A SPECTRAL SEQUENCE FOR THE HOMOLOGY
OF AN INFINITE DELOOPING

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Let $E = \{E_n : n \in \mathbf{Z}\}$ be a (-1)-connected infinite loop space:
i.e., $\Omega E_{n+1} = E_n$ for all n, and for $n \geq 0$ the space E_n is
$(n-1)$-connected. Then the *stable homology* of E is

$$H_*(E) = \varinjlim H_{*+n}(E_n)$$

under the suspension homomorphisms. One also has the *un-
stable homology* $H_*(E_0)$, which with $\mathrm{mod}\, p$ coefficients carries
a Pontrjagin product and an action of the $\mathrm{mod}\, p$ Dyer-Lashof
algebra R.

It is natural to ask how $H_*(E_0)$ determines $H_*(E)$; and
the purpose of this paper is to construct and study the general
properties of a spectral sequence whose E^2-term depends
functorially on $H_*(E_0)$ as an R-Hopf algebra and whose E^∞-
term is the associated graded module of a natural filtration
on $H_*(E)$. For simplicity we mainly treat the case $p = 2$.

The spectral sequence developed here is probably identical in the
stable range with the iterated barconstruction sequence of D. W.
Anderson.[1] We give an entirely different construction, however,
which makes evaluation of E^2 easier. In fact, if $\mathscr{A}$ is the category
of allowable R-Hopf algebras ([4] I or §2 below) then E_s^2 is the sth
derived functor of the functor $F_p \otimes_R Q(-)$, where $Q(B)$ is the R-
module of indecomposables of $B \in \mathscr{A}$.

The following heuristic description of the resulting E^2-term in
favorable cases is due to M. Mahowald. Suppose $\pi_0(E_0)$ is free-Abelian
and $H_*(E_0[1])$ is polynomial, where $E_0[1]$ is the connected component
of the identity. Then as algebras $H_*(E_0)$ is isomorphic to the homology
of a product of ΩS^n's for various n. Now $H_*(\Omega S^n)$ is an Abelian Hopf
algebra and hence has formal iterated deloopings defined inductively by

$$B^k H_*(\Omega S^n) = \mathrm{Tor}^{B^{k-1} H \cdot (\Omega S)^n}(F_2, F_2) \ .$$

Then our E^1-term is the tensor-product of the corresponding groups
$B^\infty H_*(\Omega S^n)$, and the differential d^1 is determined by the Dyer-Lashof
action on $H_*(E_0)$.

In §1 of this paper we construct the spectral sequence and outline
the algebraic identification of its E^2-term. In §2 we set up the
algebraic background required to complete this identification. We

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[1] *Added in proof.* W. Dwyer points out that our spectral sequence is *not* identical
with Anderson's, since Anderson's E_2-term depends only on the algebra structure of
$H_*(E_0)$ while ours depends on the R-module structure as well.

also exploit the fact that $Q(-)$ itself has only one derived functor on an Abelian Hopf algebra to provide a "linear" computation of the E^2-term. Then in §3 we set up an analogue of the A-algebra to make this calculation. Finally, we show in §4 that the action of the Steenrod algebra on $H_*(E_0)$ puts very strong restrictions on the differential in the spectral sequence and goes farther than one might expect towards determining the Steenrod action on $H_*(E)$.

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1. **Construction of the spectral sequence.** Let $\mathcal{T}$ be the category of compactly generated Hausdorff spaces with nondegenerate basepoint (i.e., such that the inclusion of the basepoint is a cofibration) and continuous pointed maps. For brevity call an object of $\mathcal{T}$ a *space*. Clearly Ω and Σ are adjoint functors on $\mathcal{T}$; let $\alpha_X\colon X \to \Omega\Sigma X$ and $\beta_Y\colon \Sigma\Omega Y \to Y$ be the usual adjunctions. Let $\mathcal{L}$ be the category of (-1)-connected infinite loop spaces. Thus an object E of $\mathcal{L}$ is a sequence E_n, $n \in Z$, of spaces such that $E_n = \Omega E_{n+1}$ and E_n is $(n-1)$-connected for all n. A morphism $E' \to E$ is a sequence $f_n\colon E_n' \to E_n$ of continuous maps such that $f_n = \Omega f_{n+1}$ for all n. Call an object of $\mathcal{L}$ a *spectrum*.

Let $\Omega^\infty\colon \mathcal{L} \to \mathcal{T}$ by $\Omega^\infty E = E_0$. Let $\Sigma^\infty\colon \mathcal{T} \to \mathcal{L}$ by

$$(1.1) \qquad\qquad (\Sigma^\infty X)_n = \lim_{i \to \infty} \Omega^i \Sigma^{i+n} X \ .$$

Then Ω^∞ and Σ^∞ are adjoint functors [8]; we recall their adjunction morphisms. For $X \in \mathcal{T}$, $\alpha_X^\infty\colon X \to \Omega^\infty\Sigma^\infty X$ is the inclusion of the first term in the direct limit (1.1). For $E \in \mathcal{L}$, $\beta_E^\infty\colon \Sigma^\infty\Omega^\infty E \to E$ is the sequence whose nth term is the map induced from the system

$$\Omega^i \Sigma^{i+n} E_0 = \Omega^i \Sigma^{i+n} \Omega^{i+n} E_{i+n} \xrightarrow{\Omega^i \beta^{i+n}} \Omega^i E_{i+n} = E_n \ .$$

If h_* is a connective homology theory on $\mathcal{T}$ then given $E \in \mathcal{L}$, we can form the unstable homology

$$h_*^u(E) = h_*(\Omega^\infty E)$$

and the stable homology

$$h_*(E) = \lim_{n \to \infty} \Sigma^{-n} \widetilde{h}_*(E_n) \ .$$

Our problem is to relate these groups.

For any spectrum E there exists a space X and a map $i\colon \Sigma^\infty X \to$

E such that $\Omega^\infty i$ is a split epimorphism in $\mathscr{T}$. For example, we can take $X = \Omega^\infty E$ and $i = \beta_E^\infty$, since

$$\Omega^\infty \Sigma^\infty \Omega^\infty E \xrightarrow{\Omega^\infty \beta_E^\infty} \Omega^\infty E$$

$$\alpha_{\Omega^\infty E}^\infty \Big\uparrow \qquad \nearrow \atop = $$

$$\Omega^\infty E$$

commutes.

Using this observation we can resolve a spectrum E with respect to "space-like" spectra by forming a diagram

(1.2)
$$E = E(0) \qquad E(1) \qquad E(2)$$
$$\searrow^{i_0} \quad \nearrow \quad \searrow^{i_1} \quad \nearrow \qquad \cdots$$
$$\Sigma^\infty X(0) \qquad \Sigma^\infty X(1)$$

in which (i) $\Omega^\infty i_n$ is split epi and (ii) the V-shaped segments are fibration sequence in $\mathscr{L}$.

Now applying stable homology h_* yields a spectral sequence abutting to $h_*(E)$.

In case h_* is mod p singular homology H_* for p a prime, we can express E_2 in this spectral sequence in terms of $H_*^u(E)$. We shall outline here how this is done, and develop the necessary algebra in the next section.

Apply Ω^∞ to (1.2). The fibration

$$\Omega^\infty E(n+1) \longrightarrow \Omega^\infty \Sigma^\infty X(n) \xrightarrow{\Omega^\infty i_n} \Omega^\infty E(n)$$

is principal, and $\Omega^\infty i_n$ has a section, so $\Omega^\infty \Sigma^\infty X(n)$ has the weak homotopy type of the product of fiber and base. Consequently we have a short exact sequence

$$\boldsymbol{F}_p \longrightarrow H_*^u(E(n+1)) \longrightarrow H_*^u(\Sigma^\infty X(n)) \longrightarrow H_*^u(E(n)) \longrightarrow \boldsymbol{F}_p$$

of Hopf algebras over the Dyer-Lashof algebra R (see §2 below). Splicing these together gives a long exact sequence

(1.3) $\quad \boldsymbol{F}_p \longleftarrow H_*^u(E) \longleftarrow H_*^u(\Sigma^\infty X(0)) \longleftarrow H_*^u(\Sigma^\infty X(1)) \longleftarrow \cdots$.

Now Theorem 4.2 of [4] I implies that for $X \in \mathscr{T}$, $H_*(\Sigma^\infty X) = \tilde{H}_*(X)$ is given in terms of $H_*^u(\Sigma^\infty X)$ by $\boldsymbol{F}_p \otimes_R QH_*^u(\Sigma^\infty X)$, where Q denotes the Pontrjagin algebra indecomposables. Thus if we regard $H_*^u(\Sigma^\infty X)$ as a projective object then (1.3) is a projective resolution of $H_*^u(E)$ and

$$E_s^2 = L_s(\boldsymbol{F}_p \otimes_R Q)(H_*^u(E))$$

is the left derived functor of $\boldsymbol{F}_p \otimes_R Q(-)$.

2. The homological algebra of allowable Hopf algebras over the Dyer-Lashof algebra.

2.1. *Hopf algebras.* We make the following definitions.

Let k be a field. Let (Coalg) be the category of nonnegatively graded commutative augmented k-coalgebras C such that C_0 is generated over k by its set

$$\pi C = \{x \in C_0 \colon \Delta x = x \otimes x, \; \varepsilon x = 1\}$$

of set-like elements. For brevity call an object of (Coalg) a *coalgebra*. Given coalgebras C' and C'', the coalgebra $C' \otimes C''$ is a product in (Coalg). Let (Hopf) be the category of *Abelian* group objects in (Coalg). Call an object of (Hopf) a *Hopf algebra*.

For a Hopf algebra H, πH has a natural Abelian group structure for which the identity element is the identity element 1 of H. Let $H[1]$ be the component of H containing 1: i.e., $H[1] = k \,\square_{H_0} H$. Then the natural map $k[\pi H] \otimes H[1] \to H$ is an isomorphism. It follows that (Hopf) splits as the product of the category (Ab) Abelian groups and the category (c Hopf) of connected Hopf algebras. By a theorem of J. C. Moore (unpublished, but see [5]), (c Hopf) is Abelian; so (Hopf) is Abelian. The (co) product is the tensor-product.

Let $Z\colon$ (Coalg) $\to$ (Hopf) be the "free Abelian group" functor, left adjoint to the forgetful functor; see [4] p. 23 for details. This adjoint pair induces a projective class [6] in (Hopf), so the left derived functors $L_s F$ of an additive functor F on (Hopf) are defined.

For example, let $Q\colon$ (Hopf) $\to$ (k − mod) be the module of indecomposables; thus $QH = \operatorname{Tor}_1^H(k, k)$. There is a six-term exact sequence

$$(2.1.1) \quad 0 \longrightarrow Q_1 H' \longrightarrow Q_1 H \longrightarrow Q_1 H'' \longrightarrow QH' \longrightarrow QH \longrightarrow QH'' \longrightarrow 0$$

depending functorially on the short exact sequence $k \to H' \to H \to H'' \to k$ in (Hopf). Here $Q_1 H = Q \operatorname{Tor}_2^H(k, k)$ in terms of the natural algebra structure on $\operatorname{Tor}_*^H(k, k)$. To see this first let A be an Abelian group. Let E denote the exterior algebra over k and let Γ denote the divided polynomial algebra over k. Then as algebras

$$\operatorname{Tor}^{k[A]}(k, k) = E(\Sigma(k \otimes A)) \otimes \Gamma(\Sigma^2(k * A))$$

where Σ means suspension and where $\otimes$ and $*$ are the tensor and torsion products over $\mathbf{Z}$. Thus $Qk[A] = k \otimes A$ and $Q_1 k[A] = k * A$, and (2.1.1) is the long exact sequence of the derived functors of $k \otimes -$.

In the connected case (2.1.1) is proved in [9]. The general case follows by additivity.

Now by standard techniques we have

PROPOSITION 2.1.2.   *On* (Hopf), $L_0 Q = Q$, $L_1 Q = Q_1$, *and* $L_s Q = 0$ *for* $s > 1$.

2.2.   *The Dyer-Lashof algebra.* For simplicity we fix $p = 2$. We recall some notation from [4].

Let $R(-\infty)$ be the bigraded algebra with generators Q^i, $i \geqq 0$, $\|Q^i\| = (1, i)$, and "Adem" relations

$$(2.2.1) \qquad Q^i Q^j = \sum_t \binom{t - j - 1}{2t - i} Q^{i+j-t} Q^t , \quad i > 2j .$$

$R(-\infty)$ is thus isomorphic to the opposite of the Λ-algebra [2]. We will often neglect the first gradation to obtain a graded algebra. Let s be a nonnegative integer and let I be a sequence of s nonnegative integers $i_1, \cdots, i_s$. I is *allowable* iff $i_t \leqq 2i_{t+1}$ for $1 \leqq t < s$. Let $Q^I = Q^{i_1} \cdots Q^{i_s} \in R(-\infty)$. Then $\{Q^I : I \text{ is allowable}\}$ is a basis for $R(-\infty)$.

The *excess* of an allowable sequence I of length s is $e(I) = i_1 - (i_2 + \cdots + i_s)$. The subspace spanned by $\{Q^I : e(I) < n\}$ is an ideal $B(n) \subset R(-\infty)$. Let $R(n) = R(-\infty)/B(n)$. Then $R = R(0)$ is the *Dyer-Lashof* algebra.

An $R(-\infty)$-module M is *n-allowable* iff $Q^i x = 0$ for all $x \in M$ and all $i < |x| + n$. Thus for any nonnegatively graded n-allowable $R(-\infty)$-module, the $R(-\infty)$-action factors through an $R(n)$-action. Write $(a_n R\text{-mod})$ for the Abelian category n-allowable $R(-\infty)$-modules.

For $n \in Z$, the inclusion $(a_n R\text{-mod}) \to (R(-\infty)\text{-mod})$ has a left-adjoint a_n such that

$$(2.2.2) \qquad a_n M = M/\{Q^i x : i < |x| + n, x \in M\} .$$

This adjoint pair induces a projective class in $(a_n R\text{-mod})$. Let $\mathrm{Untor}_*^R(F_2, -)$ denote the resulting derived functor of

$$F_2 \otimes_R - : (a_1 R\text{-mod}) \longrightarrow (F_2\text{-mod}) .$$

2.3.   *Hopf algebras over the Dyer-Lashof algebra.* The Dyer-Lashof algebra admits a (noncommutative) Hopf algebra structure with commutative diagonal such that $\Delta Q^n = \sum_{i+j=n} Q^i \otimes Q^j$. Hence one has a internal tensor-product on $(a_0 R\text{-mod})$ and so one can define the category $(aR\text{-Coalg})$ of allowable R-coalgebras and $(aR\text{-Hopf})$ of allowable R-Hopf algebras. (An R-Hopf algebra H is *allowable* if it is 0-allowable qua R-module and is such that $Q^{|x|} x = x^2$ for all $x \in H$.) It is easy to see that $(aR\text{-Hopf})$ is Abelian.

In [4] I §2 May constructs adjoint functors

$$(\text{Coalg}) \underset{\longleftarrow}{\overset{E}{\longrightarrow}} (aR\text{-Coalg}) \underset{\longleftarrow}{\overset{Z}{\longrightarrow}} (aR\text{-Hopf})$$

where the unnamed arrows are forgetful functors. E is such that $JEC = a_0(R \otimes JC)$, and Z (which May calls GW) covers $Z: (\text{Coalg}) \to (\text{Hopf})$. Give $(aR\text{-Hopf})$ the projective class induced by ZE.

By [4] I Theorem 4.2, $H_*^u(\Sigma^\infty X) \cong ZEH_*(X) \in (aR\text{-Hopf})$ naturally in $X \in \mathcal{T}$. Thus (1.3) is indeed a projective resolution; note in particular that it is split as a sequence of coalgebras. Thus if $\boldsymbol{Q}$: $(aR\text{-Hopf}) \to (\boldsymbol{F_2}\text{-mod})$ by $\boldsymbol{Q}H = \boldsymbol{F_2} \otimes_R QH$, then in our spectral sequence,

$$(2.3.1) \qquad\qquad E_s^2 = L_s\boldsymbol{Q}(H_*^u(E)) \ .$$

To compute $L_s\boldsymbol{Q}$ we first study $Q: (aR\text{-Hopf}) \to (a_1R\text{-mod})$. Let $H \leftarrow P_*$ be a projective resolution in $(aR\text{-Hopf})$. Clearly P_* is a projective resolution of H when regarded in (Hopf). Hence for $H \in (aR\text{-Hopf})$,

$$L_1Q(H) = Q \operatorname{Tor}_2^H(\boldsymbol{F_2}, \boldsymbol{F_2})$$

as an $\boldsymbol{F_2}$-module and $L_tQ(H) = 0$ for $t > 1$.

It is easy to see that $Q: (aR\text{-Hopf}) \to (a_1R\text{-mod})$ carries projectives to projectives. One thus has a Grothendieck spectral sequence [6]

$$(2.3.2) \qquad E_{s,t}^2 = \operatorname{Untor}_s^R(\boldsymbol{F_2}, L_tQ(H)) \underset{s}{\Longrightarrow} L_{s+t}\boldsymbol{Q}(H) \ .$$

Since $L_tQ = 0$ for $t > 1$ this spectral sequence degenerates to a long exact sequence

$$(2.3.3) \qquad
\begin{aligned}
0 &\longleftarrow \operatorname{Untor}_1^R(\boldsymbol{F_2}, QH) \longleftarrow L_1\boldsymbol{Q}(H) \longleftarrow \operatorname{Untor}_0^R(\boldsymbol{F_2}, Q_1H) \\
&\qquad\qquad \overset{\partial}{\longleftarrow} \operatorname{Untor}_2^R(\boldsymbol{F_2}, QH) \longleftarrow \cdots \ .
\end{aligned}$$

In particular, if $Q_1H = 0$ — for example, if πH is free of 2-torsion and $H[1]$ is a symmetric algebra — then

$$(2.3.4) \qquad\qquad L_s\boldsymbol{Q}(H) = \operatorname{Untor}_s^R(\boldsymbol{F_2}, QH)$$

for all s. Thus in this case $L_s\boldsymbol{Q}(H)$ is determined by the action of R on QH alone.

REMARK 2.3.5. Of course, this E_2-term may also be obtained directly, without introducing $\boldsymbol{Q}$, by observing that

$$0 \longleftarrow QH_*^u(E) \longleftarrow QH_*^u(\Sigma^\infty X(0)) \longleftarrow QH_*^u(\Sigma^\infty X(1)) \longleftarrow \cdots$$

is a projective resolution in $(a_1R\text{-mod})$ which yields E^1 on application of the functor $\boldsymbol{F_2} \otimes_R -$.

2.4. *Remark on the Adams spectral sequence.* The procedure of this section is equally applicable to the study of the E_2-term of Bousfield and Kan's mod p unstable Adams spectral sequence [2]. They prove that

$$E_2^s(X) = R^s P(H_*(X)) .$$

Here P: (unstable A-coalgebras) $\to$ (F_p-mod) is the functor $F_p \,\square_A\, P(-)$ where A is the mod p dual Steenrod algebra and $P(-)$ is the module of primitives. Then $P(C)$ is a 1-unstable A-comodule, in the sense (for $p = 2$) that $xSq^n = 0$ for all $x \in C$ and all $n \geq 1/2\,|x|$. As such, P carries injectives to injectives, and there results a Grothendieck spectral sequence

$$(2.4.1) \qquad E_2^{s,t} = \mathrm{Uncotor}_A^s(F_p, R^t P(C)) \underset{s}{\Longrightarrow} R^{s+t} P(C)$$

in which $\mathrm{Uncotor}_A^s(F_p, -)$ is the obvious right derived functor of $F_p \,\square_A\, -: (u_1 A\text{-comod}) \to (F_p\text{-mod})$.

This situation is in several ways more complicated than the Dyer-Lashof case. First, the category of unstable A-coalgebras is not additive, so one must use the theory of nonabelian derived functors to define $R^s P$ and $R^s P$. Furthermore, $R^s P$ can be nonzero for all s. Bousfield and Curtis consider the case when $R^s P = 0$ for $s > 1$, and [1] is in part an analysis of the spectral sequence (2.4.1) in this "nice" case.

Bousfield and Curtis construct a complex $\Lambda(M)$ such that

$$\mathrm{Uncotor}_A(F_2, M) = H(\Lambda(M)) .$$

$\Lambda(M)$ is closely related to the usual unstable Λ-algebra for M. Our next step here is establish an analogous complex for a 1-allowable $R(-\infty)$-module. We remark that the methods of §3 carry over to the Steenrod algebra case and give another derivation of the properties of $\Lambda(M)$.

3. A Λ-algebra for allowable R-modules. We now address the problem of computing $\mathrm{Untor}^R(F_2, M)$. As noted by S. Priddy in [10], Example 9.4, the algebra $R(-\infty)$ has a "Koszul resolution." We note here that this machinery behaves well with respect to the allowability condition, and study the resulting complex.

3.1. *The off-diagonal homology of the bar-construction.* Let

$$R = R(-\infty) \otimes -: (F_2\text{-mod}) \longrightarrow (R(-\infty)\text{-mod})$$

and

$$U = a_1(R(-\infty) \otimes -): (F_2\text{-mod}) \longrightarrow (a_1 R\text{-mod})$$

be left adjoint to the forgetful functors. The resulting cotriples on $(R(-\infty)$-mod) and $(a_1R$-mod) give rise to standard complexes $B_R(M)$ and $B_U(M)$. Here $B_R(M)$ is the usual (reduced) unnormalized bar construction with respect to $R(-\infty)$. Similarly, $B_U(M)_s = U^s(M)$. The natural surjection $R(M) \to U(M)$ induces a surjection of the corresponding bar constructions.

Given a basis X for $M \in (R(-\infty)$-mod),

(3.1.1) $\{Q^{I_1} \otimes \cdots \otimes Q^{I_s} \otimes x : I_j$ is allowable, $x \in X\}$

is a basis for $B_R(M)_s$. If $M \in (a_1R$-mod) then a basis for $B_U(M)_s$ is given by (the image of) the subset of $Q^{I_1} \otimes \cdots \otimes Q^{I_s} \otimes x$ in (3.1.1) such that in addition

$$e(I_j) > |I_{j+1}| + \cdots + |I_s| + |x|$$

for $1 \leq i \leq s$. These are the "monomial" bases.

Let $S_n \in (a_1R$-mod) have $(S_n)_n = F_2$ (with generator x) and $(S_n)_j = 0$ for $j \neq n$. Then the bigrading on $R(-\infty)$ induces a trigrading on $B_R(S_n)$ and on $B_U(S_n)$ with $(B_R(S_n))_{s,t,u}$ generated by those $Q^{I_1} \otimes \cdots \otimes Q^{I_s} \otimes x$ such that $\sum l(I_j) = t$ and $\sum |I_j| + n = u$. Clearly $B_R(S_n)_{s,t,u} = 0$ for $s > t$. The differential d preserves t and u and decreases s by 1. We now have:

PROPOSITION 3.1.2. *There is no homology in $B_R(S_n)$ or in $B_U(S_n)$ off the main diagonal — i.e., for $s \neq t$.*

For $B_R(S_n)$ this is [10] Theorem 5.3. The assertion follows for $B_U(S_n)$ upon checking that the chain-homotopy constructed by Priddy sends $B_U(S_n)$ into itself.

To compute the diagonal homology, we pass to the dual, still following [10].

3.2. *The dual of the Dyer-Lashof algebra.* We recall from [7] and [4] the structure of the dual $\hat{R}$ of the Dyer-Lashof algebra. As a commutative algebra

$$\hat{R} = \prod_{k \geq 0} \hat{R}[k]$$

where $\hat{R}[k] = F_2[\xi_{1,k}, \cdots, \xi_{k,k}]$ for $k > 0$, and $\hat{R}[0] = F_2$. Here $\xi_{i,k}$ is dual (with respect to the allowable basis) to $Q^{I_{i,k}}$, where $I_{i,k}$ is the length k sequence

$$I_{0,k} = (0, \cdots, 0)$$
$$I_{i,k} = (2^{k-i-1}(2^i - 1), \cdots, (2^i - 1), 2^{i-1}, \cdots, 1) \text{ for } 0 < i < k$$
$$I_{k,k} = (2^{k-1}, \cdots, 1) .$$

Note that $e(QI_{i,k}) = 0$ unless $i = k > 0$, when $e(QI_{k,k}) = 1$. $\xi_{0,k}$ is the identity element in $\hat{R}[k]$. $\hat{R}$ is bigraded by $||\xi_{i,k}|| = (k, 2^{k-i}(2^i - 1))$. Let Λ be a sequence of the form $(k; i_1, \cdots, i_k)$ where $k \geq 0$ and $1 \leq i_j \leq k$; then define $\xi^\Lambda = \xi_{i_1,k} \cdots \xi_{i_k,k}$. Write $l(\Lambda) = k$.

R is not a Hopf algebra, because $R_0 = F_2[Q^0]$ is not of finite type. However, the product in R induces a multiplicative unitary coproduct Δ on $\hat{R}_+$, the positive dimensional submodule of $\hat{R}$, such that

$$(3.2.1) \qquad \Delta \xi_{j,k} = \sum_{h,i} (\xi_{k-i,k-i}^{2^h-1} \xi_{j-k,k-i})^{2^{i-h}} \otimes \xi_{h,i} .$$

Also the unit in R dualizes to $\varepsilon: \hat{R} \to F_2$ by $\varepsilon(\xi_{0,0}) = 1$ and $\varepsilon(x) = 0$ otherwise.

Let $J(n)$ be the submodule of $\hat{R}$ generated by $\{\xi^\Lambda: i_k \geq n$ where $l(\Lambda) = k\}$; $J(n)$ is the ideal generated by $\{\xi_{k,k}^n: k \geq 1\}$. $J(n)$ is then the annihilator of $B(n) \subset R$.

A 1-*allowable comodule* over R is a nonnegatively graded F_2-module together with a map $\psi: N \to \hat{U}(N)$, where

$$\hat{U}(N) = \bigoplus_{n \geq 0} J(n+1) \otimes N_n \subset \hat{R} \otimes N ,$$

such that $(\Delta \otimes N)\psi = (\hat{R} \otimes \psi)\psi$ and $(\varepsilon \otimes N)\psi = id$. Notice that since $J(n+1)_0 = 0$ for $n \geq 0$, $\Delta \otimes N$ is defined on $\hat{U}(N)$.

Let $(a_1\hat{R}\text{-comod})$ denote the resulting category. It is clearly Abelian, and there are functors

$$(F_2\text{-mod}) \overset{\hat{U}}{\underset{}{\rightleftarrows}} (a_1\hat{R}\text{-comod})$$

in which the unmarked arrow is forgetful and $\hat{U}$ is its right adjoint. This adjoint pair yields an injective class in $(a_1R\text{-comod})$ and a standard complex $\Omega_{\hat{U}}(N)$. A basis Y for N determines a basis for $\Omega_{\hat{U}}(N)^s$, namely,

$$(3.2.2) \qquad \begin{aligned} &\{\xi^{\Lambda_1} \otimes \cdots \otimes \xi^{\Lambda_s} \otimes y: \xi^{\Lambda_j} \in J(|\Lambda_{j+1}| + \cdots + |\Lambda_s| + |y|) \\ &\qquad\qquad\qquad\qquad \text{for all } 1 \leq j \leq s \text{ and } y \in Y\} . \end{aligned}$$

If M is a 1-allowable R-module then $\hat{M}$ is a 1-allowable R-comodule and there is a natural isomorphism of complexes

$$B_U(M)\hat{} = \Omega_{\hat{U}}(\hat{M}) .$$

Thus if M is of finite type,

$$\text{Untor}^R(F_2, M)\hat{} = H(\Omega_{\hat{U}}(\hat{M})) .$$

3.3. *The diagonal homology of the cobar construction.* We must therefore compute $H^{s,s}(\Omega_{\hat{U}}(\hat{S}_n))$. From (3.2.2), $\Omega_{\hat{U}}(\hat{S}_n)^{s,s}$ has basis

(3.3.1) $\{\xi_{11}^{i_1} \otimes \cdots \otimes \xi_{11}^{i_s} \otimes y : i_j > i_{j+1} + \cdots + i_s + n \text{ for } 1 \leqq j \leqq s\}$,

where y generates $\hat{S}_n$. These elements are of course all cycles, since $\Omega_{\hat{U}}(\hat{S}_n)^{s+1,s} = 0$. This also follows from the equations

(3.3.2) $\Delta\xi_{11} = \xi_{11} \otimes \xi_{00} + \xi_{00} \otimes \xi_{11}$

and $\xi_{00}\xi_{11} = 0$, since then

$$\Delta\xi_{11}^i = \xi_{11}^i \otimes \xi_{00} + \xi_{00} \otimes \xi_{11}^i .$$

To compute the boundaries in $\Omega_{\hat{U}}(\hat{S}_n)^{s,s}$ note that $\Omega_{\hat{U}}(\hat{S}_n)^{s-1,s}$ has basis

(3.3.3) $\{\xi_{11}^{i_1} \otimes \cdots \otimes \xi_{11}^{i_{k-1}} \otimes \xi_{12}^i\xi_{22}^{i_k} \otimes \xi_{11}^{i_{k+1}} \otimes \cdots \otimes \xi_{11}^{i_{s-1}} \otimes y :$
$\qquad\qquad i \geqq 0 \text{ and } i_j > i_{j+1} + \cdots + i_{s-1} + n \text{ for } 1 \leqq j \leqq s - 1\}$.

Now

(3.3.4) $\Delta\xi_{12} = \xi_{12} \otimes \xi_{00} + \xi_{11} \otimes \xi_{11} + \xi_{11}^2 \otimes \xi_{01} + \xi_{00} \otimes \xi_{12}$

(3.3.5) $\Delta\xi_{22} = \xi_{22} \otimes \xi_{00} + \xi_{00} \otimes \xi_{22} + \xi_{11}^2 \otimes \xi_{11}$

so the reduced diagonal of $\xi_{12}^i\xi_{22}^j$ is

(3.3.6) $$\sum_{t=0}^{i} \binom{i}{t} \xi_{11}^{2i+2j-t} \otimes \xi_{11}^{j+t}$$

and this obviously determines the differential of an element of (3.3.3).

To describe the resulting homology, let L be the associative algebra on symbols σ_{i+1}, $i \geqq 0$, of bidegree $(1, i)$, subject to relations

(3.3.7) $$\sum_{t=0}^{i} \binom{i}{t} \sigma_{2i+2j-t+1}\sigma_{j+t+1} = 0$$

for $i, j \geqq 0$. Then $H(\Omega_{\hat{U}}(\hat{S}_n))$ maps isomorphically to the sub F_2-module of L spanned by

$$\{\sigma_{i_1+1} \cdots \sigma_{i_s+1} : i_j > i_{j+1} + \cdots + i_s + n \quad \text{for} \quad 1 \leqq j \leqq s\}$$

by sending $\xi_{11}^{i_1} \otimes \cdots \otimes \xi_{11}^{i_s} \otimes y$ to $\sigma_{i_1+1} \cdots \sigma_{i_s+1}$.

LEMMA 3.3.8. (a) Let $I = (i_1, \cdots, i_s)$ be a sequence of positive integers such that

(3.3.9) $i_j \geqq i_{j+1} + \cdots + i_s \quad for \quad 1 \leqq j < s .$

Then in L, $\sigma_{I+1} = \sigma_{i_1+1} \cdots \sigma_{i_s+1}$ is a linear combination of σ_J's with J admissible.

(b) $\{\sigma_J : J$ is admissible$\}$ is linearly independent in L.

Here "admissible" carries its usual meaning: J is admissible iff $j_k \geqq 2j_{k+1}$ for all k.

Proof. (a) Let $0 \leqq b \leqq a \leqq 2b$. Then with the substitutions $j = a - b$, $i = 2b - a$, $t = 2b - a - s$, (3.3.7) yields

$$(3.3.10) \qquad \sigma_{a+1}\sigma_{b+1} = \sum_{s=0}^{2b-a} \binom{2b-a}{s} \sigma_{a+s+1}\sigma_{b-s+1} .$$

The condition (3.3.9) guarantees the applicability of this relation to σ_{I+1} unless $I + 1$ is admissible. The reader may check that (3.3.9) is satisfied by each term of the result. If we define the *moment* of $I + 1$ as

$$m(I + 1) = \sum_{j=1}^{s} j(i_j + 1)$$

then each term of the result has strictly smaller moment. This completes the proof, since moment is positive.

(b) If we forget the first gradation, then L maps multiplicatively to the mod 2 Steenrod algebra A by sending σ_{i+1} to Sq^{i+1} for $i \geqq 0$. To see this let $\delta \colon A \to A$ be the derivation sending Sq^{i+1} to Sq^i. Then (3.3.7) maps to Leibnitz formula for $\delta^i(Sq^{2i+2j+1}Sq^{i+j+1})$, which is zero as required. Since $\{Sq^I \colon I$ is admissible$\}$ is a basis for A, the result follows.

We have proved

PROPOSITION 3.3.11. *Let $n \geqq 0$. Then*

$$\mathrm{Untor}_s^R(F_2, S_n)^{\widehat{}} = H_s(B_U(S_n))^{\widehat{}} = H^s(\Omega_U^{\widehat{}}(\widehat{S}_n)) = L^s(n)$$

where $L^s(n)$ is the F_2-module with basis

$$\{\sigma_I \colon I \text{ admissible, } l(I) = s, \ i_s > n + 1\} .$$

EXAMPLE 3.3.12. Let H be the integral Eilenberg-MacLane spectrum. Then $\Omega^\infty H = Z$ as a discrete H-space, so $H_*^*(H) = F_2[Z]$. Thus (2.3.4) applies, and since $QF_2[Z]$ is spanned by $x = [1]-[0]$, our E^2-term is $\mathrm{Untor}_*^R(F_2, F_2) = L^*(0)^{\widehat{}}$. Now $H_*(H)^{\widehat{}} = A/ASq^1$, and we have just seen that $L^*(0)$ is additively isomorphic to A/ASq^1 under the correspondence $\sigma_{i+1} \leftrightarrow Sq^{i+1}$, $i \geqq 1$. The spectral sequence therefore collapses in this case. The example of the mod 2 Eilenberg-MacLane spectrum may be handled using (2.3.3), and this provides an upper bound (which turns out to be sharp) on the size of the Steenrod algebra, derived entirely from the Dyer-Lashof algebra. We shall study the behavior of the Steenrod action in the spectral sequence in more detail in §4.

We now define a complex $L(N)$, natural in the 1-allowable $\hat{R}$-comodule N, such that

$$H(\Omega_{\hat{\iota}}(N)) = H(L(N)) \ .$$

As an F_2-module,

(3.3.13) $$L(N) = \bigoplus_{n \geq 0} L(n) \otimes N_n \ .$$

To define the differential, let $\pi_1 \colon \hat{R} \to \hat{R}[1]$ be the projection; then

(3.3.14) $$(\pi_1 \otimes 1)\psi(y) = \sum_{i > 0} \xi_{11}^i \otimes yQ^i$$

defines a right R-action on N. Let

(3.3.15) $$d(\sigma_I \otimes y) = \sum_{i > 0} \sigma_{I,i+1} \otimes yQ^i \ .$$

Now the relations in L precisely guarantee that $d^2 = 0$. There is a natural map of complexes

$$\theta \colon \Omega_{\hat{U}}(N) \longrightarrow L(N)$$

sending $\xi^{A_1} \otimes \cdots \otimes \xi^{A_s} \otimes y$ to zero unless $l(A_j) = 1$ for all j, and sending

$$\xi_{11}^{i_1} \otimes \cdots \otimes \xi_{11}^{i_s} \otimes y \longmapsto \sigma_{i_1+1} \cdots \sigma_{i_s+1} \otimes y \ .$$

Filter N by $F_n N = \bigoplus_{i \leq n} N^i$. $F_n N$ is a sub 1-allowable $\hat{R}$-comodule, and $E_n^0 N$ is a sum of copies of $\hat{S}_n$. Thus in the resulting spectral sequence, $E^1(\theta)$ is an isomorphism by the above work. Thus θ is a homology isomorphism, and we have proved

THEOREM 3.3.16. *There is an isomorphism*

$$\mathrm{Untor}^R(F_2, M)^{\widehat{}} \cong H(L(\hat{M}))$$

natural the 1-allowable $R(-\infty)$-module M of finite type.

4. The Steenrod action.

4.1. *The Nishida action.* Let A^0 be the opposite of the mod 2 Steenrod algebra, graded nonpositively. A left A^0-module M is *unstable* iff $Sq^n x = 0$ whenever $2n > |x|$. Write $(uA^0\text{-mod})$ for the resulting Abelian category. The diagonal A^0-action provides it with an internal tensor-product. A coalgebra C in $(uA^0\text{-mod})$ is *unstable* provided that $\xi x = Sq^n x$ for all $x \in C_{2n}$, where ξ is the Verschiebung.

If M is an unstable A^0-module, give $R \otimes M$ the unstable A^0-module structure determined inductively on length in R by the Nishida relation

$$(4.1.1) \qquad Sq^n(Q^r x) = \sum_t \binom{r-n}{n-2t} Q^{r-n+t} Sq^t x \ .$$

An n-allowable $R(-\infty)$-module structure on M (for $n \geq 0$) is an $A^0 R$-structure iff $R \otimes M \to M$ is an A^0-module map. The resulting category $(ua_n A^0 R\text{-mod})$ is Abelian.

Putting this together with §2.3, we have the category $(uaA^0 R\text{-Hopf})$ of allowable R-Hopf algebras with compatible unstable A^0-action. $(uaA^0 R\text{-Hopf})$ is Abelian, and the functors Z, E, and $\mathbf{Q}$ of §2.3 lift; for example, $\mathbf{Q}: (uaA^0 R\text{-Hopf}) \to (uA^0\text{-mod})$ additively. Then [4] I Theorem 4.2 asserts further that $H_*^u(\Sigma^\infty X) \cong ZEH_*(X)$ in $(uaA^0 R\text{-Hopf})$. Thus the spectral sequence is defined in $(uA^0\text{-mod})$, and $E_s^2 = L_s\mathbf{Q}(H_*^u(X))$ in $(uA^0\text{-mod})$.

Similarly, for $M \in (ua_1 A^0 R\text{-mod})$, $\mathrm{Untor}_s^R(\mathbf{F}_2, M) \in (uA^0\text{-mod})$, and the exact sequence (2.3.3) is defined in $(uA^0\text{-mod})$. When M is of finite type we can express this action in terms of the complex $L(\hat{M})$ introduced in §3.3. Give $\hat{M}$ the contragredient left A-action. There is then an A-action on $\hat{U}(\hat{M})$ for which $\psi: \hat{M} \to \hat{U}(\hat{M})$ is A-linear. We need only part of this action, namely,

$$(4.1.2) \qquad Sq^n(\xi_{11}^r \otimes x) = \sum_t \binom{r-t}{n-2t} \xi_{11}^{n+r-t} \otimes Sq^t x \ .$$

This formula follows by an easy dualization from (4.1.1). Thus one may inductively define an A-module structure on $L(\hat{M})$: if it is known on $L^{s-1}(\hat{M})$, let

$$(4.1.3) \qquad Sq^n(\sigma_{r+1} w) = \sum_t \binom{r-t}{n-2t} \sigma_{n+r-t+1} Sq^t w$$

for $w \in L^{s-1}(\hat{M})$. Then $L(\hat{M})$ is a differential-graded object in $(uA\text{-mod})$, the category of left A-modules for which $Sq^n x = 0$ whenever $n > |x|$, and

$$H(L(\hat{M})) \cong \mathrm{Untor}^R(\mathbf{F}_2, M)\hat{\ }$$

as A-modules.

EXAMPLE 4.1.4. By comparing (4.1.3) with the Adem relations in the Steenrod algebra, we have

$$(4.1.5) \qquad L^n(\hat{S}_0) = L^n(0) \cong E_0^n(A/ASq^1)$$

as A-modules, where A/ASq^1 is given the *length* filtration

$$(4.1.6) \quad F^n(A/ASq^1) = \langle Sq^I : I = (i_1, \cdots, i_s) \text{ admissible, } s \geq n, i_s > 1 \rangle \ .$$

(It follows from this, incidentally, that $E_0^n(A/ASq^1)$, given the *complementary* gradation, so Sq^I occurs in dimension $|I| - n$, is an unstable A-module. The same is true of $E^0(A)$.) In fact, the filtration on $H^*(H) = A/ASq^1$ associated with the spectral sequence is precisely the length filtration. To see this we make the following general observation.

For a left A-module M let $F^0M = M$ and inductively define sub A-modules

$$F^{s+1}M = \{Sq^n x: n + s > |x|, \, x \in F^s M\} \, .$$

For example, on A/ASq^1 this is precisely the length filtration. In general we call this the *unstable filtration*, because $E_0^s M$ (graded by *complementary* degree) is unstable, and F^*M is the largest filtration of M with this property; any other such filtration $\widetilde{F}^*M$ has $\widetilde{F}^s M \subseteq F^s M$ for all s. If M is (-1)-connected then $F^s M$ is $(2^s - 2)$-connected.

Thus in particular the filtration $\widetilde{F}^* H^*(E)$ associated to the spectral sequence for the spectrum E maps to the unstable filtration. In the example of A/ASq^1, the quotients agree, so the filtrations agree as claimed.

4.2. *The secondary Steenrod action.* Consider the filtration induced in stable cohomology $H^*(E)$ by the spectral sequence of §1. Since

$$Sq^{n+1} F^s H^{s+t}(E) \subseteq F^{s+1} H^{s+t+n+1}(E)$$

whenever $n \geq t$, we may hope to find a representative for $Sq^{n+1}\overline{x}$ in E_1 depending only on a representative for $\overline{x} \in F^s H^{s+t}(E)$ in E_1. In fact we have

THEOREM 4.2.1. *Suppose E is a spectrum such that $H_*(E)$ is of finite type.*

(a) *Consider the cohomology spectral sequence with E_1 obtained from (1.2) with $X(n) = \Omega^\infty E(n)$. Let $\overline{x} \in F^s H^{s+t}(E)$ and let $n \geq t$, so that $Sq^{n+1}\overline{x} \in F^{s+1} H^{s+t+n+1}(E)$. Let $x_0 \in \widetilde{H}^t(E(s)_0) = E_1^{s,t}$ survive to $\overline{x}$. Then*

$$\xi_{11}^n \otimes x_0 \in H^{t+n}((\Sigma^\infty \Omega^\infty E(s))_0)$$

maps to an element in $H^{t+n}(E(s+1)_0)$ surviving to $Sq^{n+1}\overline{x}$.

(b) *Suppose that $Q_1 H_*(E_0) = 0$, so that $E_2 = H(L(PH^*(E_0)))$. Let $x_0 \in L^s(PH^*(E_0))$ survive to $\overline{x} \in F^s H^{s+t}(E)$, and let $n \geq t$. Then $\sigma_{n+1} x \in L^{s+1}(PH^*(E_0))$ survives to $Sq^{n+1}\overline{x}$.*

This theorem follows immediately from Lemma 4.2.4 below.

Let E be a spectrum with $H_*(E)$ of finite type, and consider the natural fibration sequence

$$(4.2.2) \qquad F \xrightarrow{j} \Sigma^\infty E_0 \xrightarrow{k} E \xrightarrow{i} \Sigma F \ ;$$

on the 0-space level we have a fibration

$$(4.2.3) \qquad F_0 \xrightarrow{j_0} \Omega^\infty \Sigma^\infty E_0 \xrightarrow{k_0} E_0 \ .$$

Let $x \in H^t(E)$ map to $k^*x = x_0 \in H^t(\Sigma^\infty E_0) = \widetilde{H}^t(E_0)$. Let $n \geq t$, so that $Sq^{n+1}x_0 = 0$. Thus there exists $y \in H^{t+n}(F)$ such that $i^*y = Sq^{n+1}x$. Let y map to $y_0 \in \widetilde{H}^{t+n}(F_0)$. Then

LEMMA 4.2.4. $j_0^*(\xi_{11}^n \otimes x_0) = y_0$.

Proof. Let $K = \{K_t\}$ denote the mod 2 Eilenberg-MacLane spectrum, and let

$$(2.2.4) \qquad L \longrightarrow K \xrightarrow{Sq^{n+1}} \Sigma^{n+1} K$$

be a fibration sequence in $\mathscr{L}$. Let $\iota_{t+n} \in H^{t+n}(K_{t+n})$ be the fundamental class, and let $\bar{\iota}_{t+n} \in H^{t+n}(L_t)$ map to ι_{t+n} under the boundary homomorphism. Then the reduced coaction $\bar{\psi}: H^*(L_t) \to \widehat{R} \otimes H^*(L_t)$ satisfies

$$(4.2.5) \qquad \bar{\psi}\bar{\iota}_{t+n} = \xi_{11}^n \otimes \iota_t \ .$$

This formula is just dual to the well known fact [3] that the fundamental homology classes in L_t are related by Q^n.

Now, since $k^*x = x_0$ has $Sq^{n+1}x_0 = 0$, we have maps y and z in the commutave diagram of spectra

$$
\begin{array}{ccc}
F & \xrightarrow{\ y\ } & \Sigma^{t+n} K \\
{\scriptstyle j}\downarrow & & \downarrow \\
\Sigma^\infty E_0 & \xrightarrow{\ z\ } & \Sigma^t L \\
{\scriptstyle k}\downarrow & & \downarrow \\
E & \xrightarrow{\ x\ } \Sigma^t K & \xrightarrow{Sq^{n+1}} \Sigma^{t+n+1} K \ .
\end{array}
$$

At the level of 0-spaces, we have a commutative diagram

$$\begin{array}{ccccc}
K_{t+n} & \longrightarrow & L_t & \xleftarrow{\;\varphi\;} & \Omega^\infty\Sigma^\infty L_t \\
y_0 \big\uparrow & & z_0 \big\uparrow & & \big\uparrow \Omega^\infty\Sigma^\infty z_0 \\
F_0 & \xrightarrow{\;j_0\;} & \Omega^\infty\Sigma^\infty E_0 & \xleftarrow{\;\beta\;} & \Omega^\infty\Sigma^\infty\Omega^\infty\Sigma^\infty E_0 \\
& & = \qquad\qquad & & \big\uparrow \alpha \\
& & & \Omega^\infty\Sigma^\infty E_0 &
\end{array}$$

where $\beta = \Omega^\infty\beta^\infty_{\Sigma^\infty E_0}$, $\alpha = \alpha^\infty_{\Omega^\infty\Sigma^\infty E_0}$, and $\varphi = \Omega^\infty\beta^\infty_{\Sigma^t L}$. Chasing the element $\bar{\iota}_{n+t} \in H^{n+t}(L_t)$ and using (4.2.5), we have

$$\begin{array}{ccc}
\ell_{t+n} \longleftarrow\!\shortmid\; \bar{\iota}_{t+n} & \xmapsto{\;\bar{\psi}\;} & \xi^n_{11}\otimes\ell_n \\
\big\downarrow \qquad \big\downarrow & & \big\downarrow \\
y_0 \longleftarrow\!\shortmid\; y'_0 \xmapsto{\;j_0{}^*\;} & & \xi^n_{11}\otimes 1\otimes x_0 + 1\otimes\xi^n_{11}\otimes x_0 \\
= \qquad\qquad\searrow & & \big\downarrow \\
& \xi^n_{11}\otimes x_0 &
\end{array}$$

so the result follows.

EXAMPLE 4.2.6. Returning again to the case of the integral Eilenberg-MacLane spectrum, we see that Theorem 4.2.1 forces the spectral sequence to collapse. Thus we have recovered the Steenrod algebra (additively) using "only" the structure of $H_*(\Omega^\infty\Sigma^\infty X)$ and the existence of a map Sq^{n+1} satisfying (4.2.5).

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