it is a weak equivalence, suppose first that $A \to X$ in $\mathbf{A}_k$ rather than in $s\mathbf{A}_k$. Then, as in (2.12b), there is a canonical contraction for the augmentation $I\bar{S}^A_\bullet X \to IX$ as simplicial graded vector spaces. Thus

$$\pi_t(\bar{S}^A_\bullet X) = \begin{cases} X & t = 0 \\ 0 & t > 0. \end{cases} \tag{3.16}$$

Now suppose that $A \to X$ in $s\mathbf{A}_k$. Then by the Eilenberg-Zilber-Cartier theorem [19: Satz 2.9] (Theorem 4.3 below) there is a spectral sequence converging to $\pi_*(\operatorname{diag} \bar{S}^A_\bullet X)$ with

$$E^1_{s,t} = \pi_t(\bar{S}^{A_s}_\bullet X_s).$$

By (3.16), the spectral sequence collapses at

$$E^2_{s,t} = \begin{cases} \pi_s(X) & t = 0 \\ 0 & t > 0. \end{cases}$$

The map p thus induces an isomorphism at E^2, and so in homotopy.

To check that i: $X \to \operatorname{diag} \bar{S}^A_\bullet X$ is almost free, suppose first that $A \to X$ in $\mathbf{A}_k$. The images of the canonical contraction for the augmentation $I\bar{S}^A_\bullet X \to IX$

then provide a generating system which is natural in $A \to X$. The general case clearly follows.

(iii) See Quillen [37: 1.3, p. 67]. $\square$

4. Boundedness of Quillen homology

The object of this section is a proof of a boundedness assertion for Quillen homology. We assume that the field k is of characteristic $p > 0$; the characteristic 0 analogue is implied by [37: Theorem 7.3, p. 79].

Definition 4.1. A bigraded vector space M has *exponential bound* c provided that $M_{n,i} = 0$ for all $i > cp^n$.

In considering $\pi_*(X)$, $H_*^Q(X)$, etc., as bigraded vector spaces, the simplicial dimension will always be considered as the first degree.

THEOREM 4.2. *Let X be a simplicial algebra, $X \in s\mathbf{A}_k$, over a field k of positive characteristic. If $\pi_0(X) = k$ and $\pi_*(X)$ has exponential bound c, then $H_*^Q(X)$ also has exponential bound c.*

While this result is sufficient for our purposes, we remark that it can be substantially improved using the methods indicated in Remark 4.7 below.

The proof of Theorem 4.2 uses a spectral sequence, valid over any field k, converging to $H_*^Q(X)$ for $X \in s\mathbf{A}_k$, and to describe it we begin by recalling:

THEOREM 4.3 (Eilenberg-Zilber-Cartier [19: Satz 2.9]). *If A is a bisimplicial abelian group, then the total complex of the double chain complex associated to A is naturally chain homotopy equivalent to the chain complex associated to the diagonal simplicial abelian group* diag A.

Thus there is a spectral sequence (one of two) converging to $\pi_*\mathrm{diag}\, A$, with

$$E^1_{s,t} = \pi_t(A_{s,\bullet}). \tag{4.4}$$

The differential d^1 is induced by the alternating sum of the face maps in the first index s, so that

$$E^2_{s,t} = \pi_s\pi_t(A). \tag{4.5}$$

Let $\mathrm{diag}\,\bar{S}_\bullet X \to X$ be the acyclic fibration constructed in the proof of Proposition 3.4 (ii), with $A = \underline{k}$. Notice that

$$Q\,\mathrm{diag}\,\bar{S}_\bullet X = \mathrm{diag}\,Q\bar{S}_\bullet X.$$

By Theorem 4.3, there is thus a spectral sequence converging to $\pi_* Q\,\mathrm{diag}\,\bar{S}_\bullet X = H_*^Q(X)$, with

$$E^1_{s,t} = \pi_t\left(Q\bar{S}^{s+1}X\right)$$

and

$$E^2_{s,t} = \pi_s \pi_t\big(Q\bar{S}_\bullet X\big).$$

Note that $Q\bar{S} = I$: $\mathbf{A}_k \to n\mathbf{k}$, so that

$$E^1_{s,t} = \pi_t(I\bar{S}^s X). \tag{4.6}$$

Remark 4.7. It is easy to see from Dold's work [18] that $\pi_*(ISV)$ is a functor of $\pi_*(V)$ for $V \in sn\mathbf{k}$. Write $\mathscr{S}$: $nn\mathbf{k} \to nn\mathbf{k}$ for this functor; if K is the quasi-inverse to N [27: p. 95] then $\mathscr{S}(M) = \pi_*(ISK(M))$, where M is regarded as a chain complex with trivial differential. The triple structure on IS yields a triple with functor $\mathscr{S}$, and $\pi_*(IX)$ becomes an $\mathscr{S}$-algebra in a natural way for $X \in s\mathbf{A}_k$. The exact structure of $\mathscr{S}$ is very complex. Let $k = F_p$. The homotopy of a simplicial algebra is an algebra with divided powers [21], [15] (on positive even degrees if $p \neq 2$, and on degrees greater than 1 if $p = 2$), and supports operations, "higher divided powers," studied by Bousfield [8], [9] and Dwyer [20]. All this explicit information leads to an explicit form of E^2 of the spectral sequence. In the special situation considered in Section 5, the grip on E^2 becomes even tighter, as indicated in [31]. We hope to return to this topic at a later time.

There is a morphism ε of triples from IS to the identity triple, sending $IS(M)$ to M by killing products. The functor Q may be described as the coequalizer

$$ISIX \underset{\varphi}{\overset{\varepsilon}{\rightrightarrows}} IX \to QX$$

of ε and the IS-algebra structure map φ for $X \in \mathbf{A}_k$. The morphism ε induces a similar augmentation ε: $\mathscr{S} \to \mathrm{id}$, and we define a functor $\mathscr{Q}$ on $\mathscr{S}$-algebras as an analogous coequalizer. Then it is not hard to see that the spectral sequence has the form

$$E^2_{s,t} = L^{\mathscr{S}}_s \mathscr{Q}(\pi_*(IX))_t \Rightarrow H^Q_{s+t}(X). \tag{4.8}$$

In particular, E^2 depends functorially on $\pi_*(IX)$. If we take seriously the interpretation of $H^Q_*(-)$ as homology, then this appears as a "reverse unstable Adams spectral sequence." An "unstable Adams spectral sequence" was constructed and exploited by Quillen in [37]. It has the form

$$E^1_{s,t} = \mathscr{S}_s\big(H^Q_*(X)\big)_t = \pi_{s+t}(IX) \tag{4.9}$$

where $\mathscr{S}_s$ is the homogeneous component of $\mathscr{S}$ of degree s, related to S_s (as in (4.10) below) as $\mathscr{S}$ is related to IS. Our spectral sequence appears to offer a more direct approach to the study of $H^Q_*(-)$. One illustration is provided by the proof of Theorem 4.2, but again this is a subject to which we hope to return in the future.

We hasten to point out that if X is constant, then $E^1_{s,t} = 0$ for $t \neq 0$ and the spectral sequence degenerates uninterestingly at E^2. However, if $H^Q_*(-)$ is legitimately to be considered as a "homology theory," then it ought to commute up to a shift in dimension with "suspension"; and as it turns out the suspension of a constant object has interesting, and familiar, higher homotopy. We will study it in Section 5.

Theorem 4.2 follows immediately from (4.6) and the next result.

THEOREM 4.10. *Let $V \in s n\mathbf{k}$ be a simplicial graded vector space over a field k of positive characteristic p. Assume that $\pi_0(V) = 0$ and that $\pi_*(V)$ has exponential bound c. Then $\pi_*(ISV)$ again has exponential bound c.*

Remark 4.11. The proof shows that if $\pi_* V$ has exponential bound c through (simplicial) dimension n, then so does $\pi_*(ISV)$. Hence analogous restricted versions of Theorems 4.2 and 5.1 hold.

Proof of (4.10). Let N be a graded k-vector space. The graded vector space SN then breaks up naturally as a direct sum

$$SN = \bigoplus_{r \geq 0} S_r N$$

where $S_r N$ is the vector subspace generated by products of length r. In particular, $S_1 N = N$. We will show:

$(4.12)_r$ If $V \in s n\mathbf{k}$ is such that $\pi_0 V = 0$ and $\pi_* V$ has exponential bound c, then $\pi_*(S_r V)$ has exponential bound c.

We begin with a lemma, whose proof is deferred. In it

$$\mu\colon S_s N \otimes S_t N \to S_{s+t} N$$

is the multiplication map and

$$\varphi\colon S_s(S_t N) \to S_{st} N$$

is the restriction of the structure map for the triple IS on $n\mathbf{k}$.

LEMMA 4.13 [8]. *Let $V \in s n\mathbf{k}$. Let $r > 0$, and let e be the exponent of p in the prime factorization of r.*

(a) *If r is not a power of p, then*

$$\mu_*\colon \pi_*(S_{p^e} V) \otimes \pi_*(S_{r-p^e} V) \to \pi_*(S_r V)$$

is an epimorphism.

(b) *If r is a power of p, then*

$$\varphi_*\colon \pi_*\big(S_p\big(S_{p^{e-1}}V\big)\big) \to \pi_*(S_rV)$$

is an epimorphism.

If M and N are bigraded vector spaces each with exponential bound c, then $M \otimes N$ again has exponential bound c. Thus Lemma 4.13 implies:

(4.14) $(4.12)_r$ holds for $r < p$.

(4.15) If $(4.12)_p$ holds (for all V) then $(4.12)_r$ holds for all $r > 0$ (and all V).

For $M \in n\mathbf{k}$, let $K(M, n) \in sn\mathbf{k}$ be the object whose normalized chain complex is M in degree n and 0 elsewhere. Dold's theorem [18] (mentioned in Remark 4.7 above) shows that if $\pi_*(V) \cong \pi_*(\oplus_{n>0}K(M(n), n))$ then $\pi_*(S_pV) \cong \pi_*(S_p(\oplus_{n>0}K(M(n), n)))$. Since

$$S_p(V \oplus W) = \bigoplus_{i+j=p} S_iV \otimes S_jW,$$

it follows from (4.14) that we may assume that $V = K(M, n)$, with $M \in n\mathbf{k}$ such that $M_i = 0$ for $i > cp^n$. Then $(V_s)_i = 0$ for $i > cp^n$ and all s, so $(S_pV_s)_i = 0$ for all $i > cp^{n+1}$ and all s. Hence $\pi_s(S_pV)_i = 0$, for $i > cp^s$ automatically provided $s > n$, and the case $s = n$ follows from:

LEMMA 4.16. *$\pi_n(S_pK(M, n)) = 0$ if $n > 0$.*

This completes the proof of $(4.12)_p$ and hence of (4.10), by (4.15). We turn to the proofs of the lemmas.

Proof of Lemma 4.16. By the Eilenberg-Zilber theorem and the Künneth theorem together with a direct limit argument, we may assume that M is concentrated in one degree, say d, and there is one-dimensional, with generator x. If p and d are both odd, then $S_pK(M, n)_n = 0$, and the conclusion is trivial; so assume that pd is even. Then $K(M, n)_n = \langle x \rangle$, $K(M, n)_{n+1} = \langle s_0x, \ldots, s_nx \rangle$, and

$$d_k s_i x = \begin{cases} x & k = i \text{ or } i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Extend multiplicatively to S_p, and compute

$$d_k\big((s_0x)^{p-1}(s_1x)\big) = \begin{cases} x^p & k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus $d((s_0x)^{p-1}(s_1x)) = -x^p$, so that all cycles in $S_pK(M, n)_n$ are boundaries. $\square$

Lemma 4.13 rests on the following sharper result, due to Bousfield.

LEMMA 4.17 [8]. *There exist k-linear natural transformations*

$$d\colon S_{s+t}N \to S_sN \otimes S_tN,$$
$$e\colon S_{st}N \to S_s(S_tN),$$

for $N \in n\mathbf{k}$ such that the composite

$$S_{s+t}N \xrightarrow{d} S_sN \otimes S_tN \xrightarrow{\mu} S_{s+t}N$$

is multiplication by $(r+s)!/(r!s!)$, and the composite

$$S_{st}N \xrightarrow{e} S_s(S_tN) \xrightarrow{\varphi} S_{st}N$$

is multiplication by $(st)!/(s!(t!)^s)$.

Proof. Recall that if $s, t \geq 0$, then an (s,t)-*shuffle* is a permutation of $\{1, 2, \ldots, s+t\}$ preserving the order of the subsequences $\{1, \ldots, s\}$ and $\{s+1, \ldots, s+t\}$. S_rN is generated as a vector space by products of the form $x_1 \cdots x_r$, $x_i \in N$. We define d by

$$d(x_1 \cdots x_{s+t}) = \sum_\sigma \pm x_{\sigma(1)} \cdots x_{\sigma(s)} \otimes x_{\sigma(s+1)} \cdots x_{\sigma(s+t)}$$

where σ runs over the (s,t)-shuffles and $\pm$ denotes the sign

$$\prod (-1)^{|x_i||x_j|}, \tag{4.18}$$

the product running over pairs (i, j) for which $i < j$ but $\sigma(i) > \sigma(j)$. (This is of course the canonical diagonal in SN, induced by the diagonal map $N \to N \oplus N$.) This map is well-defined, since if τ is a permutation of $\{1, \ldots, s+t\}$ and σ is an (s,t)-shuffle, there is a unique (s,t)-shuffle σ' for which $\{\sigma\tau(1), \ldots, \sigma\tau(s)\} = \{\sigma'(1), \ldots, \sigma'(s)\}$ (and hence also $\{\sigma\tau(s+1), \ldots, \sigma\tau(s+t)\} = \{\sigma'(s+1), \ldots, \sigma'(s+t)\}$). This establishes a bijection between the terms of $d(x_1 \cdots x_{s+t})$ and those of $d(x_{\tau(1)} \cdots x_{\tau(s+t)})$, and the signs are arranged so that the two sums differ by exactly the sign relating the two products.

When we come to compute $\mu d(x_1 \cdots x_{s+t})$, each term rearranges itself to $+x_1 \cdots x_{s+t}$, and there are $(s+t)!/(s!t!)$ terms, so that the composition rule follows.

Define the map e by

$$e(x_1 \cdots x_{st}) = \sum_\sigma \pm (x_{\sigma(1)} \cdots x_{\sigma(t)}) \cdots (x_{\sigma((s-1)t+1)} \cdots x_{\sigma(st)}),$$

where σ runs over permutations of $\{1, \ldots, st\}$ which preserve the order of the sequences $\{1, \ldots, s\}, \ldots \{(r-1)s+1, \ldots, st\}$, and $\{1, s+1, \ldots,$

$(r-1)s+1\}$, and where the sign is as in (4.18). We leave the reader to check that this gives a well-defined map with the asserted composition property. □

Proof of Lemma 4.13. In (a), $r!/(p^e!(r-p^e)!)$ is a unit mod p, and in (b), $p^e!/(p!(p^{e-1}!)^p)$ is a unit mod p. □

5. The Eilenberg-MacLane "suspension" $\overline{W}$

Our object in this section is to prove the next theorem, and show how it implies Theorem 2.6. Again, we shall let k be a field of characteristic $p>0$, and use the notion of exponential bound defined in (4.1).

THEOREM 5.1. *Let A be a supplemented commutative graded k-algebra and assume that $\mathrm{Tor}_*^A(k,k)$ has exponential bound c. Then $L_*Q(A)$ has exponential bound pc.*

COROLLARY 5.2. *Assume $A_i=0$ for $i>a$. Then $L_*Q(A)$ has exponential bound a.*

Proof. The bar construction (or the $\overline{W}$ construction studied below) shows that if $A_i=0$ for $i>a$ then $\mathrm{Tor}_*^A(k,k)$ has exponential bound $p^{-1}a$. □

The proof of Theorem 5.1 uses the boundedness result (4.2) for Quillen homology, together with a "suspension" construction in the category $s\mathbf{A}_k$ of simplicial supplemented commutative graded k-algebras. The role of suspension, as it happens, is played by the Eilenberg-MacLane functor $\overline{W}$ [21], [26], [33]. We recall that construction in our graded setting.

Let $X\in s\mathbf{A}_k$. We define a new object $WX\in s\mathbf{A}_k$ as follows. Let

$$(WX)_n=X_n\otimes X_{n-1}\otimes\cdots\otimes X_0$$

and define simplicial operators by

$$\begin{aligned} d_i(x_n\otimes\cdots\otimes x_0) &= d_ix_n\otimes d_{i-1}x_{n-1}\otimes\cdots\otimes(d_0x_{n-i})x_{n-i-1}\\ &\qquad \otimes x_{n-i-2}\otimes\cdots\otimes x_0 \text{ if } 0\le i<n,\\ &= d_nx_n\otimes d_{n-1}x_{n-1}\otimes\cdots\otimes(d_1x_1)(\eta\varepsilon x_0) \text{ if } i=n,\\ s_i(x_n\otimes\cdots\otimes x_0) &= s_ix_n\otimes s_{i-1}x_{n-1}\otimes\cdots\otimes s_0x_{n-i}\\ &\qquad \otimes 1\otimes x_{n-i-1}\otimes\cdots\otimes x_0. \end{aligned}$$

Let $X\to W(X)$ by $x\mapsto x\otimes 1$; this is a map of simplicial algebras. Form the "quotient" simplicial algebra $\overline{W}X$ with

$$(\overline{W}X)_n=k\otimes_{X_n}(WX)_n=X_{n-1}\otimes\cdots\otimes X_0.$$

The Eilenberg-Zilber map endows the homotopy of a simplicial algebra $X \in s\mathbf{A}_k$ with a commutative bigraded k-algebra structure. We will need the classical

THEOREM 5.3. *There is a natural convergent spectral sequence*

$$E^2_{**} = \operatorname{Tor}^{\pi_*(X)}_{**}(k, k) \Rightarrow \pi_*(\overline{W}X).$$

Proof. According to the Main Theorem of Eilenberg-MacLane ([21: Theorem 20.1]; also, [34]):

$$N\overline{W}X \simeq BNX$$

where B denotes the bar construction. Filtering by homological degree, as in [35], we obtain a spectral sequence of the desired form. Alternatively, note that WX is a cofibrant simplicial module over X, and that the augmentation $WX \to k$ is an acyclic fibration, in the sense of Quillen [36: II.6.2]; so such a spectral sequence results from [36: II.6.8, Theorem 6 (b)]. □

COROLLARY 5.4. *If X is constant, with $X_n = A$, then*

$$\pi_*(\overline{W}X) = \operatorname{Tor}^A_*(k, k). \qquad \square$$

PROPOSITION 5.5. (a) *$WX \to k$ is contractible in $s\mathbf{A}_k$.*
(b) *$\overline{W}$ preserves cofibrations and acyclic fibrations.*
(c) *The sequence of simplicial vector spaces*

$$0 \to QX \to QWX \to Q\overline{W}X \to 0$$

is exact.

Proof. (a) A contraction is given, following [26], by the map $h\colon (WX)_n \to (WX)_{n+1}$ sending w to $1 \otimes w$.

(b) $\overline{W}$ clearly preserves surjections, and it follows from (5.3) that it preserves weak equivalences. To see that $\overline{W}$ preserves cofibrations, it suffices by Corollary 3.5 to show that it preserves locally free maps. This is a direct check, using the following formula. If $x_j \in X_j$, write x_j also for the element $1 \otimes \cdots \otimes x_j \otimes \cdots \otimes 1 \in (\overline{W}X)_n$, $n > j$. Then

$$s_i(x_j) = \begin{cases} x_j & \text{if} \quad i + j < n \\ s_{i+j-n}x_j & \text{if} \quad i + j \geq n. \end{cases}$$

(c) This is clear. In fact, in a fixed internal degree i, the sequence is the classifying fibration for the simplicial abelian group $(QX)_{*,i}$. □

COROLLARY 5.6. *$H_0^Q(\overline{W}X) = 0$, and, for $s > 0$, there is a natural isomorphism*

$$H_s^Q(\overline{W}X) \xrightarrow{\cong} H_{s-1}^Q(X).$$

Proof. Let $P \to X$ be an acyclic fibration from a cofibrant object, and consider the "cofibration sequence"

$$P \to WP \to \overline{W}P.$$

By (5.5) (a), WP is contractible in $s\mathbf{A}_k$, so QWP is contractible as a simplicial graded vector space. Thus the homotopy long exact sequence afforded by (5.5) (c) implies that the boundary map

$$\pi_s(Q\overline{W}P) \to \pi_{s-1}(QP)$$

is an isomorphism for all $s \geq 0$. But by (5.5) (b), $\overline{W}P \to \overline{W}X$ is an acyclic fibration from a cofibrant object, so that $\pi_*(Q\overline{W}P) = H_*^Q(\overline{W}X)$. □

Proof of Theorem 5.1. Apply Theorem 4.2 to $\overline{W}A$, using Corollary 5.4 to identify $\pi_*(\overline{W}A)$ and Corollary 5.6 to identify $H_*^Q(\overline{W}A)$. □

Finally, we translate this work back into the setting of coalgebras, as required for Theorem 2.6. Let $\mathbf{C}_k$ be the category of connected commutative graded k-coalgebras without unit. The following result is a strong form of Theorem 2.6.

THEOREM 5.7. *If k is of positive characteristic p, and $C \in \mathbf{C}_k$ is trivial above degree a, then $R^*P(C)$ has exponential bound a:*

$$R^sP(C)_i = 0 \quad \text{for } i > ap^s.$$

Proof. Any connected coalgebra is the direct limit of its finite subcoalgebras, and R^*P commutes with direct limits, so that we may assume that C is finite-dimensional. Then $A = k \oplus C^*$ is in $\mathbf{A}_k$, and

$$R^*P(C)^* \cong L_*Q(A);$$

so the result follows from Corollary 5.2. □

6. Unstable A-modules

We shall begin by proving our basic vanishing result, a restricted form of Theorem 2.7.

THEOREM 6.1. *If M is a bounded unstable right A-module, then for each $s \geq 0$,*

$$\mathrm{Ext}_{\mathrm{U}}^s\left(\overline{H}_*(BZ_p), M\right) = 0.$$

This will be proved by comparing $\overline{H}_*(BZ_p)$ with certain projective objects in $\mathbf{U}$. These are provided by the following lemma, which will emerge in the course of events or can easily be checked directly (see (7.1) below).

LEMMA 6.2. *For each $n \geq 0$, the functor $\mathbf{U} \to \mathbf{S}$ sending M to the set of elements in degree n is corepresentable: There is an object $G(n) \in \mathbf{U}$ and $\iota_n \in G(n)_n$ such that the map*

$$\mathrm{Hom}_{\mathbf{U}}(G(n), M) \to M_n$$

which sends f to $f(\iota_n)$ is an isomorphism.

Since the functor $M \mapsto M_n$ is obviously exact, $G(n)$ is a projective object in $\mathbf{U}$. We will compare $\overline{H}_*(BZ_p)$ with suitable direct limits of $G(n)$'s, constructed as follows. Map

$$G(2n) \to G(2pn) \tag{6.3}$$

by sending ι_{2n} to $\iota_{2pn}P^n$, and write

$$G_{2n} = \varinjlim \{ G(2n) \to G(2pn) \to \cdots \}.$$

LEMMA 6.4. *Let $N(1) \to N(2) \to \cdots$ be a linear system in $\mathbf{U}$. There is a natural short exact "Milnor sequence," for each s:*

$$0 \to \varprojlim{}^1 \mathrm{Ext}_{\mathbf{U}}^{s-1}(N(i), M) \to \mathrm{Ext}_{\mathbf{U}}^s\left(\varinjlim N(i), M\right)$$
$$\to \varprojlim \mathrm{Ext}_{\mathbf{U}}^s(N(i), M) \to 0.$$

Proof. Apply $\mathrm{Ext}_{\mathbf{U}}(-, M)$ to the short exact sequence

$$0 \to \bigoplus_i N(i) \xrightarrow{\text{1-shift}} \bigoplus_i N(i) \to \varinjlim N(i) \to 0,$$

defining the direct limit, break up the resulting long exact sequence into a family of short exact sequences, and use the definitions of $\varprojlim$ and $\varprojlim^1$. □

LEMMA 6.5. *For any $M \in \mathbf{U}$,*

$$\begin{aligned} \mathrm{Ext}_{\mathbf{U}}^s(G_{2n}, M) &= \varprojlim M_{2p^in}, \qquad s = 0 \\ &= \varprojlim{}^1 M_{2p^in}, \qquad s = 1 \\ &= 0, \qquad s > 1. \end{aligned}$$

The inverse system here is

$$M_{2n} \xleftarrow{P^n} M_{2pn} \xleftarrow{P^{pn}} M_{2p^2n} \leftarrow \cdots. \tag{6.6}$$

Proof. Use (6.4) and projectivity of $G(2p^in)$. □

COROLLARY 6.7. (i) $\operatorname{Ext}^s_{\mathbf{U}}(G_{2n}, M) = 0$ *for all* $s \geq 0$ *provided that* M *is bounded.*

(ii) $\operatorname{Ext}^s_{\mathbf{U}}(G_{2n}, M) = 0$ *for all* $s > 0$ *provided that* M *is of finite type.* □

LEMMA 6.8. *For any* $n \geq 0$,

$$\operatorname{Hom}_{\mathbf{U}}\big(G_{2n}, H_*\big(BZ_p\big)\big) \cong F_p.$$

Proof. Since $H^{2*}(BZ_p)$ is polynomial, P^n: $H^{2n}(BZ_p) \to H^{2pn}(BZ_p)$ is an isomorphism; so the system (6.6) is constant with value F_p. □

Let f_n: $G_{2n} \to H_*(BZ_p)$ be a generator, and form the sum

$$f = \sum_{n=1}^{p-1} f_n \colon \bigoplus_{n=1}^{p-1} G_{2n} \to \overline{H}_*\big(BZ_p\big).$$

We have the following fundamental result, due when $p = 2$ to G. Carlsson [14].

THEOREM 6.9. *The map* f *is a split epimorphism in* $\mathbf{U}$.

Theorem 6.1 follows from this and (6.7) (i). Also:

COROLLARY 6.10. $\overline{H}_*(BZ_p)$ *is a projective object in the category* $\mathbf{U}^{ft}$ *of unstable right A-modules of finite type.* □

If M is a graded F_p-vector space and $n \in \mathbf{Z}/(p-1)$, denote by $M\langle n\rangle$ the graded vector space with

$$\begin{aligned} M\langle n\rangle_i &= M_i, \qquad i \equiv 2n-1 \text{ or } 2n \bmod 2(p-1) \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Thus

$$M = \bigoplus_{n\in\mathbf{Z}/(p-1)} M\langle n\rangle. \tag{6.11}$$

LEMMA 6.12. *If* M *is a right A-module such that* $M_{2n+1}\beta = 0$ *for all* n, *then the splitting* (6.11) *occurs over the Steenrod algebra A.*

Proof. The conditions force Cartan-Serre monomials with more than one Bockstein to act trivially. All the other Cartan-Serre elements have degree congruent to 0 or $-1 \bmod 2(p-1)$, and the result follows. □

Given a right A-module M and integers $a \leq b$, let M_a^b denote the submodule of elements of degree not exceeding b modulo the submodule of elements of degree less than a.

Let

$$P = \overline{H}_*\big(BZ_p\big)$$

so that $P\langle n\rangle_a^b$ is defined. Let $\alpha(n)$ denote the sum of the digits in the p-adic expansion of n, and let $p^{\nu(n)}$ be the largest power of p dividing n.

THEOREM 6.13. *There are A-module monomorphisms*

$$\gamma_n\colon P\langle n\rangle_{2\alpha(n)-1}^{2\alpha(n)+2(p-1)\nu(n)} \to G(2n) \qquad (p>2),$$
$$\gamma_n\colon P_{\alpha(n)}^{\alpha(n)+\nu(n)} \to G(n) \qquad (p=2)$$

such that if i denotes the evident inclusion then the diagrams

$$\begin{tikzcd} P\langle n\rangle_{2\alpha(n)-1}^{2\alpha(n)+2(p-1)\nu(n)} \arrow[r, "\gamma_n"] \arrow[d, "i"] & G(2n) \arrow[d, "P^n"] \\ P\langle n\rangle_{2\alpha(n)-1}^{2\alpha(n)+2(p-1)(\nu(n)+1)} \arrow[r, "\gamma_{pn}"] & G(2pn) \end{tikzcd} \qquad (p>2),$$

$$\begin{tikzcd} P_{\alpha(n)}^{\alpha(n)+\nu(n)} \arrow[r, "\gamma_n"] \arrow[d, "i"] & G(n) \arrow[d, "Sq^n"] \\ P_{\alpha(n)}^{\alpha(n)+\nu(n)+1} \arrow[r, "\gamma_{2n}"] & G(2n) \end{tikzcd} \qquad (p=2)$$

commute.

We shall prove this presently, but we first note how it implies Theorem 6.9. In the limit, we have A-module monomorphisms

$$(6.14)\qquad g_n\colon P\langle n\rangle_{2\alpha(n)-1}^{\infty} \to G_{2n} \qquad (p>2),$$
$$g_n\colon P_{\alpha(n)}^{\infty} \to G_n \qquad (p=2)$$

such that $g_n = g_{pn}$. For $1 \le n \le p-1$ consider the composite

$$P\langle n\rangle \xrightarrow{g_n} G_{2n} \xrightarrow{f_n} P.$$

By construction, f_n is surjective in degree $2n-1$ and has image in $P\langle n\rangle$. In Corollary 6.22 we will see that $\dim(G_{2n})_{2n-1} = 1$; so Theorem 6.9 follows because:

LEMMA 6.15. *$P\langle n\rangle$ is atomic: An endomorphism over A is an isomorphism provided it is bijective in the lowest nonzero degree.*

Proof. We claim that in fact

$$\chi(\beta^e P^s)\colon H_{2np^i}(BZ_p) \to H_{2np^i-2(p-1)s-e}(BZ_p)$$

is surjective for all $0 \le s \le 2n(p^i-1)/(p-1)$ and $0 \le e \le 1$; this certainly suffices. One way to see this is to observe that the skeleton $(BZ_p)^{2np^i}$ is Spanier-Whitehead 0-dual to the stunted lens space $\Sigma(BZ_p)^{-2}_{-2np^i-1}$, and then compute the Steenrod operations on the bottom class in cohomology. We invite the reader to translate this suggestion into algebra and carry out the details. Alternatively, it is easy to check that $P\langle n\rangle$ is atomic over the subalgebra of the Steenrod algebra generated by β and P^1. □

To begin the proof of Theorem 6.13, we notice that the system $\{G(n)\}$ supports additional structure. First, there is a copairing

$$\Delta\colon G(m+n) \to G(m) \otimes G(n) \tag{6.16}$$

defined as the A-linear map sending ι_{m+n} to $\iota_m \otimes \iota_n$ in the tensor product with diagonal A-action. There is also a *left* A-action, defined as follows. For $\theta \in A^{i-j}$ define $\theta\colon G(j) \to G(i)$ as the right A-linear map sending ι_j to $\iota_j\theta \in G(i)_j$. The map (6.3) is a case of this construction.

Form a bigraded vector space $G(\cdot)$ with $G(i)_j$ in bidegree (i, j). This is an associative, commutative, unital bigraded coalgebra via (6.16). It supports a right Steenrod action with A acting vertically and decreasing degrees, and a left Steenrod action with A acting horizontally and increasing degrees. Each column of $G(\cdot)$ is right unstable, and each row is left unstable. Finally, both actions satisfy the Cartan formula with respect to Δ.

The bigraded vector space $G(\cdot)$ is of finite type. Write $G^*(\cdot)$ for its linear dual. It is a bigraded F_p-algebra with a vertical left A-action and a horizontal right A-action. Its structure is explicitly given by

THEOREM 6.17. *If $p > 2$, then $G^*(\cdot)$ is the free commutative bigraded F_p-algebra on generators*

$$\begin{aligned} &e \in G^1(1), \\ &t_i \in G^1(2p^i), \qquad i \geq 0, \\ &x_i \in G^2(2p^i), \qquad i \geq 0, \end{aligned}$$

subject to the relation $e^2 = x_0$. The left A-action is determined by

$$\begin{aligned} \beta e &= 0, & Pe &= e, \\ \beta t_i &= x_i, & Pt_i &= t_i, \\ \beta x_i &= 0, & Px_i &= x_i + x_{i-1}^p, \end{aligned} \tag{6.18}$$

where P is the total reduced power

$$P = \sum_{i \geq 0} P^i$$

and where $x_i = 0$ *and* $t_i = 0$ *if* $i < 0$. *The right A-action is determined by*

$$(6.19) \qquad e\beta = 0, \qquad eP = 0,$$

$$t_i\beta = \begin{cases} e \text{ if } i = 0 \\ 0 \text{ if } i > 0, \end{cases} \qquad t_iP = t_i + t_{i-1},$$

$$(6.20) \qquad x_i\beta = 0, \qquad x_iP = x_i + x_{i-1}.$$

If $p = 2$, *then* $G^*(\cdot)$ *is the free commutative bigraded* F_2*-algebra on generators*

$$x_i \in G^1(2^i), \qquad i \geq 0.$$

The left A-action is determined by

$$Sq\, x_i = x_i + x_{i-1}^2$$

and the right A-action is determined by

$$x_iSq = x_i + x_{i-1}.$$

Proof. In this proof we choose to return to the geometric origin of the unstable condition. Let H denote the mod p Eilenberg-MacLane spectrum, $A_* = H_*(H)$ the dual Steenrod algebra, and for a space X let $\Sigma^\infty X$ denote the associated suspension spectrum. The map [1] $\Sigma^\infty X \simeq \Sigma^\infty S^0 \wedge X \to H \wedge X$ inducing the coaction ψ: $\overline{H}_*(X) \to A_* \otimes \overline{H}_*(X)$ factors through Σ^∞ applied to η: $X \to \overline{F}_pX$. Thus ψ factors as follows:

$$\begin{tikzcd}
 & QH_*(\overline{F}_pX) \arrow[d, tail] & & \bigoplus_j QH_*(K_j) \otimes \overline{H}_j(X) \arrow[ll, "\cong"'] \arrow[d, tail] \\
\overline{H}_*(X) \arrow[r] \arrow[ur, dashed] \arrow[rr, bend right, "\psi"'] & H_*(H \wedge X) & A_* \otimes \overline{H}_*(X) \arrow[l, "\cong"'] \arrow[r, equal] & \bigoplus_j A_* \otimes \overline{H}_j(X).
\end{tikzcd}$$

Here $K_j = \overline{F}_pS^j = K(F_p, j)$. An A_*-comodule M is *unstable* provided the structure map ψ: $M \to A_* \otimes M$ factors through the inclusion $\oplus QH_*(K_j) \otimes M_j \rightarrowtail A_* \otimes M$. The homology of a space is thus automatically unstable.

The structure of QH_*K_j is well-known, and we recall it. When $j = 1$, $K_j = BZ_p$ and $QH_*(K_1)$ has generators $e, t_0, t_1, \ldots, |e| = 1, |t_i| = 2p^i$, with right A-action given by (6.19). When $p = 2$, we write x_0 for e and x_i for t_{i+1}. Now let $p > 2$. When $j = 2$, the map $Z \to Z_p$ induces $\mathbf{C}P^\infty \to K_j$, and we obtain as images of generators of $QH_*(\mathbf{C}P^\infty)$ elements $x_0, x_1, \ldots, |x_i| = 2p^i$, with A-action given by (6.20). By definition [32] these elements suspend to A_* as follows.

$$e \mapsto 1, \qquad t_i \mapsto \tau_i, \qquad x_i \mapsto \xi_i,$$

or, when $p = 2$,

$$x_i \mapsto \xi_i,$$

with $\xi_0 = 1$.

The cup-product in cohomology is induced by a pairing $K_i \wedge K_j \to K_{i+j}$, and this yields on QH_*K_* the structure of commutative bigraded algebra; in fact, it is the symmetric algebra on $\{e, t_i, x_i\}$ modulo the relation $e^2 = x_0$ if $p > 2$, and it is the symmetric algebra on $\{x_i\}$ if $p = 2$. This is well-known; for a modern treatment see [43]. It may be checked using the facts that A_* is free commutative [32] and that QH_*K_j injects into A_*. See also [23] and [25].

A Steenrod operation $\theta \in A^n$ induces an H-map θ: $K_j \to K_{j+n}$ representing the primitive cohomology class $\theta\iota_j$. This in turn yields right A-module maps $QH_*(K_j) \to QH_*(K_{j+n})$ which render $QH_i(K_*)$ a left A-module for each i. This left A-action increases degrees and is unstable in the cohomological sense:

$$P^i x = 0 \quad \text{if } |x| < 2i,$$
$$\beta P^i x = 0 \quad \text{if } |x| \le 2i.$$

The Cartan formulae in $H^*(-)$ and $H_*(-)$ guarantee that QH_*K_* is a left and a right A-module algebra. Explicit formulae are given by (6.18).

Given an A_*-comodule M we may consider the adjoint right A-module structure, φ: $M \otimes A \to M$. Let π: $A^{n-j} \to PH^n(K_j)$ be the projection dual to the inclusion $QH_n(K_j) \rightarrowtail A_{n-j}$. It is easy to check that the condition of instability is equivalent to φ factoring as

$$\begin{tikzcd} M_n \otimes A^{n-j} \arrow[rr, "\varphi"] \arrow[dr, "1 \otimes \pi"'] & & M_j \\ & M_n \otimes PH^n(K_j) \arrow[ur, dashed] & \end{tikzcd}$$

Given any $x \in M_n$, there is thus defined a right A-module map f: $PH^n(K_*) \to M$ characterized by the condition $f(\iota_n) = x$. This proves Lemma 6.2 and shows that

$$G(n) = PH^n(K_*) \tag{6.21}$$

as right A-modules. An easy check shows that the right A-module structure and the product on $G^*(\cdot)$ must agree with those on $QH_*(K_*)$ under the dual of (6.21); so Theorem 6.17 follows. □

Corollary 6.22. (i) $\Sigma G(2n) \cong G(2n + 1)$ *as A-modules.*

(ii) $G_i(2n)$ *is first nonzero for* $i = 2\alpha(n) - \mu(n)$, *where*

$$\alpha(n) = \textit{sum of digits},$$
$$\mu(n) = \textit{number of nonzero digits}$$

in the p-adic expansion of n, and is one dimensional in that degree.

Proof. (i) In effect, $\Delta\colon G(2n+1) \to G(1) \otimes G(2n)$ is an isomorphism.

(ii) The way of expressing a number n as a sum of powers of p which is most efficient, in the sense of requiring fewest terms, is given by the p-adic expansion. One may obtain an element of $G(2n)$ by using corresponding powers of x_i; the dimension of such an element is $2\alpha(n)$ if $p > 2$, and $\alpha(n) = 2\alpha(n) - \mu(n)$ if $p = 2$. For each nonzero digit, however, one may, if $p > 2$, replace one x_i with t_i, to obtain an element in degree $2\alpha(n) - \mu(n)$. This clearly gives the unique monomial in that degree. □

Proof of Theorem 6.9. If $p > 2$ we let Γ denote the subobject of $G^*(\cdot)$ obtained by restricting to even horizontal degrees. (The horizontal A-action does not respect this: $t_0\beta = e$.) If $p = 2$, let $\Gamma = G^*(\cdot)$.

We map $G^*(2n)$ to $P\langle n\rangle^*$, or $G^*(n)$ to P^* if $p = 2$, for each n, as follows. Define a new free commutative bigraded A-algebra Λ with generators s, t, x:

$$\begin{aligned} &\|s\| = (2,0), && \|t\| = (0,1), && \|x\| = (0,2),\\ &\beta s = 0, && && Ps = s,\\ &\beta t = x, && && Pt = t,\\ &\beta x = 0, && && Px = x + x^p; \end{aligned}$$

or, if $p = 2$, with generators s, x:

$$\|s\| = (1,0), \qquad \|x\| = (0,1),$$
$$\mathrm{Sq}\, x = x + x^2,$$
$$\mathrm{Sq}\, s = s.$$

Define an algebra map $\gamma\colon \Gamma \to \Lambda$ by sending

$$t_i \mapsto s^{p^i}t, \qquad x_i \mapsto s^{p^i}x,$$

or, if $p = 2$,

$$x_i \mapsto s^{2^i}x.$$

Suppose first that $p > 2$. Note that $\gamma\colon G^*(2n) \to P\langle n\rangle^*$. The map γ fails to be A-linear only because of the irregular definition of P^1x_0. We therefore look in $G^*(2n)$ for the lowest dimensional occurrence of x_0 in a monomial which maps nontrivially to Λ. This is two greater than the degree of the lowest monomial in $G(2n)$ containing at most one t_i. Reasoning as in Corollary 6.22, we find that the latter degree is $2\alpha(n-1) - 1 = 2\alpha(n) + 2(p-1)\nu(n) - 3$. Thus the first violation of A-linearity occurs in degree $2\alpha(n) + 2(p-1)\nu(n) - 1 + 2(p-1)$. Since $\alpha(n) \equiv n \bmod (p-1)$, this is killed by projecting to $(P\langle n\rangle^{2\alpha(n)+2(p-1)\nu(n)})^*$. Since $G(2n)$ is zero below degree $2\alpha(n) - 1$, we obtain A-linear maps as stated.

When $p = 2$ the discussion is analogous but simpler.

It is now straightforward to check that the maps are compatible in the desired way. $\square$

Remark 6.23. Let I denote the kernel of $\gamma\colon \Gamma \to \Lambda$. Then one may check that

$$I = \left(t_i t_j, t_i x_{j-s}^{p^s} - t_j x_{i-s}^{p^s}, x_i x_{j-s}^{p^s} - x_j x_{i-s}^{p^s} \colon i, j, s \geq 0\right) \qquad \text{if } p > 2,$$
$$I = \left(x_i x_{j-s}^{2^s} - x_j x_{i-s}^{2^s} \colon i, j, s \geq 0\right) \qquad \text{if } p = 2,$$

where again we agree that $x_i = 0$ if $i < 0$. Now I is invariant under both A-actions on Γ, where we declare if $p > 2$ that $t_0\beta = 0$. (One may reduce the number of generators:

$$I = \left(t_i t_j, t_i x_{j-s}^{p^s} - t_j x_{i-s}^{p^s}, x_i x_{j-1}^{p} - x_j x_{i-1}^{p} \colon i < j, s \geq 0\right) \qquad \text{if } p > 2,$$
$$I = \left(x_i x_{j-1}^{2} - x_j x_{i-1}^{2} \colon i < j\right) \qquad \text{if } p = 2.$$

Also, it is interesting to note that when $p > 2$, $\{t_i t_j\colon i < j\}$ generates I as a left A-ideal.)

In $G^*(2n)$, a complete list of representatives mod I may be given as follows. Let

$$n = p^e(n_e + n_{e+1}p + \cdots)$$

be the p-adic expansion of n, with $n_e > 0$ and $0 \leq n_i \leq p - 1$. Then we have, when $p \geq 2$,

$$\begin{aligned} &x_e^{n_e} x_{e+1}^{n_{e+1}} \cdots \\ &x_{e-1}^{p} x_e^{n_e - 1} x_{e+1}^{n_{e+1}} \cdots \\ &\vdots \\ x_0^p x_1^{p-1} \cdots\ &x_{e-1}^{p-1} x_e^{n_e - 1} x_{e+1}^{n_{e+1}} \cdots, \end{aligned}$$

and when $p > 2$ the corresponding monomials in which one x_i of minimal subscript is replaced by t_i. The identification of Γ/I prescribed in Theorem 6.13 is then clear, and the compatibility diagrams are just part of the compatibility with the horizontal A-action.

Remark 6.24. Many questions remain unanswered.

(a) This proof may appear rather ad hoc. Is there some divisibility criterion, akin to the Adams-Margolis theorem, for determining when an object of $\mathbf{U}^{ft}$ is projective, or when $\operatorname{Ext}_{\mathbf{U}}^*(M, N) = 0$ for all bounded N?

(b) The module $G(n)$ is, as we shall remark in Section 7, the homology of a spectrum, and the maps f_n: $G(n) \to \overline{H}_*(BZ_p)$ are induced from geometric maps. Are the g_n's? Is γ_n: $P\langle n\rangle_{2\alpha(n)-1}^{2\alpha(n)+2(p-1)\nu(n)} \to G(2n)$?

(c) Are there analogous spectra realizing the columns of $QBP_*\mathbf{BP}_*$ [43]?

(d) What is the significance of Γ/I when read horizontally? It defines, for each j, a submodule of PH^*K_j over the reduced powers (or over A if $p = 2$).

7. Dual Brown-Gitler spectra as projective covers of spheres

In this aside we shall study some of the geometry associated with the algebra of Section 6.

Grade the Steenrod algebra nonpositively. From the description of $\mathbf{U}$ given in (2.3) it is clear that

$$G(n) = \Sigma^n A/\{\beta^e P^s: 2(ps + e) > n\}A \tag{7.1}$$

provided that the right-hand side is unstable. This follows from a computation in the Steenrod algebra or by dualizing the work of Section 6. When $p = 2$

$$G(n) = \Sigma^n A/\{\mathrm{Sq}^s: 2s > n\}A.$$

In [12], Brown and Gitler constructed a spectrum $T(n)$ with this homology (at $p = 2$). It is Spanier-Whitehead n-dual to the much-studied spectrum $B[n/2]$:

$$G(n) = H_*\left(\overline{\Sigma^n B\left[\frac{n}{2}\right]}\right).$$

If we write $\overline{G(n)}$ for the linear dual of $G(n)$, made into a right A-module via χ, then this says

$$H_*B(k) = \Sigma^{2k}\overline{G(2k)}.$$

Similarly, R. L. Cohen [16] constructed spectra $B(k)$ for p odd such that

$$H_*B(k) = \Sigma^{2p(k+1)-2}\overline{G(2p(k+1)-2)}.$$

Presumably the other $G(n)$'s can be realized too.

The unstable right A-module $G(n)$ is the projective cover of $S(n) = \overline{H}_*(S^n)$. Recall, e.g. from [41: pp. 89 ff.], that a morphism p: $P \to X$ in any category is a *projective cover* provided: (a) P is projective; (b) p is an epimorphism; and (c) any f: $Y \to P$ such that pf is an epimorphism is itself an epimorphism. In general (a)–(c) characterize p (if it exists) up to a non-canonical isomorphism. In the present context the isomorphism is actually canonical, since $G(n)$ is monogenic.

One is thus compelled to ask whether the spectrum $T(n)$ admits an analogous characterization.

Let $\mathscr{S}_H$ be the full subcategory of the stable category consisting of spectra which: (a) are connective; (b) have trivial homology with $\mathbf{Z}[\frac{1}{2}]$ coefficients; and (c) admit a mod 2 homology monomorphism to the suspension spectrum of some space. Condition (b) serves to eliminate from consideration all primes (even 0) other than 2. Given a connective spectrum X there is a map $\tilde{X} \to X$ terminal among maps from $\mathbf{Z}[\frac{1}{2}]$-acyclic spectra, namely, the fiber of the localization map $X \to X[\frac{1}{2}]$. The map $\tilde{X} \to X$ is a mod 2 homology isomorphism, and is called the *cocompletion* of X at 2.

The proof of Theorem B of [13] implies that $T(n)$ lies in $\mathscr{S}_H$, provided that we replace $T(0) = S^0$ and $T(1) = S^1$ by their cocompletions.

The natural notion of "epi" in $\mathscr{S}_H$ is not the categorical one, which is, of course, what we had in mind above, but rather this: a map $f\colon X \to Y$ in $\mathscr{S}_H$ is an *H-epi* provided that $H_*(f)$ is epi. Then *H-projectives* and *H-projective covers* may be defined accordingly, and one has:

THEOREM 7.2. *$T(n)$ is an H-projective cover of $\tilde{S}^n$ in the category $\mathscr{S}_H$.*

Proof. Condition (b) of the definition is trivial, and (c) follows from the cyclicity of $H_*(T(n))$. It remains to check that $T(n)$ is projective in $\mathscr{S}_H$; and by duality this is contained in the assertion that if $f\colon X \to Y$ is an H-epi in $\mathscr{S}_H$ then $\pi_r(B(k) \wedge X) \to \pi_r(B(k) \wedge Y)$ is epi for $r \leq 2k + 1$. We leave the case $k = 0$ to the reader; remember that $B(0)$ is the 2-adically completed 0-sphere. Recall that in [12] Brown and Gitler construct, for given $k > 0$, a diagram of cofibration sequences

$$(7.3) \qquad \begin{array}{ccccccc} \cdots \to & E_2 & \to & E_1 & \to & E_0 & \to * \\ & \nwarrow & & \nwarrow & & \nwarrow & \\ & H_2 & & H_1 & & H_0 & \end{array}$$

and compatible maps $B(k) \to E_s$, in which: (a) each H_s is a mod 2 generalized Eilenberg-MacLane spectrum; (b) $B(k) \to E_s$ becomes highly connected as s becomes large; and (c) for any space K, $\pi_r(H_s \wedge K) \to \pi_r(E_s \wedge K)$ is monic for $r \leq 2k$. If $X \to \Sigma^\infty K$ is a mod 2 homology monomorphism, then clearly $\pi_r(H_s \wedge X) \to \pi_r(E_s \wedge X)$ is monic for $r \leq 2k$ as well. Thus $\pi_r(E_s \wedge X) \to \pi_r(E_{s-1} \wedge X)$ is epi for $r \leq 2k + 1$. Now let $f\colon X \to Y$ be an H-epi in $\mathscr{S}_H$. It then follows that $\pi_r(E_s \wedge X) \to \pi_r(E_s \wedge Y)$ is epi for $r \leq 2k + 1$, by induction on s using the fact that in the diagram

$$\begin{array}{ccccccc} \pi_r(H_s \wedge X) & \to & \pi_r(E_s \wedge X) & \to & \pi_r(E_{s-1} \wedge X) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \pi_r(H_s \wedge Y) & \to & \pi_r(E_s \wedge Y) & \to & \pi_r(E_{s-1} \wedge Y) & & \end{array}$$

surjectivity of the end vertical arrows implies surjectivity of the middle vertical arrow. The result now follows from property (c) of (7.3). □

The usual proof that projective covers are unique shows here that any two projective covers of X are mod 2 homology-isomorphic over X. But for $H\mathbf{Z}[\frac{1}{2}]$-acyclic spaces, this gives an integral homology equivalence, and hence, using connectivity, a homotopy equivalence, by the Whitehead theorem. Thus Theorem 7.2 provides a *characterization* of $T(n)$, and hence of its dual, $B[n/2]$.

The category $\mathscr{S}_H$ is of course very artificial. A more natural category $\mathscr{S}$ is afforded by replacing (c) in the definition by: (c′) admit a split monomorphism to the suspension spectrum of some space. If $T(n) \in \mathscr{S}$, then the analogue of Theorem 3.9 holds in $\mathscr{S}$. Thus we propose the

Conjecture 7.4. The n-dual of $B[n/2]$ is a summand of the suspension spectrum of some space.

8. The EHP spectral sequence

Our goal is to show how the restricted vanishing result Theorem 6.1 implies the general one, Theorem 2.7. We also comment on various extensions of this work. As a secondary matter, we use the results of the earlier sections to establish a lower vanishing curve for the Bousfield-Kan E_2-term of a space with bounded homology. We begin by restating (2.7).

THEOREM 8.1. (a) *If* $N \in \mathbf{U}$ *is bounded then* $\operatorname{Ext}^q_{\mathbf{U}}(\overline{H}_*(\Sigma^n BZ_p), N) = 0$ *for all* $n, q \geq 0$.

(b) *If* $N \in \mathbf{U}^{ft}$ *then* $\operatorname{Ext}^q_{\mathbf{U}}(\overline{H}_*(\Sigma^n BZ_p), N) = 0$ *for all* $q > n$.

The proof uses the right adjoint Ω of the suspension functor $\Sigma\colon \mathbf{U} \to \mathbf{U}$. The explicit construction in [10: p. 103] shows:

LEMMA 8.2. (i) *The right derived functors* $R^i\Omega$ *are trivial for* $i > 1$.

(ii) *If* N *is bounded then so are the unstable* A-*modules* ΩN *and* $R^1\Omega(N)$.

(iii) *If* N *is of finite type then so are* ΩN *and* $R^1\Omega(N)$.

Clearly the n-fold iterate Ω^n is right adjoint to the n-fold iterate Σ^n. Thus

$$\Omega^m \circ \Omega^n = \Omega^{m+n},$$

$$\operatorname{Hom}_{\mathbf{U}}(M, -) \circ \Omega^n = \operatorname{Hom}_{\mathbf{U}}(\Sigma^n M, -).$$

Write Ω^n_t for the tth derived functor of Ω^n. Since Σ^n is exact, Ω^n carries injectives to injectives; so we obtain Grothendieck spectral sequences

$$\Omega_s^m \Omega_t^n N \Rightarrow \Omega_{s+t}^{m+n} N, \tag{8.3}$$

$$\operatorname{Ext}_{\mathbf{U}}^s(M, \Omega_t^n N) \Rightarrow \operatorname{Ext}_{\mathbf{U}}^{s+t}(\Sigma^n M, N). \tag{8.4}$$

Remark 8.5. J. Neisendorfer has pointed out that the properties of Ω_t^n may be derived within the context of Section 6. Indeed,

$$\Omega N = \operatorname{Hom}_A(\Sigma G(\cdot), N)$$

where the right A-action on the right-hand side is induced from the left ("horizontal") A-action in $G(\cdot)$. Now it is easy to write down a projective resolution of $\Sigma G(n)$ of length at most 1. These resolutions are the short exact "EHP" sequences figuring prominently in the standard approach to Brown-Gitler spectra. Altogether they may be given a compatible left A-action. The resulting short exact sequence of A-bimodules, when mapped to $N \in \mathbf{U}$, gives rise to the 4-term exact sequence of [10: p. 103].

We return to preparations for our proof of Theorem 8.1. By induction using the "Singer spectral sequence" (8.3) (cf. [39]) and Lemma 8.2, we have

LEMMA 8.6. (i) $\Omega_t^n = 0$ *for* $t > n$.
(ii) $\Omega_n^n = \Omega_1 \Omega_{n-1}^{n-1}$.
(iii) *If* N *is bounded, so is* $\Omega_t^n N$ *for all* $n, t \geq 0$.
(iv) *If* N *is of finite type, so is* $\Omega_t^n N$ *for all* $n, t \geq 0$.

The proof of Theorem 8.1 (a) is now clear: in the "EHP spectral sequence"

$$E_2^{s,t} = \operatorname{Ext}_{\mathbf{U}}^s\left(\overline{H}_*(BZ_p), \Omega_t^n N\right). \tag{8.4}$$

$\Omega_t^n N$ is bounded by (8.6) (iii); so $E_2^{s,t} = 0$ by Theorem 6.1. (b) is similar. □

Theorem 8.1 can be extended somewhat.

THEOREM 8.7. *For any* $M, N \in \mathbf{U}$, *with* N *bounded*,

$$\operatorname{Ext}_{\mathbf{U}}^s\left(M \otimes \overline{H}_*(BZ_p), N\right) = 0$$

for all $s \geq 0$.

Proof. Any object of $\mathbf{U}$ is a direct limit of bounded objects of finite type, so by the Milnor sequence (6.4) we may assume that M is bounded and of finite type. Then argue by induction on the top nonzero degree n, using the short exact sequence

$$0 \to M' \to M \to M_n \otimes S(n) \to 0.$$

The Ext groups involving M' are 0 by induction, and those involving $M_n \otimes S(n)$

are 0 by Theorem 8.1 and additivity of Ext. □

Theorem 3.1 of [30], quoted from this paper, follows easily from this and Lemma 8.6 (ii).

Notice that our work shows that in certain cases the contractibility of mapping spaces from a space W reduces to a property solely of the A-module structure of $\overline{H}_*(W)$:

THEOREM 8.8. *Let W be a space such that $\overline{H}_*(W; \mathbf{Z}[1/p]) = 0$ and*

$$\mathrm{Ext}^s_{\mathbf{U}}(\overline{H}_*(W), N) = 0$$

for every bounded object $N \in \mathbf{U}$ and every $s \geq 0$. Then $\mathrm{map}_*(W, X)$ *is weakly contractible for any nilpotent space X such that $H_*(X; F_p)$ is bounded.* □

Finally, we note that the methods of this paper lead to a general lower vanishing curve for the Bousfield-Kan E_2-term.

THEOREM 8.9. *Let $C \in \mathbf{CA}$ have $C_i = 0$ for $i > c$ and $M \in \mathbf{U}$ have $M_i = 0$ for $i < m$. If $m \geq p^s c$, then*

$$\mathrm{Ext}^s_{\mathbf{CA}}(\Sigma M, C) = 0.$$

Proof. Lemma 8.2 (ii) may be made more precise: if $N \in \mathbf{U}$ has $N_i = 0$ for $i > n$, then

$$(\Omega N)_i = 0 \quad \text{for } i > n - 1,$$
$$(\Omega_1 N)_i = 0 \quad \text{for } i > pn - 1.$$

Induction using the Singer spectral sequence (8.3) then shows that

$$(8.10) \quad (\Omega^r_s N)_i = 0 \quad \text{for } i > p^s n - \left(p^{s-1} + \cdots + p + 1\right) - (r - s).$$

Since $S(0) \in \mathbf{U}$ is projective, the EHP spectral sequence (8.4) degenerates to an isomorphism (cf. [39])

$$(8.11) \qquad \mathrm{Ext}^s_{\mathbf{U}}(S(r), N) = (\Omega^r_s N)_0,$$

so that

$$(8.12) \qquad \mathrm{Ext}^s_{\mathbf{U}}(S(r), N) = 0 \quad \text{for } r > p^s n - \left(p^{s-1} + \cdots + 1\right) + s.$$

The evident induction over skeletons of $M \in \mathbf{U}$ (using the Milnor sequence (6.4) to deal with the inverse limit) shows that if $M_i = 0$ for $i < m$ then

$$(8.13) \qquad \mathrm{Ext}^s_{\mathbf{U}}(M, N) = 0 \quad \text{if } m > p^s n - \left(p^{s-1} + \cdots + 1\right) + s.$$

Now the spectral sequence of Theorem 2.5, together with the boundedness Theorem (2.6), give the result. □

Corollary 8.14. *Let $F^{\bullet}\pi_*(X)$ denote the filtration associated to the Bousfield-Kan unstable Adams spectral sequence. If $H_i(X) = 0$ for $i > c$, then for all $k > p^s c - s$,*

$$\pi_k(X) = F^{s+1}\pi_k(X).$$

Proof. By [10], the spectral sequence has

$$E_2^{s,n} = \operatorname{Ext}^s_{\mathrm{CA}}\big(S(n), \overline{H}_*(X)\big) \Rightarrow \pi_{n-s}(X). \qquad \square$$

9. Generalizing the source

In this section we address Theorem D and the deduction of Theorem A′ from Theorem A. The idea of the proof of Theorem D given here is due to M. L. Hopkins, and I am grateful to him for permission to include it. This proof replaces one sketched in [31]. While I am convinced that that line of argument can be made to work, an error was discovered in the proof of Theorem 3.2 there. (That result should probably be restricted to $\mathcal{O}_{G(p)}$ in any case.) The deduction of Theorem A′ from Theorem A is a reformulation of a proof due to A. Zabrodsky.

It will be convenient at times to work without a basepoint. The following lemma is useful in passing between pointed and unpointed hypotheses, and we will use it in the sequel without mention. Recall again that "space" means "simplicial set."

Lemma 9.1. *If X is a connected pointed fibrant space and W is any pointed space, then $\mathrm{map}_*(W, X)$ is contractible if and only if the map $X \to \mathrm{map}(W, X)$ induced from $W \to *$ is an equivalence.*

Proof. The cofibration sequence

$$*_+ \to W_+ \to W$$

(where W_+ is W with a disjoint basepoint added, and $W_+ \to W$ sends $+$ to the basepoint of W) induces a fibration sequence

$$\begin{tikzcd} & \mathrm{map}_*(W, X) \arrow[d] \\ X \arrow[r] \arrow[dr, "="'] & \mathrm{map}(W, X) \arrow[d] \\ & X \end{tikzcd}$$

The result is now clear. $\square$

We will use the following technical result, whose proof we defer to the end of the section.

PROPOSITION 9.2. *Let $W' \to W$ be a map of simplicial spaces, and let X be a fibrant space. If* $\mathrm{map}(W_n, X) \to \mathrm{map}(W'_n, X)$ *is an equivalence for each n, then*

$$\mathrm{map}(\mathrm{diag}\, W, X) \to \mathrm{map}(\mathrm{diag}\, W', X)$$

is an equivalence.

This proposition has the following well-known fact as a consequence (cf. [11: XII. 4.2 and 4.3, p. 335] and [11: X.5.2 (ii), p. 278]).

COROLLARY 9.3. *Let $W' \to W$ be a map of simplicial spaces such that $W'_n \to W_n$ is an equivalence for each n. Then* $\mathrm{diag}\, W' \to \mathrm{diag}\, W$ *is an equivalence.* □

Let $[\cdot]\colon \boldsymbol{\Delta} \to \mathbf{S}$ be the "standard" cosimplicial set, with $[\cdot]^n = [n] = \{0, 1, \dots, n\}$. Any set $S \in \mathbf{S}$ then determines a simplicial set

$$ES = \mathrm{map}([\cdot], S). \tag{9.4}$$

The space ES is clearly contractible, if S is nonempty: one may for example note that ES is the nerve of the small category with object set S and exactly one morphism from x to y for any $x, y \in S$. It also depends functorially on S, so that if S is a G-set for some group G, then ES is a G-space. If S is G-free, then so is ES, and indeed, if S is G with the translation action, then EG is the usual contractible free G-space (sometimes written WG).

To prove Theorem A′ it is useful to extend this construction to simplicial sets. For $X \in s\mathbf{S}$, let EX denote the diagonal of the bisimplicial set with EX_n in degree n. By Corollary 9.3, EX is contractible. If X is a G-space for a simplicial group G, so is EX, and if X is G-free, so is EX. We warn the reader, though, that EG is *not* the usual contractible free G-space WG (unless G is discrete—i.e. constant). The identity element of G determines a natural basepoint in EG. For G-spaces Y' and Y, write $Y' \times_G Y$ for the orbit space of the diagonal G-action on $Y' \times Y$.

PROPOSITION 9.5. *Let G be a simplicial group, let X be a connected pointed fibrant space, and assume that* $\mathrm{map}_*(G, X)$ *is contractible. Then for any G-space Y the composite $Y \to EG \times Y \to EG \times_G Y$ induces an equivalence*

$$\mathrm{map}_*(EG \times_G Y, X) \xrightarrow{\simeq} \mathrm{map}_*(Y, X).$$

Proof. $Y \to EG \times Y$ is an equivalence, so that [11: X.5.2 (ii), p. 278] it induces an equivalence in $\mathrm{map}_*(-, X)$.

$EG \times Y$ is the diagonal space of the "external product" simplicial space $W' = EG \hat{\times} Y$, with $W'_n = G^{n+1} \times Y$. $EG \times_G Y$ is the diagonal space of the simplicial space $W = EG \hat{\times}_G Y$, with $W_n = G^{n+1} \times_G Y$. We claim that $\mathrm{map}(W_n, X) \to \mathrm{map}(W'_n, X)$ is an equivalence for each n; the result then follows from Proposition 9.2. We have a commutative diagram

$$\begin{tikzcd} G^{n+1} \times Y \arrow[r] \arrow[d, "h"', "\cong"] & G^{n+1} \times_G Y \arrow[d, "k"', "\cong"] \\ G^{n+1} \times Y \arrow[r, "pr"] & G^n \times Y \end{tikzcd}$$

in which

$$h(g_0, \dots, g_n; y) = \left(g_0, g_0^{-1}g_1, \dots, g_0^{-1}g_n; g_0^{-1}y\right),$$
$$k[g_0, \dots, g_n; y] = \left(g_0^{-1}g_1, \dots, g_0^{-1}g_n; g_0^{-1}y\right),$$
$$pr(g_0, \dots, g_n; y) = (g_1, \dots, g_n; y).$$

The claim now follows from the adjointness relation

$$\mathrm{map}(G^n \times Y, \mathrm{map}(G, X)) \cong \mathrm{map}(G^{n+1} \times Y, X). \qquad \square$$

Write $BG = G \setminus EG$.

Corollary 9.6. *Let G and X be as in* (9.5). *Then*

$$\mathrm{map}_*(BG, X) \simeq *.$$

Proof. Take $Y = *$. $\square$

Corollary 9.7. *Let G and X be as in* (9.5), *and let Y be a free G-space with orbit space B. Then $Y \to B$ induces an equivalence*

$$\mathrm{map}_*(B, X) \xrightarrow{\simeq} \mathrm{map}_*(Y, X).$$

Proof. $EG \times_G Y \to B$ is an equivalence. $\square$

These corollaries clearly provide the material for a deduction of Theorem A′ from Theorem A, using the tower of connective covers of the source space W.

We turn now to a proof of Theorem D, which we restate here.

Theorem 9.8. *Let X be a nilpotent fibrant space, let G be a locally finite group, and assume that $\mathrm{map}_*(BZ_p, X)$ is contractible for every prime p occurring as the order of an element of G. Then $\mathrm{map}_*(BG, X)$ is contractible.*

To begin with, we may as well assume that G is finite. For G is the direct limit of its directed system of finite subgroups K, so that if we use [11: XII. 3.5, p. 331],

$$BG \overset{=}{\leftarrow} \underrightarrow{\lim} BK \overset{\simeq}{\leftarrow} \underrightarrow{\mathrm{holim}} BK.$$

Thus [11: X.5.2 (ii), p. 278] for a fibrant space X,

$$\mathrm{map}_*(BG, X) \overset{\simeq}{\to} \underleftarrow{\mathrm{holim}}\ \mathrm{map}_*(BK, X)$$

where we have used [11: XII.4.1, p. 334] to bring $\underleftarrow{\mathrm{holim}}$ outside. If we assume that $\mathrm{map}_*(BK, X)$ is contractible for each K, then [11: XI.5.6, p. 304] so is $\mathrm{map}_*(BG, X)$.

We next see that we may assume that X is p-complete.

LEMMA 9.9. *Let X be a nilpotent fibrant space and W any connected space with $\overline{H}_*(W; \mathbf{Q}) = 0$. Then* $\mathrm{map}_*(W, X) \simeq *$ *if and only if* $\mathrm{map}_*(W, F_{p\infty}X) \simeq *$ *for every prime p.*

Proof. Arguing as in the proof of Theorem 1.5, we find that

$$\mathrm{map}_*(W, X) \to \prod_p \mathrm{map}_*(W, F_{p\infty}X)$$

is an equivalence. Thus $\mathrm{map}_*(W, X)$ is contractible if and only if $\mathrm{map}_*(W, F_{p\infty}X)$ is for all p; so the result follows. □

The proof of Theorem 9.8 will proceed by induction. The inductive step is contained in the following.

PROPOSITION 9.10. *Let X be a connected fibrant space, G a group, and H a subgroup. Assume that* $\mathrm{map}_*(BK, X)$ *is contractible for every finite intersection K of conjugates of H in G. Then* $\mathrm{map}_*(BG, X) \simeq *$ *if and only if* $\mathrm{map}_*(G \setminus E(G/H), X) \simeq *$.

Before proving this, let us use it to complete the proof of Theorem 9.8. First assume G is a p-group, and argue by induction on the order of G. Pick a nontrivial proper normal subgroup H of G. Then $\mathrm{map}_*(BH, X) \simeq *$ by the inductive hypothesis, so by Proposition 9.10 we must show that $\mathrm{map}_*(G \setminus E(G/H), X) \simeq *$. But G acts on $E(G/H)$ through the translation action of G/H on itself; so this is just $\mathrm{map}_*(B(G/H), X)$, which, again by inductive assumption, is contractible. Of course, Theorem C starts the induction.

In the general case, take for H a p-Sylow subgroup of G. Since X may be assumed to be p-complete, it will suffice (as in the proof of Theorem 1.5) to prove

PROPOSITION 9.11. *Let H be a p-Sylow subgroup of a finite group G. Then*

$$\bar{H}_*(G \setminus E(G/H); \mathbf{Z}_{(p)}) = 0.$$

Proof. We construct a contraction for the augmented simplicial $\mathbf{Z}_{(p)}$-module $\mathbf{Z}_{(p)}(G \setminus E(G/H))$. A typical class in $G \setminus E(G/H)_n$ consists of a G-orbit of $(n+1)$-tuples of cosets, $[c_0, \dots, c_n]$, $c_i \in G/H$. Define

$$h[c_0, \dots, c_n] = \frac{1}{[G:H]} \sum_{c \in G/H} [c, c_0, \dots, c_n].$$

Then it is straightforward to check that this is a contraction. □

This completes the proof of Theorem 9.8. We now return to the deferred proofs, beginning with Lemma 9.2. For this we need the following construction. Given $W \in ss\mathbf{S}$ and $X \in s\mathbf{S}$, define $\widetilde{\mathrm{map}}(W, X) \in s^\circ s\mathbf{S}$ by

$$\widetilde{\mathrm{map}}(W, X)^n = \mathrm{map}(W_n, X).$$

LEMMA 9.12. $\mathrm{map}(\mathrm{diag}\, W, X) \cong \mathrm{tot}\, \widetilde{\mathrm{map}}(W, X)$.

Proof. This involves a direct check of definitions (cf. [11: proof of XII. 4.3, p. 335]). □

The next lemmas use the notion of fibration in $s^\circ s\mathbf{S}$; see [11: X § 4, p. 275].

LEMMA 9.13. *If $X \to Y$ is a map of fibrant cosimplicial spaces which is such that $X^n \to Y^n$ is an equivalence for each n, then the induced map* $\mathrm{tot}\, X \to \mathrm{tot}\, Y$ *is an equivalence.*

Proof. This is [11: X.5.2, p. 277], with source the cofibrant object Δ (as in Section 1 above). □

LEMMA 9.14. *Let $X \to Y$ be a fibration of simplicial sets and let W be a bisimplicial set. Then the induced map of cosimplicial spaces*

$$\widetilde{\mathrm{map}}(W, X) \to \widetilde{\mathrm{map}}(W, Y)$$

is a fibration.

Proof. This is a straightforward application of definitions. The case in which W_n is constant for each n is [11: X.4.7 (ii), p. 275]. □

An object W is *fibrant* provided that $W \to *$ is a fibration. With $Y = *$, Lemma 9.14 has as a consequence:

COROLLARY 9.15. *If the space X is fibrant then, for any simplicial space W, the cosimplicial space $\widetilde{\mathrm{map}}(W, X)$ is fibrant.* □

Proof of Proposition 9.2. According to Corollary 9.15, $\widetilde{\mathrm{map}}(W, X)$ and $\widetilde{\mathrm{map}}(W', X)$ are fibrant cosimplicial spaces. Thus Lemma 9.13 shows that $\mathrm{tot}\,\widetilde{\mathrm{map}}(W, X) \to \mathrm{tot}\,\widetilde{\mathrm{map}}(W', X)$ is an equivalence, which, by Lemma 9.12, is what we want to prove. □

Proof of Proposition 9.10. We apply Proposition 9.2 with

$$W' = E(G/H) \hat{\times}_G EG,$$
$$W = \underline{G \backslash E(G/H)}.$$

At level n we have

$$W'_n = (G/H)^{n+1} \times_G EG,$$
$$W_n = \underline{G \backslash (G/H)^{n+1}}.$$

The map $W' \to W$ is induced by $EG \to *$. As a G-set, $(G/H)^{n+1}$ breaks up into orbits of the form G/K, where K is the intersection of $(n+1)$ conjugates of H. The map $W'_n \to W_n$ is thus a disjoint union of maps of the form $BK \to *$. By hypothesis, each such map induces an equivalence in $\mathrm{map}(-, X)$, so $\mathrm{map}(W_n, X) \to \mathrm{map}(W'_n, X)$ is an equivalence for each n, and the result follows. □

10. The fundamental group

We end this paper with a proof of Theorem B, which we restate here.

THEOREM 10.1. *Let X be a finite dimensional* CW *complex and G a torsion group. Then any map BG → X induces the trivial map of fundamental groups.*

Proof. Suppose that $\sigma \in G$ is mapped nontrivially to $\pi_1(X)$. Since G is torsion, σ has finite order; say $f_{\#}\sigma$ has order m and σ order mn. Let $p\colon \tilde{X} \to X$ be the covering projection such that $p_{\#}\pi_1(\tilde{X})$ is the subgroup $\langle f_{\#}\sigma \rangle$ of $\pi_1(X)$ generated by $f_{\#}\sigma$. There results a commutative diagram

$$\begin{array}{ccccc} B\langle \sigma \rangle & \to & \tilde{X} & \to & B\langle f_{\#}\sigma \rangle \\ \downarrow & & \downarrow & & \downarrow \\ BG & \to & X & \to & B\pi_1(X). \end{array}$$

The top composite is induced by the surjective homomorphism $h\colon \langle \sigma \rangle \to \langle f_{\#}\sigma \rangle$, i.e. $h\colon Z_{mn} \to Z_m$, and we have factored Bh through the finite-dimensional CW complex $\tilde{X}$. Such a space has finite category [42: X.1.8, p. 460]; so we conclude that $(Bh)^*x$ has finite height for any $x \in \bar{E}^*(BZ_m)$ and any multiplicative generalized cohomology theory E^*. But this is false. For instance, take E^* to be complex K-theory. We may choose a CW structure on BZ_m for which

$$\begin{aligned} \overline{H}^i\left(BZ_m^{(2k)}; \mathbf{Z}\right) &= Z_m, \qquad 2|i \leq 2k, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

The Atiyah-Hirzebruch spectral sequence collapses for degree reasons. The line bundle ξ_m on BZ_m, corresponding to the representation of Z_m obtained by identifying Z_m with the complex mth roots of unity, has $\xi_m^m = 1$. Moreover $x_m = \xi_m - 1 \in \overline{K}^0(BZ_m)$ is detected in filtration 2; so we find

$$\begin{aligned} &K^0\left(BZ_m^{(2k)}\right) = \mathbf{Z}[x_m]/\left((x_m + 1)^m - 1, x_m^{k+1}\right), \\ &K^1\left(BZ_m^{(2k)}\right) = 0. \end{aligned}$$

The Milnor sequence then implies

$$K^0(BZ_m) = \mathbf{Z}[[x_m]]/\left((x_m + 1)^m - 1\right)$$

with augmentation $\varepsilon x_m = 0$. The map $Z_{mn} \to Z_m$ sends ξ_m to ξ_{mn}^n, so that $x_m \mapsto (x_{mn} + 1)^n - 1$, which is easily seen to have infinite height. See also [4]. □

Remark 10.2. Since a connected covering space of a well-pointed space of finite category again has finite category, Theorem B remains true if we require merely that X have finite category and a nondegenerate basepoint.

Remark 10.3. The assumption that the target space X is a CW complex is made entirely as a matter of convenience, and can be relaxed at the expense of listing technical features required of the space. On the other hand, in the theorems of the introduction it is essential that the source be a CW complex; this is a condition of cofibrance required to make the transition to simplicial sets. See [11: VIII, 4.1, p. 244].

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