On G and the Stable Adams Conjecture

by

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The purpose of this note is to record the results of our study of the spectrum of G, the space of stable homotopy equivalences of spheres. Because of the J homomorphism and the fibration of infinite loop spaces

$$O \xrightarrow{J} G \rightarrow G/O \tag{1}$$

one is reduced to studying G/O. We compute a summand of the cohomology of the spectrum of G/O. We also establish a fibration of infinite loop spaces

$$BU \rightarrow X \rightarrow IBO$$

where $X = G/O$ with a possibly different infinite loop space structure and IBO is the fiber of the unit map $QS^0 \rightarrow BO \times \mathbb{Z}$. Finally we formulate a stable version of the real Adams Conjecture the truth of which is shown to imply that X is G/O with the standard infinite loop space structure. Thus a proof of our conjecture will determine G in terms of more elementary infinite loop spaces.

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§1. Preliminaries

Let $QX = \lim \Omega^n\Sigma^n X$. Then QX is an infinite loop space; i.e., the zero space of the Ω-spectrum $\{Q\Sigma^n X\}$. For $X = S^0$, QS^0 has components $Q_k S^0$, $k \in \mathbb{Z}$, determined by the degree of self maps of spheres.

Stable spherical fibration theory is classified by BG where $G = Q_{\pm 1}S^0$; for oriented theory one uses BSG where $SG = Q_1 S^0$. Both G and SG are infinite loop spaces under composition. On the other hand, reduced stable cohomotopy theory is classified by $Q_0 S^0$, itself an infinite loop space under loop sum. Since SG and $Q_0 S^0$ are equivalent as spaces one would like to understand the relationship between these two basic (and apparently very different) infinite loop structures.

We remind the reader that in the case of oriented real (or complex) K-theory the zero and one components $BSO_{\oplus}$ and $BSO_{\otimes}$ are actually equivalent as infinite loop spaces when localized at any prime [AP]. Certainly nothing so simple is true for SG and $Q_0 S^0$ because their Pontryagin algebras differ.

To give all of this a focus the reader may wish to keep in mind the old problem of computing the homology of the spectrum sg associated to SG

$$H_* sg = \lim H_{*+n} B^n SG$$

Throughout this note we shall use (co-)homology with coefficients in $\mathbb{Z}/2$. All spaces will be localized at 2. The case of odd primes is fundamentally different as we shall indicate in §2.

We recall that any infinite loop space has Dyer-Lashof homology operations; in the case of QS^0

$$Q^k\colon\ H_* Q_n S^0 \to H_{*+k} Q_{2n} S^0$$

Let $[n] \in H_0 Q_n S^0 = \mathbb{Z}/2$ denote the generator. Then Browder [B] computed

$$H_*Q_0S^0 = \mathbb{Z}/2[Q^I[1]*[-2^{\ell(I)}]]$$

where $*$ denotes the Pontryagin product (under loop sum) and $I = (i_1,\ldots,i_\ell)$ runs over those sequences of positive integers with $i_j \leq 2i_{j+1}$, $i_1 > i_2 + \cdots + i_\ell$ and $\ell(I) \geq 1$. Such sequences are called allowable. The length of I, $\ell(I)$, is defined to be ℓ.

Later, Milgram [Mg] described H_*SG in terms of the Dyer-Lashof operations for QS^0 as

$$H_*SG = E[Q^k[1]*[-1]] \otimes \mathbb{Z}/2[Q^kQ^k[1] * [-3]]$$
$$\otimes \mathbb{Z}/2[Q^I[1]*[1-2^{\ell(I)}]]$$

where $k \geq 1$ and I runs over the same sequences as above except $\ell(I) > 1$. The exterior classes $Q^k[1]*[-1]$ are easily shown to come from SO under the J-homomorphism. Further, fibration (1) and the Eilenberg-Moore spectral sequence show that

$$H_*SG = H_*SO \otimes H_*G/O$$

with

$$H_*SO = E[Q^k[1]*[-1]]$$
$$H_*G/O = \mathbb{Z}/2[Q^kQ^k[1]*[-3]] \otimes \mathbb{Z}/2[Q^I[1]*[1-2^{\ell(I)}]]$$

Thus one may naively explain the difference between the Pontryagin algebras $H_*Q_0S^0$ and H_*SG by saying that the exterior classes $Q^k[1]*[-1]$ force the existence of new generators $Q^kQ^k[1]*[-3]$ to compensate for the fact that the ranks must be equal. It is important to note that these elements are decomposable in Q_0S^0, i.e.

$$Q^k Q^k[1]*[-4] = (Q^k[1]*[-2])^{*2}.$$

The rest of H_*SG looks like $H_*Q_0S^0$ (superficially at least).

By using the Dyer-Lashof operations of SG (derived from the composition product) a stronger statement is possible. We denote these operations by

$$\tilde{Q}^k\colon H_*SG \to H_{*+k}SG$$

On SO, Kochman [K] has determined these operations while on G/O one has Madsen's formula [Md]: let $x_I = Q^I[1]*[1-2^{\ell(I)}]$ then

(2) $$\tilde{Q}^k x_I = x_{(k,I)} + \sum_{\ell(J)<\ell(k,I)} x_J + \circ\text{-decomposables}$$

where $\circ$ denotes the Pontryagin product. Thus modulo lower length terms and decomposables the $\tilde{Q}$ operations correspond precisely to the Q operations. This strongly suggests some geometric relation between SG and Q_0S^0 as infinite loop spaces. We shall return to this in §3.

§2. A summand of H_*G/O

In studying G/O it is natural to consider the Adams Conjecture

(3) $$\begin{array}{ccccc} G/O & \to & BSO & \xrightarrow{BJ} & BSG \\ & \nwarrow{\scriptstyle\alpha} & \uparrow{\scriptstyle\psi^3-1} & & \\ & & BO & & \end{array}$$

According to Quillen and Sullivan [Q,S] $BJ\circ(\psi^3-1) \simeq 0$ and so one has the indicated lift α. However, Madsen [Md] using (2) has shown that no choice of α is an H-map and so α is of little use in studying G/O as an infinite loop space. It appears that the most one can say is that the infinite loop map

$$QBO(2) \xrightarrow{\bar{\alpha}} G/O$$

(induced by α restricted to BO(2)) splits up to homotopy [P2]. The deviation of α from additivity has recently been analyzed by Tonehave; it involves the Bott map $BO \xrightarrow{\eta} SO$.

In the complex case, there is no such obstruction to additivity and Friedlander and Seymour [FS] have recently solved the Stable Complex Adams Conjecture; i.e.

$$\begin{array}{ccccc} SG/U & \to & BU & \xrightarrow{BJ} & BSG \\ \nwarrow & & \uparrow \psi^3 - 1 & & \\ & \beta & BU & & \end{array}$$

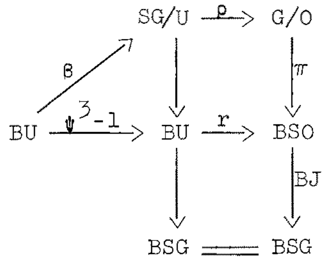
with $BJ\cdot(\psi^3-1) \simeq 0$ as infinite loop maps. (They prove the analogous assertion also at an odd prime. It follows that at an odd prime the analogue of α in (3) can be taken to be an infinite loop map). We define f to be the resulting infinite loop map

$$f\colon BU \xrightarrow{\beta} SG/U \xrightarrow{\rho} G/O$$

where ρ is the natural map. Recalling that $H_*BU = \mathbb{Z}/2[a_k]$, $\dim a_k = 2k$, we have

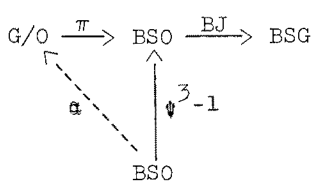
<u>Proposition</u>. $f_*(a_k) = x_{kk}$ in QH_*G/O, the module of $\circ$-indecomposables ($x_{kk} = Q^kQ^k[1]*[-3]$).

<u>Proof</u>: Consider the homotopy commutative diagram



where the vertical maps form fiber sequences and r is realification.

Let $a_2' \epsilon H_4(BU; \mathbb{Z})$ be a class which reduces (mod 2) to a_2. Then in integral homology, $(\psi^3-1)_*(a_2') = 8a_2'$ [A2]. Let $b_2 \epsilon H_4(BSO; \mathbb{Z})$/Torsion be a generator. Then $r_*(a_2') = n \cdot b_2$ where n is odd [C]. Let $c_2 \epsilon H_4(G/O; \mathbb{Z})$/Torsion be a generator. Then using a solution of the real Adams Conjecture [Q,S]



one can deduce that $\pi_*(c_2) = 8k \cdot b_2$, k odd. Hence $f_*(a_2') = \ell \cdot c_2$, ℓ odd. Thus, reducing mod 2 and using a standard Bockstein argument we find $f_*(a_2) = x_{22}$ in QH_*G/O (in mod 2 homology). Also
$f_*(a_1) = f_*(Sq_*^2 a_2) = Sq_*^2 x_{22} = x_{11}$.

Using this fact and examining the diagonal map it is easy to see that in QH_*G/O

$$f_*(a_k) = x_{k,k} + \Sigma\ x_I$$

for some (possibly empty) set of allowable sequences I with $\ell(I) > 2$ and $|I| = \Sigma i_j$ even. We wish to show that $\Sigma\ x_I = 0$.

$$0 = Sq_*^1 f_*(a_k) = Sq_*^1(\Sigma\ x_I) = \Sigma(i_1-1)x_{I-\Delta_1}$$

where $I = (i_1, \ldots)$ and $\Delta_1 = (1,0,\ldots,0)$. If i_1 is even then $x_{I-\Delta_1}$ is allowable, hence I appears only if i_1 is odd. However, let

$$m = \max\{\ell(I) \mid x_I \text{ is a summand of } f_*(a_k)\}$$

and suppose $(i_1, i_2, \ldots, i_m)$ occurs. Then

$$0 = f_*(0) = f_*(Q^{2i_1-1} a_k) = \tilde{Q}^{2i_1-1} f_*(a_k)$$

$$= \sum_{\ell(I)=m} \tilde{Q}^{2i_1-1} x_I + \sum_{\ell(I)<m} \tilde{Q}^{2i_1-1} x_I = \Sigma\, x_{(2i_1-1,I)} +$$

terms of length $\leq m$ (using (2)). Since each of the terms $x_{(2i_1-1,I)}$ is allowable this completes the proof.

This proposition has immediate implications for the (co-)homology of the Ω-spectrum g/O with zero space G/O. Let bu denote the Ω-spectrum with zero space BU; i.e. connective reduced complex K-theory. Adams [A1] has computed

$$H^* bu = \Sigma^2 A/A(Sq^1, Sq^3)$$

where A denotes the mod 2 Steenrod algebra as usual. Now f induces a map of spectra

$$f\colon bu \to g/O$$

and we have

Corollary. $H_* bu$ is a $\mathbf{Z}/2$-summand of $H_* g/O$.

Proof: Equivalently we show f^* is surjective. Since $H^* bu$ is monogenic over A we need only show that the generator in dimension 2 is in the image. But this is the Hurewicz dimension so the result follows from the Proposition.

Remark: H^*bu is not a summand over A because the 2 and 3 dimensional classes of H^*g/O are connected by Sq^1.

§3. Two Conjectures

In this section we study the cokernel of the infinite loop map $f: BU \to G/O$ of §2.

Let ko denote the Ω-spectrum representing connective unreduced real K-theory; i.e. the zero space of ko is $BO \times \mathbb{Z}$. The unit map $S \to ko$ gives rise to an infinite loop map

$$u: QS^0 \to BO \times \mathbb{Z}$$

Recalling that $H_*BO = \mathbb{Z}/2[\overline{e}_k]$ and $u_*(Q^k[1]*[-2]) = \overline{e}_k$ [P2], we have an exact sequence of $\mathbb{Z}/2$-modules

$$0 \to QH_*BU \xrightarrow{f_*} QH_*G/O \xrightarrow{\gamma} QH_*Q_0S^0 \xrightarrow{u_*} QH_*BO \to 0$$

where $Q(\cdot)$ is the algebra indecomposables functor. The map γ is defined on basis elements by $x_I \to Q^I[1]*[-2^{\ell(I)}]$. Since f_* and u_* are induced by infinite loop maps, both preserve Dyer-Lashof operations. By (2), γ preserves Dyer-Lashof operations up to a length filtration. Of course, a priori, γ is just an algebraic map, but the first author has a spectral sequence for computing the homology of a spectrum and the E^2 term depends on the homology indecomposables of the zero space as an unstable module over the Dyer-Lashof algebra. Thus it seems plausible to make the following conjecture. Let IBO denote the fibre of $u: QS^0 \to BO \times \mathbb{Z}$.

Conjecture A. There exists a fibration

$$BU \xrightarrow{f} G/O \to IBO$$

of infinite loop spaces.

One consequence of this conjecture is a complete calculation of H^*g/O. From the cofibration sequence

$$\cdots \to \Sigma^{-1}(ko/S) \to S \to ko \to ko/S \to \cdots$$

and Stong's calculation [St], $H^*ko = A/A(Sq^1, Sq^2)$ we have

$$H^*\Sigma^{-1}(ko/S) = \Sigma^{-1}I(A/A(Sq^1, Sq^2))$$

Hence from the Corollary of §2 we have

Corollary of Conjecture A

$$H^*g/O = \Sigma^2(A/A(Sq^1, Sq^3)) \oplus \Sigma^{-1}I(A/A(Sq^1, Sq^2))$$

with the 2 and 3 dimensional generators connected by Sq^1.

We now construct a candidate for a solution to Conjecture A. Since BSpin is connected, the composite

$$QS^0 \xrightarrow{u} BO \times \mathbf{Z} \xrightarrow{\psi^3 - 1} BSpin$$

is null homotopic as an infinite loop map. Hence there is an induced map of infinite loop space fibrations

(4)
$$\begin{array}{ccccc} IBO & \rightarrow & QS^0 & \rightarrow & BO \times \mathbb{Z} \\ \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \psi^3 - 1} \\ Spin & \rightarrow & * & \rightarrow & BSpin \end{array}$$

Let X be the fiber of the composite IBO $\xrightarrow{\psi}$ Spin $\xrightarrow{c}$ SU where c is complexification. Then from the Bott sequence BSO $\xrightarrow{\eta}$ Spin $\xrightarrow{c}$ SU we have an induced map of infinite loop space fibrations

(5)
$$\begin{array}{ccccccc} BU & \longrightarrow & X & \longrightarrow & IBO & \longrightarrow & SU \\ \Big\| & & \Big\downarrow{\scriptstyle \varepsilon} & & \Big\downarrow{\scriptstyle \psi} & & \Big\| \\ BU & \xrightarrow{r} & BSO & \xrightarrow{\eta} & Spin & \xrightarrow{c} & SU \end{array}$$

where r is realification.

We have been unable to prove that

(6)
$$BU \rightarrow X \rightarrow IBO$$

is a solution to Conjecture A. However in §4 we show that as a space X is equivalent to G/O and so X provides some delooping of G/O (possibly non standard). We also show that both X and G/O provide infinite loop space factorizations of r. Thus (6) seems a very good candidate for Conjecture A.

First we show how Conjecture A relates to the Stable Adams Conjecture. Consider the diagram

(7)
$$\begin{array}{ccccccc} BU & \to & X & \overset{\varphi}{\dashrightarrow} & G/O & & \\ & \searrow^{r} & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \pi} & & \\ & & BSO & \xrightarrow{\psi^3-1} & BSO & \xrightarrow{BJ} & BSG \end{array}$$

where r is realification and π is inclusion of the fiber of BJ. By the Adams Conjecture $BJ\cdot(\psi^3-1) \simeq 0$ as maps of spaces but not H-spaces. By the Stable Adams Conjecture $BJ\cdot(\psi^3-1)\cdot r \simeq 0$ as infinite loop space maps (see §2). We propose the intermediate conjecture.

Conjecture B. $BJ\cdot(\psi^3-1)\cdot\varepsilon \simeq 0$ as infinite loop space maps.

This immediately implies Conjecture A

Lemma. Any infinite loop map φ completing diagram (7) is an equivalence (at 2).

Proof: In §4 we show $X \simeq G/O$ as a space, hence it is enough to prove φ_* is surjective in mod-2 homology. Over the Dyer-Lashof algebra, QH_*G/O is generated by the coalgebra C with basis $\{x_{a,b}\colon a \leq 2b,\ a \geq b \geq 0\}$ (see [Md]). Thus it suffices to show $x_{a,b} \in \operatorname{Im} \varphi_*$ modulo terms of higher length. As an algebra $C^* = \mathbb{Z}/2[x,y]$ where x and y are dual to x_{11} and x_{21} respectively. Since $X \simeq G/O$ as a space, H^*X is a polynomial algebra and thus it suffices to show $x_{11}, x_{21} \in \operatorname{Im} \varphi_*$. By the argument of the Proposition of §2, $x_{11} \in \operatorname{Im} \varphi_*$. The relation $Sq^1 x = y$ implies $x_{21} \in \operatorname{Im} \varphi_*$. This completes the proof.

§4. Properties of X.

Proposition. $X \simeq G/O$ as spaces.

Let $\operatorname{Im}J \times \mathbb{Z}$ denote the fiber of $\psi^3-1\colon BO \times \mathbb{Z} \to BSpin$. Then from diagram (4) and the 3 x 3 Lemma for infinite loop spaces (or spectra) we have the following homotopy commutative diagram of infinite loop spaces and maps.

(8)

$$\begin{array}{ccccc}
C_{\oplus} & \to & QS^0 & \xrightarrow{u} & ImJ \times \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
IBO & \to & QS^0 & \xrightarrow{u} & BO \times \mathbb{Z} \\
\downarrow{\scriptstyle \psi} & & \downarrow & & \downarrow{\scriptstyle \psi^3 - 1} \\
Spin & \to & * & \longrightarrow & BSpin
\end{array}$$

where the vertical and horizontal sequences are fibrations and where the common fiber, $C_{\oplus}$, is called the (additive) coker J. A (multiplicative) coker J, $C_{\otimes}$, is defined as the fiber of the unit map $u\colon QS^0 \to ImJ \times \mathbb{Z}$ restricted to the 1-components. As spaces $C_{\oplus} \simeq C_{\otimes}$.

<u>Proof of Proposition</u>: Combining diagrams (5) and (8) we have

(9)

$$\begin{array}{ccccc}
C_{\oplus} & \longrightarrow & C_{\oplus} & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & IBO & \longrightarrow & SU \\
\downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \psi} & & \| \\
BSO & \xrightarrow{\eta} & Spin & \xrightarrow{c} & SU
\end{array}$$

Let $IBO_{\otimes}$ be the fiber of $u\colon QS^0 \to BO \times \mathbb{Z}$ restricted to the 1-components. May [My] has shown $IBO_{\otimes} \simeq C_{\otimes} \times Spin$ as infinite loop spaces. Since $IBO_{\otimes} \simeq IBO$ as spaces and since $KO^*(C_{\otimes}) = 0$ [Sn] we have a splitting $IBO \xrightarrow{\psi} Spin$ (as spaces) and thus from (9) a splitting $X \underset{\leftarrow - -}{\xrightarrow{\epsilon}} BSO$;

i.e. $X \simeq C_{\oplus} \times BSO$ as spaces. However $G/O \simeq C_{\otimes} \times BSO$ [MST] and so this completes the proof.

From diagram (5) we see that X factors realification. Next we show that G/O shares this property. The Atiyah-Bott-Shapiro orientation of Spin bundles defines a KO-characteristic class $e\colon G/O \to BSO_{\otimes}$ which is an infinite loop map [MST]. The Adams cannibalistic class $\rho_3\colon BSO \to BSO_{\otimes}$ is an infinite loop equivalence [MST].

<u>Proposition</u>. G/O factors realification; i.e.

$$\begin{array}{ccc} BU & \xrightarrow{f} & G/O \\ & {\scriptstyle r}\searrow & \Big\downarrow{\scriptstyle \rho_3^{-1}\cdot e} \\ & & BSO \end{array}$$

is homotopy commutative as infinite loop maps.

First we establish

<u>Lemma</u>.

$$\begin{array}{ccc} G/O & \xrightarrow{e} & BSO_{\otimes} \\ {\scriptstyle \pi}\Big\downarrow & & \Big\downarrow{\scriptstyle \rho_3^{-1}} \\ BSO & \xleftarrow{\psi^3-1} & BSO \end{array}$$

is homotopy commutative as infinite loop maps.

<u>Proof</u>: The diagram

$$\begin{array}{ccc} G/O & \xrightarrow{e} & BSO_{\otimes} \\ {\scriptstyle \pi}\Big\downarrow & & \Big\downarrow{\scriptstyle \psi^3/1} \\ BSO & \xrightarrow{\rho_3} & BSO_{\otimes} \end{array}$$

is homotopy commutative on the space level [MST]. Let $d = [(\psi^3/1)\cdot e]/(\rho_3 \cdot \pi)\colon G/O \to BSO_\otimes$. Since the set of homotopy classes of infinite loop maps $C_\otimes \to BSO_\otimes$ is trivial [MST] there is an infinite loop map δ factoring d

$$\begin{array}{ccccc} C_\otimes & \to & G/O & \xrightarrow{e} & BSO_\otimes \\ & & {\scriptstyle d}\downarrow & \swarrow{\scriptstyle \delta} & \\ & & BSO_\otimes & & \end{array}$$

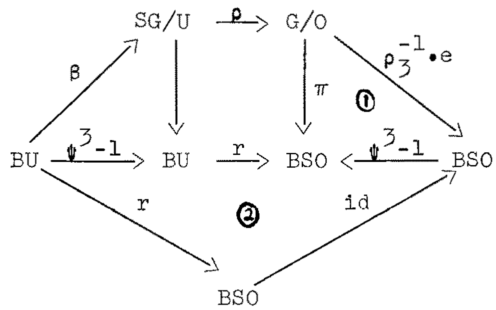
An infinite loop self map of BSO is determined by its restriction to the zero space [MST]. ($BSO_\otimes$ is equivalent to BSO [AP].) Similarly a self map of BSO is determined by its induced morphism in rational homology [My]. Since $d \simeq 0$ and e is a rational equivalence it follows that $\delta \simeq 0$ as an infinite loop map.

The lemma now follows from the homotopy commutativity of

$$\begin{array}{ccc} BSO_\otimes & \xrightarrow{\psi^3/1} & BSO_\otimes \\ \uparrow{\scriptstyle \rho_3} & & \uparrow{\scriptstyle \rho_3} \\ BSO & \xrightarrow{\psi^3-1} & BSO \end{array}$$

as infinite loop maps [My]. This completes the proof.

<u>Proof of Proposition:</u> Consider the diagram

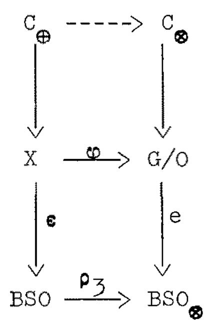


By definition the unlabeled squares commute up to homotopy as infinite loop maps. Similarly for square 1 by the preceding lemma and for square 2 by Adams [A2]. This completes the proof.

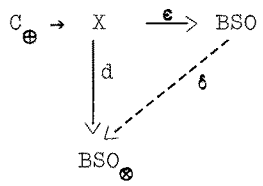
Finally we mention another

<u>Corollary of Conjecture B</u>: $C_\oplus \simeq C_\otimes$ as infinite loop spaces.

<u>Proof</u>: By the Lemma of §3, φ is an equivalence. Since ρ_3 is also an equivalence, it suffices to show that φ fits into a map of infinite loop space fibrations

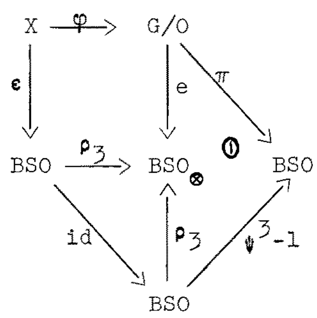


Let $d = e \cdot \varphi/\rho_3 \cdot \varepsilon$: $X \to BSO_\otimes$. Since the set of homotopy classes of infinite loop space maps $C_\oplus \to BSO_\otimes$ is trivial [MST] there is an infinite loop map δ factoring d



Now as in the proof of the preceding lemma it suffices to show $\delta_* = 0$ in rational homology.

Consider the diagram



Square 1 commutes up to homotopy by the preceding lemma. The outer diagram commutes up to homotopy by definition of φ. Since ρ_3 and $\psi^3 - 1$ are rational equivalences the result follows.

<u>Remark</u>: P. May has made some low dimensional calculations with homology operations which support this corollary.

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