Massey-Peterson Towers and Maps from Classifying Spaces

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The "Sullivan conjecture" [10] asserts that, given a finite-dimensional connected CW complex $X$ and a finite group $G$, the space $X^{BG}$ of pointed maps from the classifying space $BG$ to $X$ has the weak homotopy type of a point. This conjecture was resolved in the affirmative in [9]. It is then natural and important to ask about the mapping space $X^{BG}$ for infinite dimensional spaces $X$. The situation then appears to be far more complex, even, for instance, when we take $X$ to be the classifying space of a connected topological group. In this paper I shall stage a raid into this area. As proof of the riches to be found there, I offer the following:

Theorem A. For any elementary Abelian 2-group $E$, the classifying space functor $B$ induces a weak homotopy equivalence

$$\mathrm{Hom}(E,SU_2) \to BSU_2^{BE}$$

from the discrete space of group homomorphisms from $E$ to $SU_2$ to the indicated pointed mapping space. In particular, $\mathrm{Hom}(E,SU_2) \to [BE,BSU_2]$ is bijective.

The techniques used actually depend only on $H^*(X;\mathbb{F}_2)$, but operate only under the assumption that $X$ is simply connected. Notice that if $X$ is a simply connected CW complex whose mod 2 cohomology is polynomial on a single 4-dimensional generator, then $\Omega X$ is 2-locally equivalent to $SU_2$. A natural question arises: is $X$ 2-locally equivalent to $BSU_2$? The following result shows that as far as maps from $BE$ are concerned $X$ and $BSU_2$ are indistinguishable.

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Theorem B. Let $X$ be a simply connected CW complex whose mod 2 cohomology is a polynomial algebra on a single 4-dimensional class. Then $[B\mathbb{Z}/2,X]$ contains exactly two elements, one of which, call it $f$, induces a nontrivial map in mod 2 cohomology. Moreover for any elementary Abelian 2-group $E$, the map

$$\bar{H}^1(E;\mathbb{Z}/2) = [BE,B\mathbb{Z}/2] \to [BE,X]$$

induced by $f$ is a bijection.

I will use the obstruction theory of Massey and Peterson ([8], [1], [7]). This theory applies to simply connected spaces whose mod p cohomology is "very nice" [4]. An unstable algebra B over the mod p Steenrod algebra A is very nice provided that it is of finite type and admits a simple system of generators whose vector space span is closed under the action of A. This is admittedly an awkward condition, but it does include the classical Lie groups and the complex and quaternonic Stiefel varieties, and at 2, the real Stiefel varieties as well. The Massey-Peterson theory should be regarded as a piece of light artillery, with which one can move quickly and execute small ambushes before wheeling in the heavy simplicial guns of Bousfield and Kan [5]. I note that a very elementary application of this theory shows that the algebraic theorem from [9], quoted below as (3.1), yields the Sullivan conjecture for elementary Abelian p-groups and simply connected spaces whose mod p cohomology is finite and very nice; see (3.2) below. This result is in part contained in:

Theorem C. Let $E$ be an elementary Abelian p-group. Evaluation of mod p cohomology induces a map

$$[BE,X] \to \mathrm{Hom}(H^*X,H^*BE)$$

to the indicated set of A-algebra maps. If $X$ is simply connected and $H^*X$ is very nice, then this map is bijective.

Conjecture: This is still true if $X$ is any simply connected space whose mod $p$ homology is of finite type.

The Massey-Peterson theory will be reviewed in Section 1, with some improvements, due largely to J. R. Harper and A. Zabrodsky. A couple of technical results are proved in Section 2, and a convergence theorem, due to A. K. Bousfield, appears in Section 4. The theorems stated above are proved in Section 3, by application of an algebraic result from [9].

I am very grateful to John Harper, who tutored me patiently on Massey-Peterson theory, and to Alex Zabrodsky, who suggested a proof of the key Lemma 1.11 in conversation at Aarhus and later proposed the marvellous property (1.7) of Massey-Peterson towers used here to prove (1.11). I am also indebted to Pete Bousfield, Gunnar Carlsson, Mark Mahowald, and Jeff Smith, for their help. Finally, I thank the Mathematics Departments of Northwestern University and the University of Cambridge for their hospitality.

## §1. Obstruction theory.

I shall begin by recalling briefly the theory of Massey and Peterson [8], [1], with improvements due to Harper [7] and Zabrodsky. Unless otherwise specified, $H^*(X)$ denotes the mod $p$ cohomology of $X$, $p$ an arbitrary prime.

Mod $p$ cohomology in its richest form is a functor from pointed spaces to the category $\mathcal{A}^*$ of augmented unstable algebras over the Steenrod algebra $A$. Let $\mathcal{A}^*_{ft}$ denote the full subcategory of those of finite type. Formation of the augmentation ideal gives a functor $I$ to the category $\mathcal{U}^*_{ft}$ of unstable left $A$-modules of finite type, and this functor has a left adjoint $U$ [8], [1]. It is easy to verify that an object of $\mathcal{A}^*_{ft}$ is very nice in the sense of the introduction iff it is of the form $U(M)$ for some $M \in \mathcal{U}^*_{ft}$.

The functor $U$ helps to relate algebra to geometry. The category $\mathcal{U}^*_{ft}$ has enough projective objects, and there is a contravariant association $P \mapsto K(P)$ of a mod p generalized Eilenberg-MacLane space to a projective in $\mathcal{U}^*_{ft}$, equipped with compatible natural isomorphisms

(1.1) $$\pi_t(K(P)) \cong \mathrm{Hom}_A(P,S(t))$$

(1.2) $$H^*(K(P)) \cong U(P),$$

where $S(t) = \bar{H}^*(S^t)$.

There is a functor $\Omega : \mathcal{U}^*_{ft} \to \mathcal{U}^*_{ft}$ left adjoint to suspension $\Sigma$ :

(1.3) $$\mathrm{Hom}_A(\Omega M,N) \cong \mathrm{Hom}_A(M,\Sigma N).$$

Since $\Sigma$ is exact, $\Omega$ carries projectives to projectives, and the isomorphism (1.2) is naturally compatible with $\Omega$:

(1.4) $$K(\Omega P) \simeq \Omega K(P).$$

Now let $X$ be a simply connected space such that $H^*(X) \cong U(M)$, and let $M \leftarrow P_\bullet$ be a projective resolution of $M$ in $\mathcal{U}^*_{ft}$. There is a tower of principal fibrations under $X$:

(1.5)
$$\begin{tikzcd}
& \vdots \arrow[d, phantom] & \\
& X_2 \arrow[r, "k_2"] \arrow[d] & K(\Omega^2 P_3) \\
& X_1 \arrow[r, "k_1"] \arrow[d] & K(\Omega P_2) \\
X \arrow[r, "j_0"] \arrow[ur, "j_1"] \arrow[uur, "j_2"] & K(P_0) \arrow[r, "k_0"] & K(P_1)
\end{tikzcd}$$

such that

(1.6) $$\ker(H^*(X_s) \to H^*(X)) = \ker(H^*(X_s) \to H^*(X_{s+1}));$$ and

(1.7) $k_s$ is induced by a null-homotopy of $d_s k_{s-1}$. That is, there exists a commutative square

$$\begin{array}{ccc} X_{s-1} & \xrightarrow{h_s} & PK(\Omega^{s-1}P_{s+1}) \\ \Big\downarrow{\scriptstyle k_{s-1}} & & \Big\downarrow{\scriptstyle \pi} \\ K(\Omega^{s-1}P_s) & \xrightarrow{d_s} & K(\Omega^{s-1}P_{s+1}), \end{array}$$

in which $\pi$ is the path-space fibration, such that the induced map $X_s \to K(\Omega^s P_{s+1})$ of homotopy fibers is homotopic to $k_s$. Here and below I write $d_s$ for any map induced by $d_s$.

Property (1.7) was suggested by Zabrodsky. It appears to be a fundamental feature of Massey-Peterson towers, and it may be possible to give a treatment of the subject in which it occupies a central position. For the present, however, I give a derivation of it from other known properties in the next section, and treat it as an axiom in this section.

By applying $\pi_*$ to (1.5) one obtains a spectral sequence with

$$E_2^{s,t} = \mathrm{Ext}^s(M, S(t)) \Rightarrow \pi_{t-s}(X).$$

The Ext group here, and below, is computed in the category $\mathcal{U}^*_{ft}$, or, equivalently, in $\mathcal{U}^*$. The goal of the present paper is to show that under certain circumstances, the Massey-Peterson machinery allows one to draw conclusions about $[Y,X]$ for $Y$ not even a suspension, given the assumptions one expects to demand by analogy with this spectral sequence.

<u>Theorem 1.8.</u> Let $Y$ be a connected CW complex such that $\tilde{H}_*(Y;Z)$ is of finite type and p-torsion, and let $X$ be a simply connected space such that $H^*(X)$ is of finite type and isomorphic to $U(M)$. Consider the map

$$H^* : [Y,X]_* \to \mathrm{Hom}_A(M,\bar{H}^*(Y)).$$

Then $H^*$ is (a) monic if $\mathrm{Ext}^s(M,\bar{H}^*(\Sigma^s Y)) = 0$ for all $s > 0$ and (b) epic if $\mathrm{Ext}^{s+1}(M,\bar{H}^*(\Sigma^s Y)) = 0$ for all $s > 0$.

This theorem and the method of proof presented below are for the most part due to Harper ([7] 2.2.1, for example). I have chosen a different set of convergence conditions. Moreover, an improvement will be noticed in part (a), for Harper proves only that, under the stated assumptions, $f \simeq *$ if $f^* = 0$. That proof is easier, requiring, aside from (1.6), only the elementary fact that $k_s i_s \simeq d_s$, where $i_s : K(\Omega^s P_s) \to X_s$ is the inclusion of the fiber over $*$. This restricted form of Theorem 1.8(a) is in fact all that is needed to prove the cases of the Sullivan conjecture considered here, Theorems 3.2 and 3.3. The full strength of (1.8) is required, however, to prove the theorems stated in the introduction.

Before starting the proof of Theorem 1.8, it is convenient to record a couple of consequences of Zabrodsky's observation (1.7). They both involve principal actions, for which I need some notation. Given a map $k : X \to B$, I shall write $\alpha_k : \Omega B \times E_k \to E_k$, or just $\alpha$, for the action of $\Omega B$ on the homotopy fiber $E_k$ of $k$. Also, given $f : Y \to E_k$ and $h : Y \to \Omega B$, I shall write $h * f$ for the composite

$$Y \xrightarrow{\Delta} Y \times Y \xrightarrow{h \times f} \Omega B \times E_k \xrightarrow{\alpha} E_k$$

The following lemma is a restatement of "primitivity of the principal action" [7] 1.2.6.

<u>Lemma 1.9.</u> The k-invariants are linear over the algebraic differential. That is, the following diagram is homotopy commutative.

$$\begin{tikzcd} K(\Omega^s P_s) \times X_s \arrow[r, "\alpha"] \arrow[d, "d_s \times k_s"] & X_s \arrow[d, "k_s"] \\ K(\Omega^s P_{s+1}) \times K(\Omega^s P_{s+1}) \arrow[r, "\mu"] & K(\Omega^s P_{s+1}). \end{tikzcd}$$

<u>Proof</u>. Use naturality of the principal actions resulting from (1.7), and the fact that $\mu \simeq \alpha_\pi$ . $\square$

<u>Corollary 1.10</u>. Let $f : Y \to X_s$ and $h : Y \to K(\Omega^s P_s)$. Then

$$k_s(h * f) \simeq d_s h * k_s f. \quad \square$$

<u>Lemma 1.11</u>. The following diagram is homotopy-commutative.

$$\begin{tikzcd} K(\Omega^s P_{s-1}) \times X_s \arrow[r, "d_s \times 1"] \arrow[dr, "pr_2"'] & K(\Omega^s P_s) \times X_s \arrow[d, "\alpha"] \\ & X_s \end{tikzcd}$$

It is in order to prove Lemma 1.11 that property (1.7) was introduced here. However, the proof of this lemma involves a technical result about compatibility of various principal actions which, in order not to further delay presentation of the proof of Theorem 1.8, I have placed in the next section.

<u>Proof of Theorem 1.8</u>. I shall prove part (a), and leave the proof of part (b), which is similar and somewhat easier, to you. So let $Y$ be a connected CW

complex such that $H_*(Y)$ is of finite type, let $X$ and $M$ be as in the statement of the theorem, and suppose that $f,g : Y \to X$ induce the same map in cohomology. Then the composites $f_0, g_0 : Y \to X \to K(P_0)$ are homotopic. I will now show that $f_s, g_s : Y \to X \to X_s$ are homotopic provided $f_{s-1}$ and $g_{s-1}$ are. By principality of $X_s \to X_{s-1}$, there is a map $h : Y \to K(\Omega^s P_s)$ such that $g_s \simeq h * f_s$. Thus by (1.10), $k_s g_s \simeq d_s h * k_s f_s$. Now $f_s$ and $g_s$ both lift to $X_{s+1}$, so $k_s f_s$ and $k_s g_s$ are both null-homotopic, and since $[Y, K(\Omega^s P_{s+1})]$ is a group under $*$, it follows that $d_s h \simeq *$. Thus $h^*|\Omega^s P_s \in \mathrm{Hom}_A(\Omega^s P_s, \bar{H}^*(Y)) = \mathrm{Hom}_A(P_s, \bar{H}^*(\Sigma^s Y))$ is a cocycle. By assumption, it is therefore also a coboundary; that is, $h$ factors through $d_{s-1} : K(\Omega^s P_{s-1}) \to K(\Omega^s P_s)$. Lemma 1.11 then implies that $h * f_s$ is homotopic to $f_s$, as claimed.

Now the issue of whether the homotopies $f_s \simeq g_s$ together yield a homotopy $f \simeq g$ is a question of convergence, and will be dealt with in Section 4. This finishes my treatment of Theorem 1.8. $\square$

## §2. Two proofs.

It is now time to prove (1.7). The proof is based on:

__Lemma 2.1.__ The composite

$$\delta : K(\Omega^s P_s) \times X \xrightarrow{1 \times j_s} K(\Omega^s P_s) \times X_s \to X_s$$

induces a monomorphism in cohomology.

__Proof.__ This follows from a comparison of the "fundamental sequences" [7] associated to the vertical fibration sequences in the homotopy commutative diagram

$$\begin{tikzcd}
K(\Omega^s P_s)\times X \arrow[r] \arrow[d] & K(\Omega^s P_s)\times X_s \arrow[r, "\alpha"] \arrow[d] & X_s \arrow[d] \\
PK(\Omega^{s-1} P_s)\times X \arrow[r] \arrow[d] & PK(\Omega^{s-1} P_s)\times X_{s-1} \arrow[r, "pr_2"] \arrow[d] & X_{s-1} \arrow[d] \\
K(\Omega^{s-1} P_s) \arrow[r, "in_1"] & K(\Omega^{s-1} P_s)\times K(\Omega^{s-1} P_s) \arrow[r, "\mu"] & K(\Omega^{s-1} P_s),
\end{tikzcd}$$

by analogy with the proof of [7] 1.2.6. □

Lemma 2.2. The k-invariant $k_s$ may be characterized as the unique map $k$ such that (a) the diagram

$$\begin{tikzcd}
K(\Omega^s P_s)\times X_s \arrow[r, "\alpha"] \arrow[d, "d_s\times k"] & X_s \arrow[d, "k"] \\
K(\Omega^s P_{s+1})\times K(\Omega^s P_{s+1}) \arrow[r, "\mu"] & K(\Omega^s P_{s+1})
\end{tikzcd}$$

is homotopy commutative and (b) $kj_s : X \to K(\Omega^s P_{s+1})$ is null-homotopic.

Proof. The k-invariant $k_s$ satisfies (a) by virtue of primitivity of the principal action, [7], and (b) since $j_s$ lifts to $j_{s+1}$. On the other hand, (a) and (b) together allow one to compute that
$\delta^* k = d_{s+1} pr_1 : K(\Omega^s P_s)\times X \to K(\Omega^s P_{s+1})$; but $\delta^*$ is monic by Lemma 2.1. □

Proof of (1.7). It follows easily from the compatability of the splitting of the fundamental sequence with the k-invariant $k_{s-1}$ that $d_s k_{s-1} \simeq *$. Pick a null-homotopy $h$, and look at the commutative diagram

$$\begin{array}{ccccc} & & X_s & \xrightarrow{k} & K(\Omega^s P_{s+1}) \\ & \nearrow^{j_s} & \downarrow & & \downarrow \\ X & \xrightarrow{j_{s-1}} & X_{s-1} & \xrightarrow{h} & PK(\Omega^{s-1}P_{s+1}) \\ & & \downarrow k_{s-1} & & \downarrow \pi \\ & & K(\Omega^{s-1}P_s) & \xrightarrow{d_s} & K(\Omega^{s-1}P_{s+1}). \end{array}$$

Any such $k$ satisfies (a) of Lemma 2.2, as noted in the proof of Lemma 1.9. To complete the proof, it therefore suffices to alter $h$ to another null-homotopy $h_s$ such that the map $k'$ induced on homotopy fibers satisfies $k'j_s \simeq *$. Since $j_{s-1}$ is epic in cohomology, $kj_s$ factors as $\ell j_{s-1}$ for some $\ell : X_{s-1} \to K(\Omega^s P_{s+1})$. If $\chi$ reverses paths and $*$ juxtaposes them, then $h_s = \chi\ell * h$ has the desired property. □

Proof of Lemma 1.11. This is based on the following technical result about principal actions.

Proposition 2.3. Let $h$ be a null-homotopy of a composite $gf$, and construct homotopy fibers to produce a commutative diagram

$$\begin{array}{ccccc} & & F & \xrightarrow{k} & \Omega Z \\ & & \downarrow & & \downarrow \\ H & \to & X & \xrightarrow{h} & PZ \\ \downarrow \ell & & \downarrow f & & \downarrow \pi \\ G & \to & Y & \xrightarrow{g} & Z \end{array}$$

Then the homotopy fibers of $k$ and of $\ell$ are identical, and if we call this common space $E$, then the following diagram is homotopy commutative.

$$\begin{tikzcd} \Omega^2 Z\times E \arrow[r] \arrow[rr, bend left, "\Omega i\times 1"] \arrow[dr, "\simeq"'] & \Omega^2 Z\times\Omega G\times E \arrow[r, "\Omega\alpha_g\times 1"] \arrow[d, "1\times\alpha_\ell"] & \Omega G\times E \arrow[d, "\alpha_\ell"] \\ & \Omega^2 Z\times E \arrow[r, "\alpha_k"] & E \end{tikzcd}$$

To prove this proposition, draw pictures of the elements of the spaces involved; you will see that the homotopy required is similar to the one showing that a double loop space is homotopy commutative. It is convenient to remember that when $\Omega Z$ is regarded as the homotopy fiber of $\pi$, it maps to $PZ$ by sending a loop to the reverse of its second half.

By including $*$ into $\Omega G$, we find that $\alpha_k$ factors as $\alpha_\ell(\Omega i\times 1)$ where $i : \Omega Z \to G$ is the natural map. Since $i \circ \Omega g \simeq *$, this implies:

<u>Corollary</u> 2.4. Let $h$ be a null-homotopy of a composite $gf$, and construct homotopy fibers to produce a commutative diagram

$$\begin{tikzcd} E \arrow[r] & F \arrow[r, "k"] \arrow[d] & \Omega Z \arrow[d] \\ & X \arrow[r, "h"] \arrow[d, "f"] & PZ \arrow[d, "\pi"] \\ & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Then

$$\begin{tikzcd} \Omega^2 Y\times E \arrow[r, "\Omega^2 g\times 1"] \arrow[dr, "pr_2"'] & \Omega^2 Z\times E \arrow[d, "\alpha_k"] \\ & E \end{tikzcd}$$

is homotopy-commutative. □

Lemma 1.11 follows from an application of this Corollary to (1.7). $\square$

## §3. Applications.

To apply Theorem 1.8 when $Y$ is a suspension of the classifying space $BE$ of an elementary Abelian p-group, I recall from [9] a basic vanishing theorem.

Theorem 3.1. Let $M$ be an unstable left A-module of finite type. Then

$$\mathrm{Ext}^s(M, \bar{H}^*(\Sigma^n BE)) = 0$$

(a) for any $s > n \geq 0$ and for $s = n > 0$; and

(b) for any $s, n \geq 0$ if $M$ is finite.

Theorem C follows immediately from this and Theorem 1.8. Notice, by the way, that since

$$\pi_n(X^{BE}, *) = [\Sigma^n BE, X],$$

these theorems also imply:

Theorem 3.2. If $E$ is an elementary Abelian p-group and $X$ a simply connected space whose mod p cohomology is finite and very nice, then $X^{BE}$ is weakly contractible.

Moreover:

Theorem 3.3. The Sullivan conjecture is valid for elementary Abelian p-groups and spheres.

Proof. Since $[\Sigma^n BG, S^1] = \bar{H}^1(\Sigma^n BG; \mathbb{Z}) = 0$ for any $n \geq 0$ and any finite group $G$, the Sullivan conjecture for $G$ arbitrary is trivial for $X = S^1$. The case of $X = S^m$ for $m > 1$ with $m$ odd or $p = 2$ is covered by Theorem

3.1. The remaining case is dealt with using the following trick, which I owe to J. R. Harper.

Let $J_{p-1}S^{2k}$ denote the skeleton of the James construction on $S^{2k}$ for which $H^*(J_{p-1}S^{2k}) = U(S(2k))$. Let $F_{2k}$ be the homotopy fiber of the natural map $S^{2k} \to J_{p-1}S^{2k}$. Then an easy computation shows that $H^*(F_{2k}) = U(M)$ where

$$M = \langle x_{4k-1}, y_{(2pk-2)p^i} : i \geq 0 \rangle$$

with trivial A-action. Since Ext is additive, we find that

$$\mathrm{Ext}^s(M, \bar{H}^*(\Sigma^n BE)) = 0, \qquad n,s \geq 0,$$

and so, from Theorem 1.8, $F_k^{BE}$ is weakly contractible. Since $(J_{p-1}S^{2k})^{BE}$ is too, from Theorem 3.2, the result follows from the homotopy long exact sequence of a fibration. □

Many other spaces which are not $U(M)$'s may be handled by analogous tricks.

To prove Theorem B, let $M_k$ be the A-module generated over $\mathbb{F}_2$ by $\{x_i : i \geq k\}$, with $|x_i| = 2^i$ and $Sq^{2^i} x_i = x_{i+1}$. Then $U(M_2)$ is the unique A-algebra which as an $\mathbb{F}_2$-algebra is polynomial on a single 4-dimensional generator. Thus Theorem C shows that

$$[BE, X] \xrightarrow{\cong} \mathrm{Hom}_A(M_2, \bar{H}^* BE).$$

With $E = \mathbb{Z}/2$, the latter set clearly has order two, proving the first assertion. Note that $H^*(B\mathbb{Z}/2) \cong U(M_0)$, and that the nontrivial map $B\mathbb{Z}/2 \to X$ induces $U(i)$ in cohomology, where $i : M_2 \to M_0$ is the inclusion. Now the rest of Theorem B follows from the commutative diagram

$$\begin{array}{ccc} [BE,BZ/2] & \xrightarrow{\cong} & \mathrm{Hom}_A(M_0,\bar{H}^*(BE)) \\ f\downarrow & & i\downarrow\cong \\ [BE,X] & \xrightarrow{\cong} & \mathrm{Hom}_A(M_2,\bar{H}^*(BE)) \end{array}$$

in which the bottom arrow is iso by Theorem C. □

Theorem 3.4. In the situation of Theorem B, the component of $X^{BE}$ which contains the trivial map is weakly contractible.

Proof. There are short exact sequences

$$0 \to M_k \to M_0 \to M_0^{k-1} \to 0$$

of A-modules, with $M_0^{k-1}$ finite. The long exact sequence induced in $\mathrm{Ext}^*(-,\bar{H}^*(\Sigma^n BE))$, together with Theorem 3.1, shows that

$$\mathrm{Ext}^s(M_k,\bar{H}^*(\Sigma^n BE)) \to \mathrm{Ext}^s(M_0,\bar{H}^*(\Sigma^n BE))$$

is an isomorphism. But $M_0$ is projective in the category $\mathcal{U}^*$, so we conclude that for all $s > 0$, $n \geq 0$, and $k \geq 0$,

$$\mathrm{Ext}^s(M_k,\bar{H}^*(\Sigma^n BE)) = 0.$$

It is easy to see that for $n > 0$, this group is also zero when $s = 0$; so, putting $k = 2$, Theorem 1.8 gives

$$\pi_n(X^{BE},*) = [\Sigma^n BE,X] = 0. \qquad \square$$

It would be interesting to get information on the homotopy type of the other components of $X^{BE}$. When $X = BSU_2$, one may argue as follows. Since the center of $SU_2$ is $Z_2$, there is a group homomorphism $Z_2 \times SU_2 \to SU_2$, inducing a pointed map $BZ_2 \times BSU_2 \to BSU_2$. Pass to spaces of pointed maps from $BE$; the Abelian group $BZ_2^{BE}$ acts on $BSU_2^{BE}$. If $h : E \to Z_2$ is a homomorphism, then the action by $Bh$ provides a homotopy equivalence from the component of $BSU_2^{BE}$ containing the trivial map to the component containing $hf$, where $f : BZ_2 \to BSU_2$ is induced by the inclusion. This completes the proof of Theorem A.

§4. Convergence.

The final task is to prove a convergence theorem. While Massey and Peterson [8] did important work on this issue, it seems better to appeal to the now standard work of Bousfield and Kan [5]; so move to the simplicial framework by passing to singular simplicial sets. To relate a Massey-Peterson tower for $X$ to the p-adic completion $(\mathbb{Z}/p)_\infty X$ of [5], we have:

Lemma 4.1. Let $X$ be a simply-connected space such that $H^*(X)$ is of finite type and very nice. Let (1.5) be a Massey-Peterson tower for $X$. Then $\{X_i\}$ and $\{(\mathbb{Z}/p)_i X\}$ are weakly equivalent prosystems.

Proof. By [5] III §5.5, p. 84 and induction, each $X_i$ is $\mathbb{Z}/p$-nilpotent. By (1.6), the first image prosystem $\{\mathrm{Im}(H_*(X_i) \to H_*(X_{i-1}))\}$ is the constant system $\{H_*(X)\}$. Thus $\{X_i\}$ is a $\mathbb{Z}/p$-tower for $X$, so the result follows from [5] III §6.4, p. 88. □

According to [5] VIII §3, homotopy classes of maps agree in the categories of CW complexes and of simplicial sets; so the following theorem is sufficient for our purpose.

Theorem 4.2. Suppose that $X$ is connected and nilpotent and that $Y$ is connected with $\bar{H}_*(Y;\mathbb{Z}[\frac{1}{p}]) = 0$. Then the map $X \to (\mathbb{Z}/p)_\infty X$ induces an equivalence of pointed mapping spaces

$$X^Y \to ((\mathbb{Z}/p)_\infty X)^Y.$$

The statement of the theorem in this generality and the proof given here are both due to A. K. Bousfield, and I am grateful to him for allowing me to reproduce them.

Proof. Recall from [6] that there is, up to homotopy, a fiber square

$$\begin{array}{ccc} X & \to & \Pi_\ell X_{\mathbb{Z}/\ell} \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \to & (\Pi_\ell X_{\mathbb{Z}/\ell})_{\mathbb{Q}} \end{array}$$

where $X_A$ denotes the Bousfield $H_*(-;A)$-localization of $X$ [2]. Thus there is, up to homotopy, an analogous fiber square of pointed function spaces with source space $Y$. Now Proposition 12.2 of [2] easily implies that $C^B$ is contractible whenever $B$ is $h_*$-acyclic and $C$ is $h_*$-local. Taking $h_*(-) = H_*(-;\mathbb{Z}[\frac{1}{p}])$, it follows that $(Z_A)^Y \simeq *$ for any space $Z$, where $A = \mathbb{Q}$ or $A = \mathbb{Z}/\ell$ with $\ell$ prime to $p$. Thus the fiber square implies that the map

$$X^Y \to (X_{\mathbb{Z}/p})^Y$$

is an equivalence, and the proposition follows since $X_{\mathbb{Z}/p} \simeq (\mathbb{Z}/p)_\infty X$ by §4 of [2]. $\square$

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