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Mark Mahowald's work on the homotopy groups of spheres

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In this paper we attempt to survey some of the ideas Mark Mahowald has contributed to the study of the homotopy of spheres. Of course, this represents just a portion of Mahowald's work; some other aspects are described elsewhere in this volume. Even within the restricted area of the homotopy of spheres, this survey can only touch on some of Mahowald's most seminal contributions, and will leave aside many of his ideas on the subject. On the other hand we will try to set the stage upon which Mahowald has acted, so we give brief reviews of certain parts of homotopy theory in Sections 1 and 4. This includes the Image of J, the EHP-sequence, and the Adams spectral sequence. Of course we will not attempt an exhaustive survey of the relevant history of homotopy theory; for more information, the reader may look at G. W. Whitehead's "Fifty years of homotopy theory" [**88**] or at Chapter One of the second author's book [**76**].

We will base our account on a discussion of three of Mahowald's most influential papers: *The Metastable Homotopy of* S^n (1967), *A new infinite family in* ${}_2\pi_*^S$ (1977), and *The Image of J in the EHP sequence* (1982).

One of Mahowald's jokes is that in his world there are only two primes: 2, and the "infinite prime." We will always work localized at 2, unless obviously otherwise.

Both authors are happy to have this occasion to thank Mark for the many exciting and fruitful interactions we have had with him.

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1. The Context, I

Understanding of the homotopy groups of spheres in 1967 could be gathered under three rather separate heads: the Image of J, the EHP sequence, and the Adams spectral sequence.

1.1. Bott periodicity and the J-homomorphism. Raoul Bott's proof [12] of the periodicity theorem was published in 1959. This is a computation of the homotopy groups of the classical groups, in a range of dimensions which increases linearly with the rank. It can be phrased so as to give all the homotopy groups of the "stable group," which is the evident union. This represented the first time all the homotopy groups of a space with infinitely many nontrivial k-invariants had been computed; and it is still the starting point for most such knowledge. For example, the homotopy of the stable orthogonal group $O = \bigcup O(n)$ is of period 8 and is given by the following table, which may be sung (from the bottom up) to the tune of a well-known lullaby [26].

i mod 8	$\pi_i O$
7	$\mathbf{Z}$
6	0
5	0
4	0
3	$\mathbf{Z}$
2	0
1	Z_2
0	Z_2

Generators for these groups may be constructed explicitly, for example using Clifford algebras.

The relevance of these groups to the homotopy groups of spheres was well-known from the construction of the "J-homomorphism" by G. W. Whitehead [87] in 1942. Since $O(n)$ acts by proper maps on R^n, its action extends to S^n viewed as the one-point compactification of R^n. This action has the point at infinity as a fixed-point, and so we get an inclusion of $O(n)$ into the monoid $F(n)$ of all pointed self-equivalences of S^n. $F(n)$ embeds into $\Omega^n S^n$ as the components of degree ± 1, but we desire a *pointed* map (with the constant map as basepoint of $\Omega^n S^n$) and so instead we map $\alpha \in O(n)$ to the loop-difference $\alpha - 1 \in \Omega^n S^n$. These maps are compatible up to homotopy as n increases, and in the limit they give a map $j : O \longrightarrow QS^0$, which in homotopy induces the *J-homomorphism*

$$J : \pi_i O \longrightarrow \pi_i^S.$$

Great geometric interest attaches to the image of this map; in terms of framed bordism, for example, the image is the set of elements which can be represented by framed spheres. It is very unusual that one has at hand explicitly constructed homotopy classes, defined without the indeterminacy usually associated with a spectral sequence. Despite the tremendous importance of these classes, they seem to have no standard names. Following Mahowald,[1] we will label them so that a specific generator of the jth nonzero homotopy group in O maps to the element β_j. The first few of these elements are the well-known *Hopf maps*: $\beta_1 = \eta$, $\beta_2 = \nu$, $\beta_3 = \sigma$.

In 1966 Adams's *On the groups $J(X)$—IV* [4] was published. This paper consolidated and organized a number of ideas, in part due independently to M. G. Barratt [6] and H. Toda [86], which were to become increasingly important in later years. In it he determined the order of the image of J up to a factor of 2 in dimensions of the form $8k-1$. The remaining factor of 2 led Adams to propose the celebrated Adams Conjecture. We will return to these ideas in more modern form in §4.

Adams's method was also important. He constructed a certain KO-theory analogue of the Hopf invariant, replacing Steenrod operations by Adams operations, and housed it in a suitable Ext^1 group. This is the e-invariant, and it detects the image of J. A byproduct of this method was that (if the Adams conjecture holds) the image of J is in fact a direct summand in π_*^S. One may also consider the "zeroth-order" KO-theory invariant, assigning to a map its induced map in KO-theory. Adams showed that this detects elements as well, in dimensions congruent to 1 and 2 mod 8. These are the μ and $\eta\mu$ elements, and Adams constructed them by means of Toda brackets. He also reformulated these constructions in terms of iterates of a K-theory self-equivalence of a Moore space; and this idea of periodic families was to become one of Mahowald's principal interests.

1.2. Toda's suspension-theory. In 1962, Hirosi Toda's book *Composition Methods in Homotopy Groups of Spheres* [84] appeared. This book is based upon the "EHP-sequence" of I. M. James [38], which Toda had extended in suitable form to odd primes. James had obtained this sequence by constructing a map, the "Hopf invariant" $H : \Omega S^n \to \Omega S^{2n-1}$, whose homotopy fiber is (localized at 2) S^{n-1} included into ΩS^n by the suspension map. The long exact sequence in homotopy has the form

$$\cdots \xrightarrow{H} \pi_{i+1}S^{2n+1} \xrightarrow{P} \pi_{i-1}S^n \xrightarrow{E} \pi_i S^{n+1} \xrightarrow{H} \pi_i S^{2n+1} \xrightarrow{P} \cdots.$$

Toda wrote Δ for the map P, which Mahowald always calls the "Whitehead product"; the precise relationship with the Whitehead product is ([87], p. 548) that for $\gamma \in \pi_{i-n-1}S^{n-1}$,

$$P(E^{n+2}\gamma) = [\iota_n, E\gamma].$$

In particular, $P(\iota_{2n+1})$ is the *Whitehead square*

$$w_n = [\iota_n, \iota_n] \in \pi_{2n-1}S^n.$$

[1]This is the convention used in [47] and in [10], but in [50] the numbering differs by 1, so that there $\beta_1 = \nu$, etc.

Toda explored this sequence as an inductive mechanism for computing unstable homotopy groups of spheres. Given a nonzero element $\alpha \in \pi_i S^{n+1}$ one may inquire after its "EHP history":

- What sphere was α born on, and what is the Hopf invariant γ of a maximal desuspension?
- What sphere will α die on (or is it immortal, that is, stable), and what element β has $P\beta$ equal to the maximal nonzero suspension?

The class γ is the "birth certificate" of α, and β is its "death certificate."

The *k-stem* consists of the groups $\pi_{k+n}S^n$ together with the suspension maps between them. It is also convenient to indicate the Hopf invariant of an element when it is nonzero, and a class β such that $P\beta = \alpha$ if $E\alpha = 0$.

The EHP-sequence, together with the connectivity of S^{2n+1}, implies Freudenthal's theorem that $E : \pi_{k+n}S^n \longrightarrow \pi_{k+n+1}S^{n+1}$ is an isomorphism for $n > k+1$. This is the *stable range* of the k-stem. The EHP sequence also shows that when $n > \frac{1}{3}(k+1)$, the obstructions to $E : \pi_{k+n}S^n \longrightarrow \pi_{k+n+1}S^{n+1}$ being an isomorphism lie in the stable range; this is called the *metastable range* of the k-stem.

In his book Toda computed the k-stem for $k \leq 19$, with some ambiguity about the behavior of P and H. His method involved the construction of elements with specified Hopf invariant by means of compositions and Toda brackets, or by means of an extension of the EHP sequence in the metastable range, in which the suspension map E is replaced by the iterated suspension. In particular when E is replaced by the *stabilization* map

$$\pi_k(X) \longrightarrow \varinjlim_i \pi_{k+i}(\Sigma^i X) = \pi_k^S(X)$$

then one has the *Toda stabilization sequence*: for $k < 3n-3$, there is an exact sequence

$$\cdots \longrightarrow \pi_{k+1}^S P_n \xrightarrow{T} \pi_{n+k}S^n \xrightarrow{E^\infty} \pi_k^S \xrightarrow{I} \pi_k^S P_n \longrightarrow \cdots.$$

Here π_*^S is the stable homotopy ring $\pi_*^S(S^0)$ and P_n is the *stunted projective space*

$$P_n = RP^\infty / RP^{n-1}.$$

The map T is a form of the Whitehead product map P, but we refrain from using this letter yet again. In modern terms, this exact sequence arises from identifying the homotopy fiber of $\Omega^n S^n \longrightarrow QS^0$ with QP_n through a range of dimensions. It is related to the EHP sequence in the metastable range by the commutative ladder ([47], p. 51)

$$\begin{tikzcd}
\cdots & \pi_{k+1}^S S^n \arrow[r] & \pi_{k+1}^S P_n \arrow[r] \arrow[d, "T"] & \pi_{k+1}^S P_{n+1} \arrow[r] \arrow[d, "T"] & \pi_k^S S^n & \cdots \\
\cdots & \pi_{n+k+2}S^{2n+1} \arrow[r, "P"] \arrow[u, "\cong"] & \pi_{n+k}S^n \arrow[r, "E"] & \pi_{n+k+1}S^{n+1} \arrow[r, "H"] & \pi_{n+k+1}S^{2n+1} \arrow[u, "\cong"] & \cdots
\end{tikzcd}$$

in which the top sequence is associated to the evident cofibration sequence $S^n \longrightarrow P_n \longrightarrow P_{n+1}$.

1.3. The Adams spectral sequence. In 1958 Frank Adams [**1**] revolutionized the way stable homotopy groups were studied, with his introduction of the Adams spectral sequence. This spectral sequence effectively broke the problem of computing the p-primary part of these groups into three parts: First, one constructs an estimate of the stable homotopy which depends functorially on the action of Steenrod operations on the mod p cohomology—this is the E_2-term in the spectral sequence. It is expressed homologically in terms of the algebra $\mathcal{A}$ of all Steenrod operations:

$$E_2^{s,t}(X) = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X), F_p).$$

This E_2-term is an upper bound, and the second step is to cut it down to the correct size (via differentials in the spectral sequence). Finally, one "solves the extension problems," passing from the associated graded group back to the actual stable homotopy, with as much structure as desired.

This spectral sequence opened the door to a vast computational effort. The theses of J. P. May [**65**] (1964, under J. C. Moore) and of M. C. Tangora [**82**] (1966, under Mahowald) resulted in extensive computations of the E_2-term and considerable information about higher differentials. Work by Mahowald and Tangora [**59**] contemporaneous with the Memoir resulted in a rather complete picture of the Adams spectral sequence through the 45-stem. A few oversights were corrected by Barratt, R. J. Milgram [**67**], Barratt, Mahowald and Tangora [**11**], G. W. Whitehead, and (in 1983) by R. Bruner [**22**]. One of the great resources Mahowald quickly came to be able to draw on was an intimate familiarity with this large family of homotopy classes—knowing them by their first name, so to speak: their position in the Adams spectral sequence, their relationships with each other, and, as we shall presently describe, their unstable behavior.

The Adams spectral sequence offered more than such stem-by-stem calculations, however; and indeed, Adams' motive for constructing it, and his first and most celebrated application of it, was the resolution in all dimensions of the Hopf invariant one problem ([**2**], 1960). The Adams spectral sequence gave a decreasing filtration of stable homotopy groups, providing some measure of the complexity of a class. In the stable homotopy of a sphere, only odd multiples of the fundamental class are not in filtration 1. Adams interpreted the Hopf invariant as picking out F^1/F^2. At E_2^1 there is for each i a nonzero class h_i in the (2^i-1)-stem, corresponding the the indecomposable Steenrod operation Sq^{2^i}. The element h_0 survives to the element 2ι; the next three survive to the Hopf maps η, ν, and σ; and Adams showed there were nonzero differentials on all the rest. Besides the well-known interpretation involving division algebras, this proved that the only parallelizable spheres are S^1, S^3 and S^7.

Mahowald has credited this paper with arousing his interest in Homotopy Theory: it served as his textbook on the subject.

2. The Memoir, *The Metastable Homotopy of S^n*

In his Introduction to this remarkable paper [**47**] Mahowald says, "Our object is to bring to Toda's theorem [the stabilization sequence described above] the power of stable methods developed by Adams," and later in the Introduction he describes his results on the EHP history of a family of classes which he conjectured (correctly, as it turned out) to be the Image of J. Mahowald's aim was to unite the three principal strands of contemporary research in Homotopy Theory. He perceived in Toda's iterated-suspension EHP sequence a means of trapping the metastable homotopy groups of a sphere between two stable groups, each of which could be attacked using the Adams spectral sequence.

Before we examine this work in detail, it is worthwhile pointing out what Mahowald did *not* use.

- The mod p lower central series spectral sequence of [**15, 78**], with its attendant Λ-algebra model, appeared in 1966, and the Massey-Peterson theory in 1967 [**64**], but neither were at this point part of Mahowald's methodology. The Adams spectral sequence was still for him an exclusively stable tool.
- The splitting theorems of Barratt and P. J. Eccles [**7**] and of V. P. Snaith [**81**] did not appear until 1974.
- The Adams conjecture was not resolved until 1970 [**80, 75**]. (In fact in 1970 Mahowald himself [**48**] offered a line of argument resolving the ambiguous factor of 2 in the Image of J. The details of a variant of this argument were filled in by Mahowald and D. M. Davis in [**34**].)

The first order of business in the Memoir was the construction of a map

$$I : E_r^{s,t}(S^0) \longrightarrow E_r^{s-1,t}(P_n)$$

of spectral sequences (in the range $t - s \leq 2n - 2$) compatible in the abutment with Toda's "Hopf invariant" I. This map is of course not induced by a stable map of spaces. Constructions of this sort are today a standard part of the technique. Mahowald's construction was generalized by Milgram [**68**]. For a modern setting for results of this sort, see [**37**].

This spectral version of I allowed Mahowald to do computations using the Adams spectral sequence. The next problem was to discover means of computing part of $E_2(P_n)$ from the known structure of $E_2(S^0)$. For this Mahowald devised his "pre-spectral sequence," associated to the filtered complex arising from the skeleton filtration of P_n. This is the Adams E_2 version of the Atiyah-Hirzebruch spectral sequence for $\pi_*^S(P_n)$. A long central section carries out this computation, and displays the result through dimension 29 in sixteen impressively concise tables. Mahowald asserts that "The use of a large computer was important in this work," and this seems to be the first such mechanization of a problem in homological algebra—certainly the first in algebraic topology. Mahowald has maintained an active interest in computer-aided computation; for a recent effort see [30].

From this it was an easy step to use Toda's stabilization sequence (in its spectral form) and its relationship with the EHP sequence to obtain the metastable homotopy of spheres through about π_{40}, and good information on the entire k-stem for $k \leq 28$.

Another contribution of the Memoir is the construction of classes in the stable homotopy ring which behave as the Adams conjecture predicted that the Image of J should behave. Mahowald produced such elements in a novel way, via a stable map $\lambda : RP^\infty \longrightarrow S^0$ obtained using James periodicity . In modern terms, he wrote down the transfer associated to the universal double cover over RP^∞. This is a stable map $P_0^\infty \longrightarrow S^0$ (where P_0^∞ denotes the suspension spectrum of RP^∞ with a disjoint basepoint adjoined), and he composed this with the 0-connected covering map $P^\infty \longrightarrow P_0^\infty$. To our knowledge this represents the first published occurrence of this map. Mahowald was careful to distinguish λ from another map from RP^∞, namely the J-homomorphism composed with the usual inclusion of RP^∞ into SO. He conjectured that λ is surjective to the 2-component in positive stems; this is the Kahn-Priddy theorem ([40], 1972).

The virtue of replacing O (which has simple homotopy) by P^∞ (which has complicated stable homotopy) is that one can describe elements of $\pi_*^S P^\infty$, and their images in π_*^S, by means of the (stable) Adams spectral sequence—and, as indicated above, this computation is the heart of the Memoir. Mahowald proved that certain classes in high filtration survive in the Adams spectral sequence for the stunted projective space P_n, by an ingenious inductive argument. He was then able to compute the e-invariant of the image of these classes under λ, to see that they remain essential in π_*^S. This produced summands in π_*^S of order equal to the order of the image of J specified by the Adams conjecture, together with elements resembling the other elements constructed by Adams. Mahowald conjectured that these elements were in fact in the image of J, and this was later established in [48, 34].

In any case, the result was that in the Adams spectral sequence for a sphere, certain elements along the Adams vanishing line survive. The pattern is displayed in Figure 1. (The case $k = 1$ is exceptional and we leave it aside.)

In this portion of the Adams spectral sequence "chart" for the sphere, vertical lines denote multiplication by h_0 and diagonal lines represent multiplication by h_1. We have denoted classes in E_2 by the names of elements to which they survive in homotopy. Thus for example β_{4k-1} is a class in the $(8k-1)$-stem of order $2^{\nu(k)+4}$, which is the 2-primary part of $16k$. There is also an "exotic extension," $\eta\beta_{4k-1} = \beta_{4k}$. The elements of E_2 labeled α_s survive to well-defined classes in stable homotopy, since they are in the smallest Adams filtration subgroup of their stable stem which contains any nontrivial 2-torsion.

Toda's stabilization sequence, in its spectral form, allowed Mahowald now to compute many of the Whitehead products of these elements. This work extended [46], which dealt with the Hopf maps. There are two kinds of results at this point.

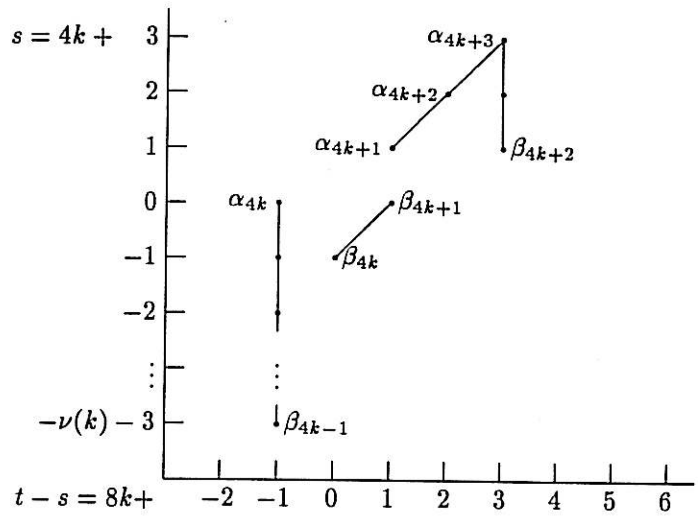

FIGURE 1. Mahowald's "Im J"

First, on most spheres, the elements in Im J have as Whitehead product (with the fundamental class) other elements in Im J (possibly the zero element!). Second, there is the beautiful theorem of Toda [**85**] to the effect that the Hopf invariant of the Radon-Hurwitz desuspension of the Whitehead square is an appropriate generator of the image of J. That generator therefore has trivial Whitehead product. Mahowald was able to verify this for his elements. His approach to this, via the Adams spectral sequence, led him to make the following conjecture, which describes the behavior of most of the remaining elements. We assume we are in the metastable range.

Conjecture: Assume $n + |\beta_j| \equiv -2(\bmod 2^{j+1})$.

- If $n + |\beta_j| \neq 2^{j+1} - 2$, $[\iota_n, \beta_j] \neq 0$.
- Otherwise, $[\iota_n, \beta_j] = 0$ if and only if h_j^2 is a permanent cycle in the Adams spectral sequence. Moreover, if $\theta_j \in \pi^S_{2^{j+1}-2}$ is represented by h_j^2, then $I(\theta_j) = i_*(\beta_j)$ in the diagram

$$\pi^S_{2^{j+1}-2} \xrightarrow{I} \pi^S_{2^{j+1}-2} P_n \xleftarrow{i_*} \pi^S_{2^{j+1}-2} S^n,$$

where $i : S^n \longrightarrow P_n$ denotes the inclusion of the bottom cell.

The way in which this conjecture fits into the EHP sequence will become clearer in §5.

After Browder's work [**18**], we would paraphrase the second part of this conjecture by saying that the stable Hopf invariant of a class of Kervaire invariant one is a generator of the image of J. A passion for the Kervaire invariant [**49, 8**] is another of Mahowald's distinguishing characteristics; and one may trace the origin of this fixation to this point. Notice that he remained neutral about whether such elements θ_j exist or not.

In conversation Mahowald will often refer to this paper as a reference for the source of some idea: "It's all in the Memoir." When we turn to his later work in the sections below, this will be amply borne out: many of the ideas are refinements or "recasts" of ones discussed above. As a final example of this, we note the rather obscure definition (on p. 24) of the "root" of a stable homotopy class α, written $\sqrt{\alpha}$. This was the original mention of the Mahowald invariant, which we discuss further in §5.2.

3. *An infinite family in* ${}_2\pi_*^S$

The solution to the Hopf invariant one problem showed that there are only four nonzero permanent cycles in the "one-line" $\mathrm{Ext}^1_{\mathcal{A}}(F_2, F_2)$ of the Adams spectral sequence for the sphere. In homological dimension 2, the most interesting classes are the h_j^2's; they represent framed manifolds of Kervaire invariant 1. The construction of such classes seems a very case-by-case affair [9], and the Hopf invariant one story would seem to support the belief that only finitely many h_j^2's survive. More strongly, there was the "Doomsday Conjecture": In every homological degree there are only finitely many survivors. This conjecture had its adherents in the early 1970's, but the paper we review in this section demolished it.

Work of Mahowald and Tangora [59] put an upper bound on the survivors in $\mathrm{Ext}^2_{\mathcal{A}}(F_2, F_2)$; besides the h_j^2's, there are h_0h_2, h_0h_3, h_2h_4 (which survive respectively to 2ν, 2σ, and a class called ν^* by Toda), and the classes h_1h_j for $j \geq 3$. These were known to survive for $i \leq 5$ by explicit computation. (h_1h_4 survives to a class named η^* by Toda.) Mahowald's theorem is that they all survive, to classes $\eta_j \in \pi^S_{2^j}$.

The class η_j is constructed as a composite

$$\begin{tikzcd} S^{2^j} \arrow[r, "f"] \arrow[dr, "\eta"'] & X_j \arrow[r, "g"] \arrow[d, "p"] & S^0 \\ & S^{2^j-1} & \end{tikzcd}$$

where X_j is a suitably chosen finite spectrum. The map f gives us the h_1, and Sq^{2^j} is nonzero on the bottom class of the cofiber of g, so the composite will be represented in Ext by h_1h_j. The two maps are constructed by entirely different procedures.

The construction of g begins by regarding the Hopf class σ as a map $S^7 \to SG$ and using the Boardman-Vogt infinite loop space structure on SG—actually only the double loop space structure—to extend to a map

$$\Omega^2 S^9 \longrightarrow SG \longrightarrow QS^0,$$

where the second factor is the shift map used in the construction of the J-homomorphism. This is adjoint to a stable map

$$\hat{\sigma} : \Omega^2 S^9 \longrightarrow S^0.$$

Mahowald used the iterated loop structure to blow the low-dimensional class σ up to high-dimensional homotopy on the cells of the infinite complex $\Omega^2 S^9$.

The relevant part of this homotopy is picked out using the Snaith splitting of $\Omega^2 S^9$. This splitting has the form

$$\Omega^2 S^9 \simeq \bigvee_{k \geq 0} \Sigma^{7k} B_k,$$

where B_k is a certain finite spectrum with bottom cell in degree 0 and top cell in degree $k - \alpha(k)$. Mahowald could compute the cohomology of B_k, and on the basis of this and other evidence he conjectured that it was precisely the spectrum B_k constructed by E. H. Brown and S. Gitler [**19**]. Those authors used an elaborate Postnikov system approach in their construction, and had in mind entirely different applications of the B_k's: they play a key role in the Brown-Peterson approach to the immersion conjecture, which was finally implemented by R. L. Cohen.

Mahowald's guess was verified soon after by Brown and F. P. Peterson [**20**], and although Mahowald found ways to avoid it we will use this fact as a matter of convenience.

Let us write $k = 2^j$. Then Mahowald took for X_j the spectrum $\Sigma^{7k} B_k$; for g, the restriction of $\hat{\sigma}$ to that factor of $\Omega^2 S^9$; and for p, the collapse map to the top cell.

Mahowald then analyzed the Adams spectral sequence for B_k, to find the class $f \in \pi_k B_k = \pi_1(\Sigma^{1-k} B_k)$. The E_2-term could be computed in this stem by the Λ-algebra machinery developed by Brown and Gitler in their construction of the spectrum. Keeping the notation $k = 2^{j-3}$, one finds for each $j \geq 3$ a class

$$u_j \in \mathrm{Ext}_{\mathcal{A}}^{1,2}(H^*(\Sigma^{1-k} B_k), F_2)$$

lifting $h_1 \in \mathrm{Ext}_{\mathcal{A}}^{1,2}(F_2, F_2)$. The problem is that there are also lots of classes in the zero-stem ready to accept differentials starting on u_j.

Mahowald's ingenious solution to this uses the fact that the B_k's fit into an inverse system

$$\cdots \xrightarrow{q} \Sigma^{-4} B_8 \xrightarrow{q} \Sigma^{-2} B_4 \xrightarrow{q} \Sigma^{-1} B_2 \xrightarrow{q} B_1$$

in such a way that the u_j's match up. Again using the Λ algebra machinery he was able to show that in the zero-stem, each of these maps induces the zero map in Ext. This implies that these maps also induce the zero map in $E_r^{s,s}$ for all r and s, and this forces all the u_j's to be permanent cycles: $d_2 u_j = d_2 q u_{j+1} = q d_2 u_{j+1} = 0$, so u_j survives to $E_3(B_k)$. In this computation j was arbitrary, so u_{j+1} survives to E_3 as well and $u_j \in E_3(B_k)$ is hit by $u_{j+1} \in E_3(B_{2k})$. We can then run the same argument over again to show that each u_j survives to $E_4(B_k)$! By induction, they are all permanent cycles.

This approach to constructing homotopy classes has had wide-spread fallout in homotopy theory. The Brown-Gitler spectra and this sort of application of them was extended to odd primes by R. L. Cohen [27] in his thesis. Other classes have been constructed by similar methods by Mahowald [52], Bruner [21], and W. H. Lin [45].

More generally, the inverse system used above, with its nice mapping properties, was used by G. Carlsson [23] in his work on the Segal conjecture for elementary abelian 2-groups. He showed that $\overline{H}^*RP^\infty$ splits off of the direct limit of the cohomologies of the Brown-Gitler spectra. The first author formulated Carlsson's work as saying that H^*RP^∞ is an injective *unstable* $\mathcal{A}$-module (actually the work was done in homology), and used it in his proof of the Sullivan conjecture [71].

4. The Context, II

In this section we will briefly review some of the developments in the study of homotopy groups of spheres in the years between 1967 and 1982. We update the three areas discussed in §1.

4.1. The Adams Conjecture and the Space J. We briefly describe the standard picture (see [66]) associated with a solution of the Adams conjecture. Recall that we always work locally at 2.

The Adams operation ψ^3 is a map $BO \to BO$. In homotopy it behaves as follows: In dimension $4k$ it multiplies by 3^{2k}, and in other dimensions it is the identity. Thus (using the H-space structure on BO to form the difference), $\psi^3 - 1$ lifts to a map

$$\varphi : BO \longrightarrow BSpin.$$

The fiber of this map is a space J, whose homotopy can easily be computed using the fact that $\nu(3^{2k} - 1) = 3 + \nu(k)$ (where $2^{\nu(k)}$ is the largest power of 2 dividing k). The generators of $\pi_* Spin$ map by the boundary map to elements $\beta_j \in \pi_* J$. There are in addition distinguished elements of order 2 in $\pi_* J$, denoted α_j. α_{4k-1} and α_{4k} denote respectively the elements of order 2 in $\pi_{8k-5}J$ and $\pi_{8k-1}J$; these are multiples of β_{4k-2} and β_{4k-1}, respectively. α_{4k+1} and α_{4k+2} map to generators of $\pi_{8k+1}BO$ and $\pi_{8k+2}BO$ respectively. The low dimensional behavior of this fibration sequence is set out in Figure 2.

The Adams conjecture asserts that the composite of $\psi^3 - 1$ with the map $Bi : BO \to BG$ is trivial, where Bi is induced by the inclusion of O into the composition-monoid G of stable self-maps of spheres. A solution to the Adams conjecture is a null-homotopy; and this gives us the following diagram:

$$\begin{tikzcd}
Spin \arrow[r] \arrow[ddr] & J \arrow[r] \arrow[d] & BO \arrow[r, "\varphi"] \arrow[d] & BSpin \arrow[d, "k"] \\
 & SG \arrow[r] \arrow[d, "*[-1]"] & * \arrow[r] & BSG \\
 & Q_0S^0 & &
\end{tikzcd}$$

$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$
11	α_7			Z_8	β_6	0	0
10	α_6	Z_2				Z_2	Z_2
9	α_5	Z_2	$\oplus$	Z_2	β_5	Z_2	Z_2
8				Z_2	β_4	Z	Z
7	α_4			Z_{16}	β_3	0	0
6			0			0	0
5			0			0	0
4			0			Z	Z
3	α_3			Z_8	β_2	0	0
2	α_2	Z_2				Z_2	0
1	α_1	Z_2				Z_2	0
n			$\pi_n J$			$\pi_n BO$	$\pi_n BSpin$

FIGURE 2. The Homotopy Long Exact Sequence for $\pi_* J$

in which k is the composite $BSpin \to BSO \to BSG$. The diagonal composite is the J-homomorphism composed with the covering map from $Spin$.

This puts an upper bound on the size of the image of J: the J-homomorphism (restricted to $Spin$) factors through $\pi_* J$. To obtain a lower bound, one works on the level of spectra. The map φ is the restriction to the zero-component of an infinite-loop map $Z \times BO \to BSpin$, which is associated to a map of spectra. The spectrum associated to $Z \times BO$ represents connective real K-theory, and Mahowald denotes it by bo. Its three-connected cover is $bo\langle 4\rangle$, with zero-space $BSpin$. (Mahowald often writes $\Sigma^4 bsp$ for this spectrum.) Thus

$$\varphi : bo \longrightarrow bo\langle 4\rangle.$$

Write j for the fiber of this map; it is a spectrum with zero-space $Z \times J$. The unit map $S^0 \to bo$ clearly lifts uniquely to a map

$$e : S^0 \longrightarrow j,$$

which induces on the zero-components of zero-spaces a map $e : Q_0 S^0 \to J$. This is a geometric version of the (real) e-invariant. The beautiful fact is that the composite

$$J \longrightarrow Q_0 S^0 \xrightarrow{e} J$$

is a homotopy equivalence. This gives a product splitting

$$Q_0 S^0 = J \times C$$

in which C is a space called "Coker J". In particular, the homotopy of J sits in the stable homotopy groups of spheres as a direct summand. It is the "linear" part of the stable homotopy ring, and we will denote this subgroup by $\pi_* J$. C is 5-connected. η is in $\pi_* J$, though it declines "for personal reasons" (to quote

Mahowald) to be in the image of the version of the J-homomorphism adopted here, since $Spin$ is 2-connected.

In [48] Mahowald announced a resolution of the conjecture made in the Memoir, identifying the high Adams filtration classes he studies there (Figure 1 above) with the summand $\pi_* J$. For more detail, see [34].

4.2. The EHP spectral sequence. This is merely a more sophisticated point of view on the classical EHP sequence.

The EHP sequence is an exact couple, so there is an associated spectral sequence. We can regard this as the homotopy spectral sequence associated to a filtration of $QS^0 = \bigcup \Omega^n S^n$:

$$
\begin{tikzcd}
* \arrow[r] & \Omega S^1 \arrow[r] \arrow[d, "H"] & \Omega^2 S^2 \arrow[r] \arrow[d, "H"] & \Omega^3 S^3 \arrow[r] \arrow[d, "H"] & \cdots \arrow[r] & QS^0 \\
& \Omega S^1 & \Omega^2 S^3 & \Omega^3 S^5 & &
\end{tikzcd}
$$

Let us agree that $F_n QS^0 = \Omega^{n-1} S^{n-1}$. Then if we use the usual Serre indexing for the spectral sequence,

$$E^1_{s,t} = \pi_{s+t}(\Omega^{s+1} S^{2s+1}) = \pi_{2s+1+t} S^{2s+1}$$

—that is, the t-stem on the $(2s+1)$-sphere. The spectral sequence converges to $\pi_*(QS^0) = \pi_*^S$. If we arrest the filtration at $\Omega^n S^n$, we get a spectral sequence converging instead to $\pi_* S^n$, whose differentials are simply the differentials in this range of the full spectral sequence; this is the "restriction principle" for spectral sequences. This offers the prospect of an inductive computation of the homotopy groups of spheres; notice that the portion of the E^1-term contributing to the n-stem—where $s + t = n$—involves the homotopy of (odd) spheres in stems less than n.

From the general theory of spectral sequences we can make the following comments about the EHP spectral sequence:

- Elements of the abutment are represented at E^1 by the Hopf invariant of a maximal desuspension.
- Each differential represents a Whitehead product (with the fundamental class), and gives the EHP history of an unstable homotopy class.

In more detail, to compute a differential on a class whose representative at E^1 is $x \in E^1_{s,t} = \pi_{2s+1+t} S^{2s+1}$, one applies P to get to $\pi_{2s-1+t} S^s$, pulls this class back as far as possible under suspension—say to $z \in \pi_{2s-r+t} S^{s-r+1}$—and then applies the Hopf invariant: $Hz \in \pi_{2(s-r)+t+r} S^{2(s-r)+1} = E^1_{s-r,t+r-1}$:

$$
\begin{tikzcd}
z \in & \pi_{2s-r+t} S^{s-r+1} \arrow[r, "E^{r-1}"] \arrow[d, "H"] & \pi_{2s-1+t} S^s & \ni Px \\
d^r z \in & \pi_{2(s-r)+t+r} S^{2(s-r)+1} & \pi_{2s+1+t} S^{2s+1} \arrow[u, "P"] & \ni x
\end{tikzcd}
$$

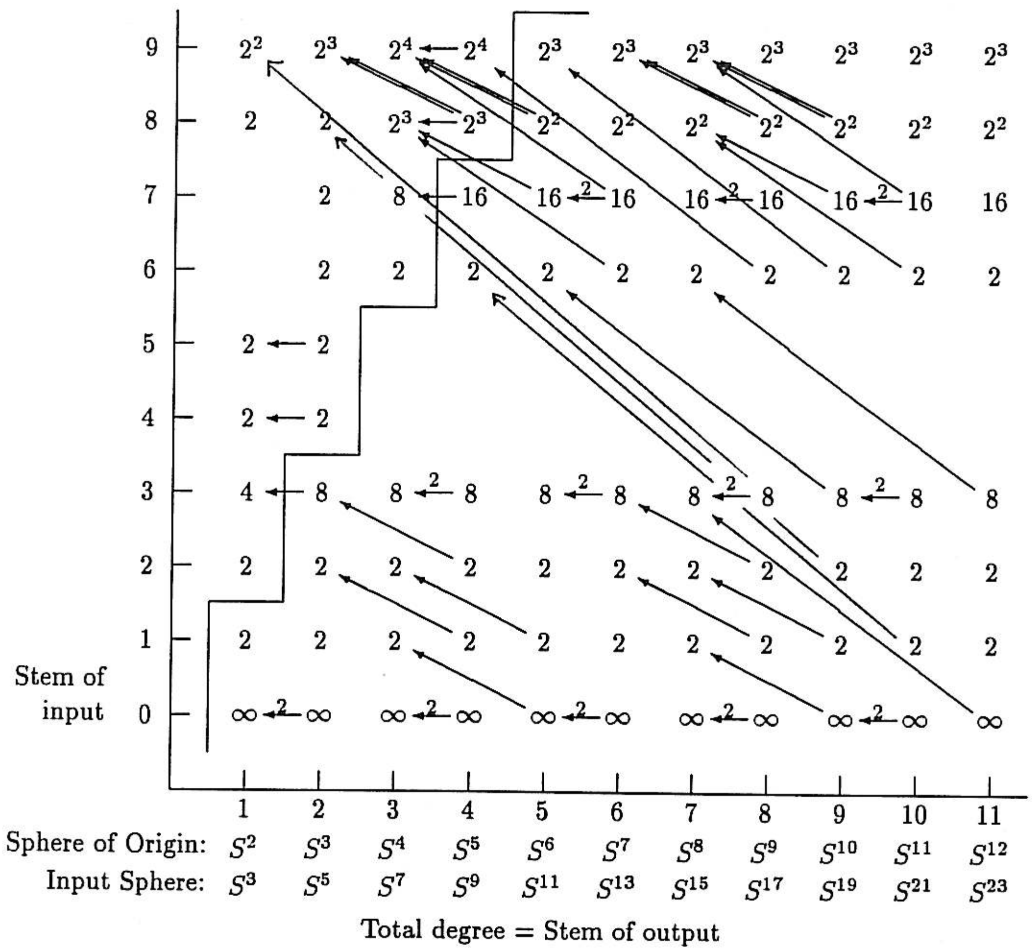

FIGURE 3. The EHP spectral sequence

Figure 3 displays a portion of the EHP spectral sequence. This is easily gleaned from Toda's book. We print it here because we will want to compare the EHP spectral sequence to other spectral sequences. We have used Toda's convention, writing 2^3 for example for the group $(Z/2)^3$. We indicate only the differentials whose source and target are both in the range displayed. A double arrow indicates a rank 2 map (between elementary abelian 2-groups); an arrow with a 2 over it indicates that the image has index 2. We have also indicated the edge of the metastable range, by a staircase.

The Whitehead square w_n is related to the issue of finding independent vector fields on a sphere by the fact that it is J composed with the clutching function $\tau : S^{n-1} \to SO(n)$ of the tangent bundle of S^n. Giving k independent vector fields is equivalent to giving a compression of τ through a map $\overline{\tau} : S^{n-1} \to SO(n-k)$. J composed with such a compression is a desuspension of w_n. When $n = 1, 3$ or 7, w_n desuspends all the way to the zero-sphere: it is null, and the sphere is parallelizable. That this never happens again is the content of the Hopf invariant one problem.

Radon and Hurwitz gave one such "linear" desuspension, which we will call $\overline{w}_n$. Toda **[85]** proved that the Hopf invariant of $\overline{w}_n$ is a generator $\beta_{\nu(n+1)}$ of the image of J. (Thus for the Radon-Hurwitz desuspension, $k = |\beta_{\nu(n+1)}|$. A check of the numbers shows that this Hopf invariant is in the stable range, so we don't have to worry about desuspending elements of Im J.)

This shows that the fundamental class $\iota_{n-1} \in E^1_{n-1,0}$ survives to E^{k+1} in the EHP spectral sequence, and it computes the differential $d^{k+1}\iota_{n-1}$; but it is not yet clear that the proposed target has not been hit by an earlier differential in the spectral sequence. This is in effect what Adams showed in his solution **[3]** of the vector field problem; this differential *is* nonzero.

An alternative and completely stable proof of the vector field problem, not using Adams operations *per se*, is given in **[55]** by Mahowald and Milgram.

4.3. The Generalized Adams Spectral Sequence. In 1967 S. P. Novikov published a long and intriguing paper **[74]** which initiated a study of homotopy theory "from the viewpoint of [complex] cobordism." He analyzed the spectral sequence obtained from the complex cobordism spectrum MU in a manner analogous to the construction of the Adams spectral sequence. Mahowald independently considered the case in which the ground spectrum is bo. Very simply, the stable Adams spectral sequence for the homotopy of X, based on the spectrum E with unit $S^0 \to E$, is the homotopy spectral sequence associated to the diagram

$$\begin{array}{ccccccc} X & \longleftarrow & \overline{E}\wedge X & \longleftarrow & \overline{E}\wedge\overline{E} & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ E\wedge X & & E\wedge\overline{E} & & E\wedge\overline{E}\wedge\overline{E} & & \end{array}$$

which is made by splicing and smashing the cofibration sequence

$$\overline{E} \longrightarrow S^0 \longrightarrow E$$

defining $\overline{E}$.

The Adams spectral sequence based on MU has had a profound impact on stable homotopy theory. Localized at a prime p, MU splits into a wedge of suspensions of the Brown-Peterson spectrum BP, with coefficient ring

$$BP_* = Z_{(p)}[v_1, v_2, \ldots], \quad |v_n| = 2(p^n - 1).$$

The generator v_1 encodes Bott periodicity, and the other generators provide analogous "higher periodicity" operators. See the second author's book **[76]** for more information about the generalized Adams spectral sequence and BP point of view.

Much of Mahowald's work has focused on v_1-periodic phenomena; but by the mid 1970's he was a strong advocate of the BP viewpoint, and has contributed much to our understanding of higher periodicity (see **[32]**, for example).

5. *The Image of J in the EHP-sequence*

This paper **[54]** represents the culmination of Mahowald's work continuing the line of inquiry started in the Memoir. The principal unstable results had been announced in **[50]**, and the stable methodology was set out in **[53]**. Our account of the Annals paper includes a partial description of that work, and is influenced also by later refinements and clarifications **[36, 43, 31, 60, 34, 61]**.

5.1. Stable v_1-periodic homotopy. Let M denote the Moore spectrum $S^0 \cup_2 e^1$. As described in §1.1, Adams had constructed a map $A : \Sigma^8 M \to M$ which induces an isomorphism in K-theory. One may formally invert A in the homotopy of M; the result is the homotopy of the mapping telescope of the system

$$M \xrightarrow{\Sigma^{-8}A} \Sigma^{-8} M \xrightarrow{\Sigma^{-16}A} \Sigma^{-16} M \longrightarrow \cdots$$

which we denote by $A^{-1}M$. The main stable result of this work is a complete computation of this "v_1-periodic homotopy" of the Moore spectrum. Following the set-up used in **[54]**, we will describe the answer for a different spectrum, namely

$$Y = M \wedge (S^0 \cup_\eta e^2).$$

There is now a K-theory isomorphism $v_1 : \Sigma^2 Y \to Y$ (a "fourth root of A"), and one of the principal results of the Annals paper is that the localized homotopy is given by

$$v_1^{-1}\pi_* Y = F_2[v_1^{\pm 1}] \otimes E[a]$$

where a is a class in dimension 1. A free $F_2[v_1]$-module of rank 2 can be found inside $\pi_* Y$ by direct construction; the force of the result is that any class differs from one of these by v_1-torsion.

This theorem was interpreted by A. K. Bousfield [13] as implying that the map $M \to A^{-1}M$ is the terminal K-theory equivalence out of M: $A^{-1}M$ is the *K-theory localization* of M. As a corollary he obtains an effective characterization of the K-local spectra in terms of their mod p homotopy groups: the Adams map A must act isomorphically on them for each prime p. These results form the cornerstone and guiding example for much contemporary work in stable homotopy theory.

The method of proof may be described roughly as follows. Mahowald considers the generalized Adams spectral sequence based upon the spectrum bo, converging to the homotopy (localized at 2) of a spectrum X. If X is the spectrum Y, the self-map v_1 acts on the spectral sequence. Since v_1 is detected by K-theory, it has filtration 0 and acts "horizontally" in the spectral sequence. Mahowald now hopes to compute the localization of the E_2-term and prove that the localized spectral sequence converges to the localized homotopy of Y. This convergence is not automatic; it may fail in either of two ways: there may be an infinite string of v_1-torsion summands in E_∞, in higher and higher filtration, which become linked together by extensions to produce a torsion-free summand in $\pi_*(v^{-1}Y)$; or there may be a torsion-free summand in E_2 which is eaten away bite by bite by infinitely many v_1-torsion-valued differentials, and which thus does not contribute to $\pi_*(v^{-1}Y)$ despite consisting entirely of permanent cycles in the localized spectral sequence.

Mahowald proves three things about the (unlocalized) bo-spectral sequence, which together prove convergence and give the computation. We state them in very weak form, so as not to become embroiled in detailed numerology. We state them for $X = Y$.

- Low-degree computation: $E_2^0 = F_2[v_1]$ and E_2^1 is free of rank one over $F_2[v_1]$.
- Vanishing line: $E_2^{s,t} = 0$ for $s > m(t-s)+b$, for certain numbers m and b. In fact, the slope m can be taken to be $1/5$ [31].
- Bounded torsion: In filtration 2 and larger, multiplication by v_1 raises filtration enough so that the vanishing line is eventually intersected.

The result clearly follows from these facts. The method used is to analyze the stages in the bo Adams resolution, and the maps between them, by means of the classical Adams spectral sequence. One must thus understand the homotopy type of smash-powers of bo. One of the main tools Mahowald relies on is a splitting theorem for $bo \wedge bo$, first announced in [48]; for proofs see [69] and [53].

At an odd prime it is possible [70] to analyze the classical Adams spectral sequence directly; there the map v_1 acts parallel with the vanishing line, so convergence of the localized spectral sequence is automatic. In the case of the mod p Moore spectrum there is one nontrivial differential, which can be computed by comparing the classical Adams spectral sequence with the Adams spectral sequence based on BP. The result is that the localized homotopy of a Moore spectrum is exterior on one generator over $F_p[v_1^{\pm 1}]$.

Recent work of the second author [77] extends this method to study the homotopy groups of mapping telescopes of v_2-self-maps. Surprisingly, it turns out that the potential failures of convergence mentioned above do in fact occur in that context.

Another approach to the computation of the localized homotopy of a Moore space, using the Adams conjecture, has been given by M. C. Crabb and K. Knapp [28].

5.2. The stable EHP-sequence and the Mahowald invariant. In the Annals paper Mahowald replaces the Toda suspension sequence with the "Snaith maps." In [81] Snaith had displayed a stable splitting of $\Omega^n\Sigma^n X$, for any connected space X. By projecting to the factors, one obtains maps to the zero-spaces of the factors. In the case $X = S^1$, the first of these factors is ΣP^n, where P^n denotes the real projective space (or its suspension spectrum). Looping this map, we obtain $s : \Omega^{n+1}S^{n+1} \to QP^n$. Later work by N. J. Kuhn [41] and by F. R. Cohen, L. R. Taylor, and May [25, 24], shows that these maps can be chosen to be compatible under suspension, so that the following diagram commutes:

$$\begin{array}{ccccc} \Omega^n S^n & \longrightarrow & \Omega^{n+1}S^{n+1} & \longrightarrow & \Omega^{n+1}S^{2n+1} \\ \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle E^\infty} \\ QP^{n-1} & \longrightarrow & QP^n & \longrightarrow & QS^n \end{array}$$

This gives a relation between the EHP sequence and the stable cell-structure of projective space in *all* dimensions, not just in the metastable range. In fact, one can assemble the homotopy long exact sequences of the bottom fibration sequences into an exact couple, whose associated spectral sequence is the Atiyah-Hirzebruch spectral sequence for the stable homotopy of $P = P^\infty$; so one obtains a map

$$s : \Big(EHP \text{ Spectral Sequence}\Big) \longrightarrow \begin{pmatrix} \text{Atiyah-Hirzebruch} \\ \text{spectral sequence for } \pi_*^S P \end{pmatrix}.$$

This map is more convenient than the Toda's Hopf invariant because it provides stable invariants for unstable classes even outside the metastable range.

Notice that in the metastable range, the map is an isomorphism at E^1. In fact, the map in E^1_s is the suspension map $\pi_* S^{2s+1} \to \pi_*^S$, and the target spectral sequence is sometimes called the *stable EHP spectral sequence.* It is a certain localization of the EHP spectral sequence.

We interject here some comments on another idea of Mahowald's (see [5]). Another way to construct the Atiyah-Hirzebruch spectral sequence for $P = P^\infty$ is via the sequence of collapse maps to stunted projective spaces:

$$P_+ = P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots.$$

This sequence may be continued backwards as well—

$$\cdots \longrightarrow P_{-2} \longrightarrow P_{-1} \longrightarrow P_0 \longrightarrow P_1 \longrightarrow \cdots$$

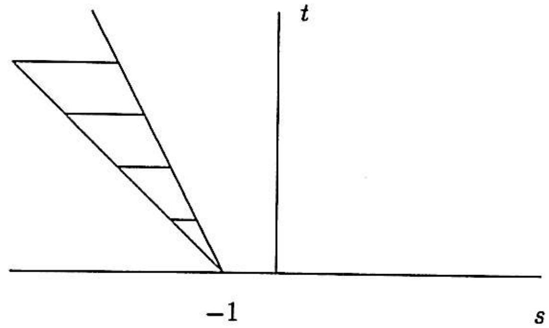

FIGURE 4. E^∞ of the $\pi_*^S P_{-\infty}$ spectral sequence

—where for any n we define P_n as the Thom spectrum of n times the Hopf bundle over RP^∞ (which is a virtual bundle for $n < 0$). This inverse system of spectra goes by the name, "$P_{-\infty}$." There results a spectral sequence occupying the upper half-plane. James periodicity shows that for each r, E_s^r is periodic in s, of period which increases with r; and that the periodicity isomorphisms are compatible as r changes. Thus the half-plane spectral sequence may be regarded as a localization of the first-quadrant spectral sequence.

This localized spectral sequence converges to the homotopy groups of the homotopy inverse limit of the tower of stunted projective spaces. Mahowald conjectured that this inverse limit is none other than the 2-adic completion of the spectrum S^{-1}. This is a strong form of a conjecture made later by Graeme Segal about the stable cohomotopy of the classifying space of an arbitrary finite group, and it was proved by W. H. Lin **[44]**.

This shows that the E^∞-term of the spectral sequence is concentrated on and above the line $s+t=-1$. A theorem of J. D. S. Jones **[39]** asserts that $E^\infty_{s,t} = 0$ for $2s+t > -1$. (Jones's result, and proof, were given earlier in different language by G. E. Bredon **[16]**,[2] and again later in **[72]**.) The E^∞-term thus has the form shown in Figure 4.

Any nonzero stable homotopy class $\alpha \in \pi_n^S(S^0{}^\wedge_2) = \pi^S_{n-1}(S^{-1}{}^\wedge_2)$ has a coset of representatives in $E^1_{st} = \pi_t^S$ in this spectral sequence, where $s+t = n-1$. In the Memoir, Mahowald called this coset the *root invariant*, $\sqrt{\alpha}$; we prefer to call it the *Mahowald invariant*, $R(\alpha)$. $R(\alpha)$ is a subset of π_t^S, and the theorem of Bredon and Jones shows that $t \geq 2n$. $R(\alpha)$ is nonempty and never contains zero, so this gives an interesting way of creating higher and higher dimensional homotopy classes. A consequence of the work in the Annals paper, for example, is **[56, 72]** that for each s

$$\alpha_s \in R(2^s\iota),$$

where α_s is the element of π_*J considered above. This exhibits the tendency of R to increase the level of periodicity—from $2 = v_0$ to v_1 in this case; and an

[2] See formula (12) on page 291. We thank N. Minami for bringing this paper to our attention. For related ideas, and a conjecture closely related to the Sullivan conjecture, see also **[17]**.

important current conjecture [56] in homotopy theory asks for a general result like this.

5.3. Unstable v_1-periodic homotopy. Many of Mahowald's ideas found better expression as the subject developed. One, which may be traced back to the Memoir, is that the metastable projective space invariants should carry the "K-theoretic" part of the unstable homotopy of spheres in all dimensions. In *The image of J in the EHP sequence* Mahowald expresses this in the following terms.

Instead of applying the functor π_* to the Snaith map $s : \Omega^n S^n \to QP^{n-1}$, he applies the functor $[\Sigma^n Y, -]$. Here Y denotes the four-cell spectrum $C(2\iota) \wedge C(\eta)$ described above; its bottom cell is in degree 0. $\Sigma^3 Y$ is the suspension spectrum of $RP^2 \wedge CP^2$, and by $\Sigma^n Y$ we mean the space $\Sigma^{n-3} RP^2 \wedge CP^2$; so $n \geq 3$. The square brackets denote pointed homotopy classes. This is a "homotopy theory"; it converts fibration sequences to long exact sequences. For $n > 4$ it is an abelian group. The K-theory isomorphism v_1 desuspends also, and we may form a localization by inverting it. This gives a module over the graded field $F_2[v_1^{\pm 1}]$. One should think of this as an analogue of the rational homotopy of a space; K-homology (or mod 2 K-homology) plays the role of rational homology.

Mahowald proves that $s : \Omega^{2n+1} S^{2n+1} \to QP^{2n}$ induces an isomorphism in this theory. This can be thought of as an analogue of the theorem of Serre, that the rational homotopy of an odd sphere maps isomorphically to the stable homotopy of QS^0; the "zero-th" Snaith map is the stabilization map $\Omega^n S^n \to QS^0$.

Notice that inverting v_1 in $[\Sigma^* Y, -]$ is not a stabilization; quite to the contrary, it brings high-dimensional—so very unstable—classes down to low degrees. Mahowald's result is that nonetheless the result can be computed in stable terms.

This result follows from considering the map of fibrations:

$$\begin{array}{ccccc}
\Omega^{2n-1} S^{2n-1} & \xrightarrow{E^2} & \Omega^{2n+1} S^{2n+1} & \longrightarrow & \Omega^{2n-2} C(n) \\
\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\
QP^{2n-2} & \longrightarrow & QP^{2n} & \longrightarrow & QP^{2n}_{2n-1}
\end{array}$$

Here $C(n)$ (which Mahowald always denotes by $W(n)$) is the fiber of the double suspension map, and of course P^{2n}_{2n-1} is a mod 2 Moore space. Then in fact the right-hand vertical arrow induces an isomorphism in $v_1^{-1}[\Sigma^* Y, -]$: from the point of view of v_1-periodic homotopy the fiber of the double suspension is a stable mod 2 Moore space. This result is proven by considering certain modified unstable Adams towers, and showing that the localization of the induced map is an isomorphism.

These two sequences of fibrations link together over n to give exact couples in a homotopy theory, and so spectral sequences. We have just said that the map induces an isomorphism in E^1, so by the five-lemma the Snaith maps also induce isomorphisms.

The issue of the relationship between the condition that a map induce an isomorphim in v_1-periodic homotopy and that it induce an isomorphism in mod p K-homology is an interesting one. Snaith and the first author [73] showed that the limiting Snaith map $s : Q_0S^0 \to QP$ *is* a K-homology isomorphism. In very recent work, A. K. Bousfield [14] shows that the theorem of Mahowald mentioned above, to the effect that $s : \Omega^{2n+1}S^{2n+1} \longrightarrow QP^{2n}$ induces an isomorphism in $v_1^{-1}[\Sigma^*, -]$, implies that it is a K-theory-isomorphism also. This gives a calculation of $K_*(\Omega^{2n+1}S^{2n+1})$. These methods have been extended further by L. Langsetmo in [42], using work of Mahowald and R. D. Thompson [63].

5.4. The j_*P spectral sequence. The Annals paper considers two further issues. One wants to know about the image of the true homotopy in the localizations computed here; and one really is primarily interested in the ordinary homotopy groups, not in the Y-homotopy groups. These questions may be addressed in the following way, which brings us back again to ideas nascent in the Memoir.

We noted above that the unit of the spectrum j is a version of the Adams e-invariant. It induces a map $\pi_*^S X \to j_*X$, and a corresponding map of Atiyah-Hirzebruch spectral sequences. Mahowald applies this in particular to the case of $P = RP^\infty$:

$$e : \begin{pmatrix} \text{Atiyah-Hirzebruch} \\ \text{spectral sequence for } \pi_*^S P \end{pmatrix} \longrightarrow \begin{pmatrix} \text{Atiyah-Hirzebruch} \\ \text{spectral sequence for } j_*P \end{pmatrix}.$$

Mahowald computes the latter spectral sequence in detail. The result is quite complex; but after all, via the Snaith map s from the EHP spectral sequence (described above in Section 5.2), it is the most accurate image of the global behavior of unstable homotopy theory we have, and we believe that it illuminates and unifies many of Mahowald's ideas and insights. We describe it here, and then comment on the behavior of the composite es.

In the spectral sequence, $E_s^1 = j_*$ for $s \geq 1$. (We apologize for the re-use of the letter s.) The differential d^1 comes from the cellular chain complex for P, so is alternately zero and multiplication by 2. E_s^2 is thus $\ker(2|j_*)$ for s even and $\operatorname{coker}(2|j_*)$ for s odd. We will use certain graphical conventions to depict these two graded vector spaces. In Figure 5, the vertical lines denote η-multiplications. The generators are represented by dots, which are offset to the left or right in accordance with their origin in the long exact sequence depicted in Figure 2.

Figure 6 depicts the spectral sequence from E^2 on. Here we have omitted the vertical η multiplications for clarity, but preserved the horizontal offset. We have also omitted the heads of the arrows depicting the differentials; they point north-west.

Further analysis of the cell-structure of P leads to further "short" differentials, d^2's and d^3's, which are more easily drawn than described. Generally it may be said that α's hit α's and β's hit β's. What remains after they have been executed is for the most part concentrated along diagonals where $s + t \equiv -1, -2 \bmod 4$.

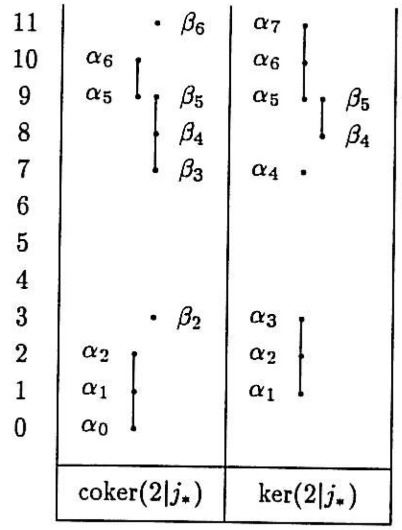

FIGURE 5. The columns of E^2 of the j_*P spectral sequence

Here there are longer differentials, starting with the "vector field differential"

$$d^r \iota_{4k-1} = \beta_{\nu(k)+2}\iota_{4k-1-r},$$

where $r = |\beta_{\nu(k)+2}| + 1$. The rest of the differentials along this total degree line slide up one notch at a time; the general formula is

$$d^*(\alpha_i \iota_{4k-|\alpha_i|-1}) = \beta_{\nu(k)+i+2}\iota_*,$$

where each $*$ denotes a number whose value can be determined from the rest of the formula. In the interest of clarity we have drawn only the differentials starting on the fundamental classes ι_{4k-1}.

This completes the description of the spectral sequence depicted in Figure 6. The composite es sends the EHP spectral sequence (Figure 3) into this one, and we invite the reader to make the comparison at this point. Remember that we have drawn the EHP spectral sequence from E^1, and the j_*P spectral sequence from E^2.

There is a clear periodicity in this spectral sequence. It may be extended periodically to the left, by replacing the spectrum j with the K-homology localization S_K of the sphere spectrum, of which j is the (-1)-connected cover. It may also be continued downwards, by replacing P by the inverse system $P_{-\infty}$. If both of these changes are made, the only survivors occur along the total degree lines $s+t = -1, -2$, where the length of the "long differentials" becomes infinite. This leads to the conclusion that

$$\operatorname*{holim}_{n\to\infty}(P_{-n} \wedge S_K) \simeq \Sigma^{-1}SQ_p \vee \Sigma^{-2}SQ_p.$$

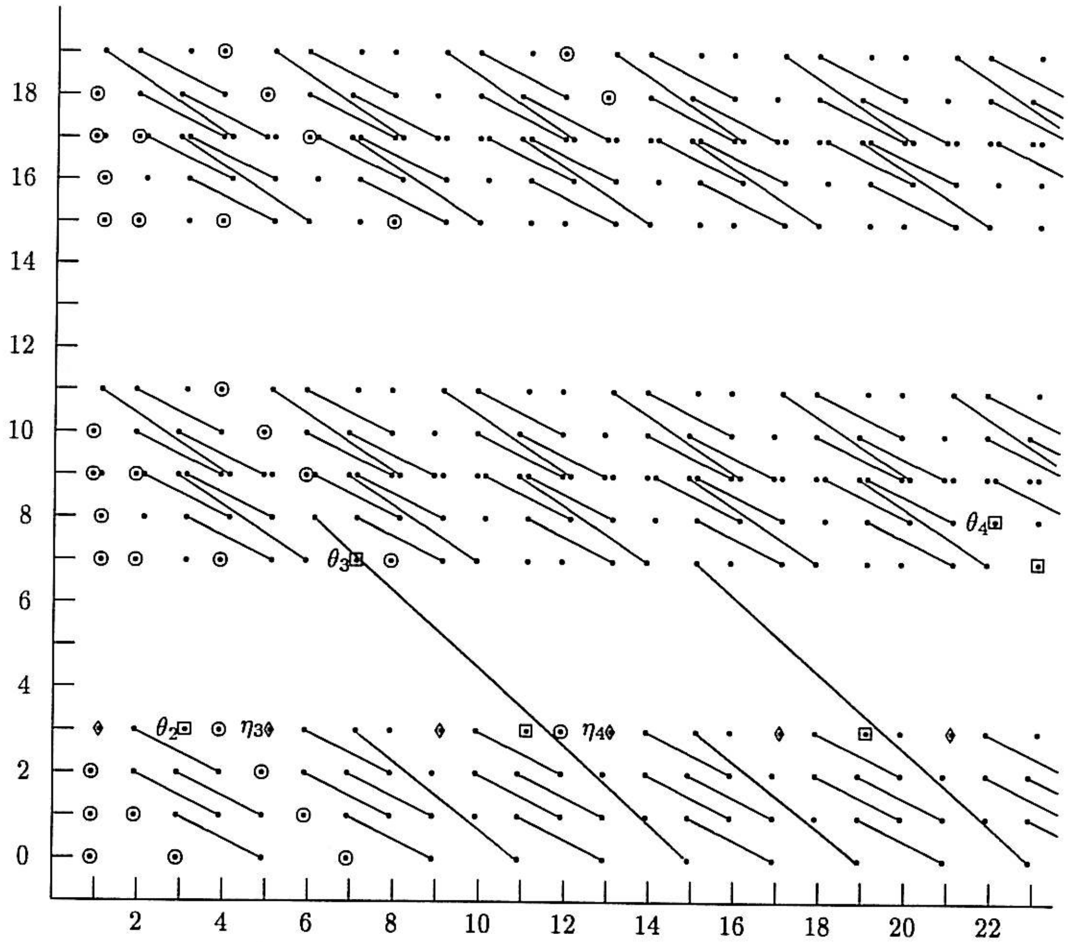

FIGURE 6. The j_*P spectral sequence

where Q_p denotes the p-adic rational numbers. For us $p = 2$; but the same result holds at odd primes, despite the rather different way j_* and the spectral sequence look in the two cases [**83**].

We mention this extended spectral sequence because it gives an interpretation of the survivors in the original spectral sequence: they are all "edge-effects." This occurs in the Atiyah-Hirzebruch spectral sequence for $\pi_*^S P$ as well; by Jones's theorem, all this homotopy supports differentials if we allow the negative cells to play their part.

The survivors are of three types. First there are the classes which would be the sources of long differentials, except that these differentials end in $s \leq 0$. These are circled. Next, there are the classes which would be hit by long differentials, except that these differentials would start in $t < 0$. They are boxed. Finally, there are classes along $t = 3$ which occur because the homotopy of the spectrum j is irregular in low dimensions. They are enclosed in diamonds.

These permanent cycles compute $j_*(P)$. The circled classes in degrees congruent to -1 mod 4 assemble to cyclic groups, as do the boxed classes. Mahowald analyzes the composite

$$\pi_*^S \cong \pi_*(Q_0S^0) \xrightarrow{s} \pi_*^S P \xrightarrow{e} j_*P.$$

The summand of π_*J in π_*^S maps isomorphically to the lifts of the circled survivors. Mahowald proved that the only diamond classes in the image are those in dimension which is a power of 2 and at least 8. They are the images of the classes η_j whose construction was reviewed in §3.

Of the boxed classes, only the classes of order two (that is, of lowest filtration) and in dimension of the form $2^j - 2$ can be in the image; and they are hit by elements of Kervaire invariant one and only by such elements. This is the first way these classes enter the story. The reader should reconcile this with the conjecture recalled in §2 from the Memoir.

The next question to address is: to what extent are the Snaith map and the e-invariant split-epimorphisms, as maps of spectral sequences? This might be regarded as a sort of unstable Adams conjecture, if we think of the Adams conjecture as asserting that the e-invariant is split by the J-homomorphism.

As a start, consider the permanent cycles. The map es puts a lower bound on the sphere of origin of the elements in $\pi_*J \subset \pi_*^S$; these elements cannot be born before their images on RP^∞. These lower bounds are usually attained [**29, 34**], and the Hopf invariants are the elements of π_*J whose e-invariant is the representative in the j_*P spectral sequence. The only exception is that β_{4k} is not born on S^2 with Hopf invariant β_{4k-1} (which is itself not born till much later), but rather on S^3 with an unstable Hopf invariant (except for $\beta_4 = \eta\sigma$, which is born on S^6 with Hopf invariant ν).

The elements of π_*J are thus born very early, with correspondingly high dimensional Hopf invariant. (On their sphere of birth, they are far from being in the "unstable image of J": the elements of π_*SO are born rather late [**33**].)

The elements η_j, on the other hand, are born very late, with Hopf invariant ν, as one expects from the position of their e-invariant in the j_*P spectral sequence.

The Kervaire classes should also be born "as late as possible," and one hopes that there is a maximal desuspension of $\theta_j \in \pi^S_{2^{j+1}-2}$ with Hopf invariant β_j, since this is the behavior of their e-invariants. A theorem of Barratt, Jones, and Mahowald [10] asserts that if $\theta \in \pi^S_{2^{j+1}-2}$ has nonzero image in j_*P, then it has Kervaire invariant 1; and conversely if it has Kervaire invariant 1 and order 2, then its image is nonzero.

Next turn to the differentials. The most important one is the "vector-field differential,"

$$d^r \iota_{4k-1} = \beta_{\nu(k)+2}\iota_{4k-r-1},$$

where $r = |\beta_{\nu(k)+2}| + 1$. This differential of course holds in the EHP spectral sequence and the $\pi^S_* P$ spectral sequence just as in the j_*P spectral sequence. In the $\pi^S_* P$ spectral sequence, it can be interpreted as giving the "stable relative attaching map" of the top cell in RP^∞; and of course this is the context in which Adams solved the vector field problem. A glance at Figure 6 shows the relationship between the maximal desuspension of the Whitehead square (which is recorded by this differential) and the Kervaire class. On its sphere of birth the Radon-Hurwitz desuspension $\overline{w}_{4k}$ of the Whitehead square has rather high order if $\nu(k)$ is large. If k is a power of 2, then one, two, or four suspensions later (depending upon k), it should become divisible by 2, and equal to a maximal desuspension of a Kervaire class. The order is then decreased as one suspends further, until S^{4k} is reached and the Whitehead square dies. The Kervaire class then lives on as a stable class.

The best general information Mahowald gives us about the other unstable classes is Theorem 1.5 in the Annals paper (cf. (4.7) of [50]). This asserts that in dimensions bigger than $2n$, a class in j_*P^{2n} is in the image of $\pi_*(\Omega^{2n+1}S^{2n+1})$ provided only that its image in j_*P is in the image of π^S_*—an image specified above, up to the Kervaire invariant problem. This result gives a large family of unstable homotopy classes with tightly controlled behavior in the EHP sequence.

These classes are constructed by first manufacturing suitable homotopy classes in projective space, and then using the map $P^{2n} \to \Omega^{2n+1}S^{2n+1}$ to convert these to classes in the sphere. This method harks back to the Memoir, and indeed the proof uses the calculations of the Memoir.

It is interesting to ask the fates of the remaining elements of the image of J in the π^S_* spectral sequence and in the EHP sequence. These are the survivors in the j_*P spectral sequence which are not in the image of e or es. In [50] (Proposition 4.8) a relation is given between the Whitehead products of certain of these classes and the Kervaire invariant classes. This has not appeared with detailed proof.

The announcement [50] gives certain compositions among these unstable classes, and details can be found in [61].

6. Conclusion

We have ended this account of Mahowald's contributions to the study of the homotopy groups of spheres with a description of a paper published some ten years ago. One should not infer from this that he has in any way reduced the intensity of his work. On the contrary, as this or almost any current conference proceedings shows, he is involved in many aspects of contemporary homotopy theory. We mention for example:

- The comparison of unstable v_1 and K-theory localization. This work **[63]** was mentioned above in Section 5.3. It is an integral part of current foundational work by Bousfield **[14]** and E. Dror Farjoun **[35]**, and of computational work by Langsetmo **[42]** and others.
- The Mahowald invariant (Section 5.2). Its relationship to periodicity is a guiding idea in contemporary stable homotopy theory. See **[56]**, **[79]**.
- The homotopy groups of localizations and telescopes. With the failure of the telescope conjecture **[77]**, these computations take on new significance, and Mahowald has been involved in both ends: with the second author and P. Shick **[57]** and with N. Shimomura **[58]**.
- The development of M. J. Hopkins's ideas about generalized dimensions in localizations of the stable homotopy category.
- The analysis of higher analogues of the EHP-sequence **[62]**.

We could extend the list further, but we prefer to wish Mark a very happy birthday and many more years of mathematical leadership!

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