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Looping Massey-Peterson towers

J. R. Harper and H. R. Miller

Introduction

In this paper we begin a study of the mod p cohomology of function spaces by means of an Adams resolution of the target space. For simplicity and concreteness we will require the source space to be a sphere, so we are dealing with iterated loop spaces. We will use the "classical" unstable Adams towers constructed by Massey and Peterson [4] and Barcus [1]. This technique restricts us to consideration of target spaces X with "very nice" cohomology: $H^*(X) = U(M)$ for some unstable module M over the Steenrod algebra $\mathcal{A}$, where U is the Steenrod-Epstein enveloping unstable $\mathcal{A}$-algebra functor. The class of such spaces includes Stiefel manifolds over $\mathbf{C}$ and $\mathbf{H}$ (and over $\mathbf{R}$ too if $p = 2$), but excludes nontrivial wedges and suspensions.

Our method is simply to compute, to the extent possible, the cohomology of the spaces in the k-fold loop space of a Massey-Peterson tower. By construction, the cohomology of each space E_s in such a tower for X surjects to $H^*(X)$. This fails for the looped tower, and there results a filtration of $H^*(\Omega^k X)$ in the category of $\mathcal{A}$-Hopf algebras. Our main result determines the associated quotient of $\mathcal{A}$-Hopf algebras, under rather restrictive connectivity assumptions, in terms of certain homological functors on the category $\mathcal{U}$ of unstable $\mathcal{A}$-modules. To describe these, let $\Omega^k M$ denote the maximal unstable quotient of the $\mathcal{A}$-module $\Sigma^{-k}M$. This is a right-exact functor from $\mathcal{U}$ to $\mathcal{U}$, and has left-derived functors Ω^k_s.

Theorem. Let X be a simply connected finite complex such that $H^*(X) = U(M)$ for some $M \in \mathcal{U}$. Let b and t be integers such that $M^i = 0$ unless $b < i \le t$, and assume that $k - 1 \le b - p^{-2}t$. Then there is a natural filtration of $H^*(\Omega^k X)$ by $\mathcal{A}$-Hopf algebras A_s such that $A_{-1} = \mathbf{F}_p$, $\bigcup A_s = H^*(\Omega^k X)$, and

$$A_{s+1}//A_s \cong U(\Omega\Omega^{s+k}_{s+1}M).$$

A more precise version of this theorem is stated in Section 3 as Theorem 3.9.

We remark that, while examples (like 4.10 below) show that some restrictions on M are necessary, the filtration by images of $H^*(\Omega^k E_s)$ seems to satisfy 3.9 in much more general circumstances than we have been able to prove here. For example, computations of W.M. Singer [5] show that $H^*(\Omega^k S^{n+k})$ has this form for all $k \geq 0$ and $n \geq 1$, at least for $p = 2$. Indeed, the derived functors he computes are precisely the ones entering into 3.9 if M is a k-fold suspension in $\mathcal{U}$.

We begin by reviewing certain elementary facts about derived functors of Ω^k. In Sections 2 and 3 we give a self-contained account of the Massey-Peterson-Barcus theory, using the Eilenberg-Moore spectral sequence. Finally, in Section 4, we prove the theorem.

We adopt the following convention on primes. When the case $p = 2$ can be handled by the substitution $\beta = \mathrm{Sq}^1$, $P^i = \mathrm{Sq}^{2i}$, we make no further comment; only when something special happens for $p = 2$ do we take notice.

Most of the work on this paper was done before 1980. The intervening years have seen the exploitation by Jean Lannes and others of more sophisticated variants of this approach, but using source spaces with more convenient cohomology. We offer our apologies for the long delay in publication, and express our gratitude for the opportunity to submit it to these proceedings in honor of Ioan James.

1 Some algebra

We begin with the algebraic loop functor Ω and its derived functors [4,5]. For any unstable $\mathcal{A}$-module M, define a vector space ΦM by

$$\begin{aligned} (\Phi M)^{2pn} &= M^{2n}, \\ (\Phi M)^{2pn+2} &= M^{2n+1}, \\ (\Phi M)^i &= 0 \quad \text{otherwise.} \end{aligned}$$

Write $\overline{x}$ for the element of ΦM corresponding to $x \in M$. It is direct to check that

1.1 ΦM becomes an unstable $\mathcal{A}$-module if we declare

$$\begin{aligned} &P^{pj}\overline{x} = \overline{P^j x}, \\ &P^{pj+1}\overline{x} = \overline{\beta P^j x} \quad \text{if } |x| \text{ is odd,} \\ &\beta^\epsilon P^j \overline{x} = 0 \quad \text{otherwise.} \end{aligned}$$

1.2 The map $\lambda\colon \Phi M \to M$ defined by

$$\lambda\overline{x} = \begin{cases} P^n x & \text{if } |x| = 2n, \\ \beta P^n x & \text{if } |x| = 2n+1, \end{cases}$$

is $\mathcal{A}$-linear, and its kernel and cokernel are suspensions in $\mathcal{U}$.

Define functors Ω and Ω_1 by means of the resulting natural exact sequence

$$0 \longrightarrow \Sigma\Omega_1 M \longrightarrow \Phi M \xrightarrow{\lambda} M \longrightarrow \Sigma\Omega M \longrightarrow 0. \tag{1.3}$$

Since Φ is exact and λ is monic on projectives, standard homological methods imply that Ω_1 is the first left derived functor of Ω and that Ω has no higher derived functors.

The functor $\Omega: \mathcal{U} \to \mathcal{U}$ is right adjoint to Σ. Since the Σ is exact, Ω carries projectives to projectives. Consequently there is a Grothendieck spectral sequence associated to the composite $\Omega^j \Omega^k = \Omega^{j+k}$:

$$\Omega^j_s \Omega^k_t M \overset{s}{\Longrightarrow} \Omega^{j+k}_{s+t} M. \tag{1.4}$$

In particular, with $j = 1$, we obtain the *Singer exact sequence* [5]

$$0 \longrightarrow \Omega\Omega^k_s M \longrightarrow \Omega^{k+1}_s M \longrightarrow \Omega_1 \Omega^k_{s-1} M \longrightarrow 0. \tag{1.5}$$

One sees also by induction that

$$\Omega^k_s M = 0 \quad \text{if } s > k, \quad \text{and} \tag{1.6}$$

$$\Omega^s_s M = (\Omega_1)^s M. \tag{1.7}$$

We will need estimates on the connectivity and coconnectivity of $\Omega^{s+k}_s M$. If N is a graded vector space let

$$\begin{aligned} \operatorname{conn} N &= \min\{i : N_i \neq 0\} - 1 \\ \operatorname{coconn} N &= \max\,\{i : N_i \neq 0\}. \end{aligned}$$

1.8 Lemma. *If $M \in \mathcal{U}$, then*

$$\begin{aligned} \operatorname{conn} \Omega^{s+k}_s M &\geq p^s(\operatorname{conn} M - k), \\ \operatorname{coconn} \Omega^{s+k}_s M &\leq p^s \operatorname{coconn} M - k. \end{aligned}$$

Proof. From (1.3) we have

$$\begin{aligned} \operatorname{conn} \Omega_1 M &\geq p \operatorname{conn} M, \\ \operatorname{coconn} \Omega_1 M &\leq p \operatorname{coconn} M. \end{aligned}$$

(If p is odd, both inequalities can be improved by dividing according to parity, and in any case the second can be improved by at least 1. But these suffice for our purposes.) (1.7) then gives the result if $k = 0$. If $s = 0$ the result is clear, and the rest follows by induction using the Singer sequence (1.5). Notice that the cokernel determines the connectivity and the kernel determines the coconnectivity. ∎

The above algebra is connected with the geometry by the contravariant functor $K(\,)$ which associates to any projective unstable $\mathcal{A}$-module F of finite type a generalized Eilenberg-MacLane space $K(F)$ with homotopy $\pi_t K(F) \cong \operatorname{Hom}_{\mathcal{A}}(F, \Sigma^t \mathbf{F}_p)$. Then $H^*(F) \cong U(F)$; and to a morphism $f: F' \to F$ we associate the unique H-map

$K(f): K(F) \to K(F')$ inducing $U(f)$ in cohomology. The functor K is in a certain sense adjoint to H^*: (spaces) $\to \mathcal{U}$:. there is a natural bijection

$$[X, K(F)] \cong \mathrm{Hom}_{\mathcal{U}}(F, H^*(X)).$$

There is a natural homotopy equivalence $K(\Omega^k F) \simeq \Omega^k K(F)$, so $H^*(\Omega^k K(F)) \cong U(\Omega^k F)$. For more general spaces X with $H^*(X) = U(M)$, $H^*(\Omega^k X)$ is much larger than $U(\Omega^k M)$; the isomorphism in this case is, as we shall see, a reflection of the triviality of the higher derived functors of Ω^{k+s} on F.

2 The cohomology theory of induced fibrations

The basic homological tool in the study of pull backs of fibrations is the Eilenberg-Moore spectral sequence. Suppose we have a pull-back diagram

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "p"] & PB_0 \arrow[d, "p_0"] \\ B \arrow[r, "f"] & B_0 \end{tikzcd} \tag{2.1}$$

with p_0 the path-loop fibration. Under certain restrictions, an analysis of (2.1) was carried out by Massey and Peterson [3] and by Barcus [1], using the Serre spectral sequence. In this section we offer a proof of a variant of their results, using the Eilenberg-Moore machinery. We state all our results and comment on them first, and then sketch proofs.

To express the result conveniently we need the following universal construction. Given an unstable $\mathcal{A}$-algebra R and a map $g: R \to G$ of unstable $\mathcal{A}$-modules, we seek the initial unstable $\mathcal{A}$-algebra B accepting an $\mathcal{A}$-linear map from G such that the composite $R \to B$ is an $\mathcal{A}$-algebra homomorphism. Write $\overline{U}_R(G)$ for this B; it is a reduced version of a construction denoted U_R by Massey and Peterson. It is easily seen that if $U(R) \to R$ is the unique $\mathcal{A}$-algebra map extending the identity map $R \to R$, then

$$R \otimes_{U(R)} U(G) \xrightarrow{\cong} \overline{U}_R(G).$$

From this we see:

2.2 Lemma. *If g is monic then so is $R \to \overline{U}_R(G)$; and the extension*

$$0 \longrightarrow R \longrightarrow G \longrightarrow Q \longrightarrow 0$$

in $\mathcal{U}$ induces an extension

$$\mathsf{F}_p \longrightarrow R \longrightarrow \overline{U}_R(G) \longrightarrow U(Q) \longrightarrow \mathsf{F}_p$$

of $\mathcal{A}$-algebras: $\overline{U}_R(G)$ is free over R, and $U(Q) = \overline{U}_R(G) \otimes_R \mathsf{F}_p = \overline{U}_R(G)//R$. ∎

2.3 **Theorem.** *Consider the fiber square*

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & PB_0 \arrow[d] \\ B \arrow[r, "f"] & B_0. \end{tikzcd}$$

Assume that B and B_0 are simply connected, that $H^*(B)$ and $H^*(B_0)$ are of finite type, and:

(i) $H^*(B)$ is a free module over $\operatorname{Im} f^*$.

Assume further that there is given a sub $\mathcal{A}$-module Q of $H^*(B_0)$ such that

(ii) $\Omega_1 Q = 0$;

(iii) $H^*(B_0)$ is a free module over $U(Q)$; and

(iv) $\ker f^*$ is the ideal generated by Q.

Let $R = H^*(B) \otimes_{H^*(B_0)} \mathsf{F}_p$. There is then a short exact sequence in $\mathcal{U}$, the "fundamental sequence"

$$0 \longrightarrow R \longrightarrow G \longrightarrow \Omega Q \longrightarrow 0, \tag{2.4}$$

and an extension $i\colon G \to H^*(E)$ of the natural $\mathcal{A}$-algebra map $R \to H^*(E)$, such that the induced map

$$\overline{U}_R(G) \longrightarrow H^*(E)$$

is an isomorphism, and

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \Omega Q \arrow[d, "\omega"] \\ H^*(E) \arrow[r, "\partial^*"] & H^*(\Omega B_0) \end{tikzcd} \tag{2.5}$$

commutes. Here ∂ is the Barratt-Puppe boundary map and ω is the natural "suspension" map.

Thus R embeds in $H^*(E)$, so $R \cong \operatorname{Im} p^*$.

As a special case, take $B = *$, so $E = \Omega B_0$. The theorem yields the

2.6 **Corollary.** *Let X be a simply connected space such that $H^*(X)$ is of finite type and $H^*(X) = U(Q)$, $\Omega_1 Q = 0$. Then the suspension map induces an isomorphism*

$$U(\Omega Q) \xrightarrow{\cong} H^*(\Omega X).$$

∎

If X is k-connected and $\Omega_1 \Omega^i Q = 0$ for all i with $0 \leq i < k$, then by induction we have

$$U(\Omega^k Q) \xrightarrow{\cong} H^*(\Omega^k X).$$

The Singer sequence (1.5) shows that this condition is in fact equivalent to requiring $\Omega^{s+k}_{s+1}Q = 0$ for all $s \geq 0$; so our main theorem holds in this case without dimension assumptions.

2.7 Example. Typically one obtains the conditions of Theorem 2.3 by starting with B_0 such that $H^*(B_0) = U(N)$ for some $N \in \mathcal{U}$ for which $\Omega_1 N = 0$; for instance, $B_0 = K(N)$ for N projective in $\mathcal{U}$. Any $Q \subseteq N$ then has $\Omega_1 Q = 0$ as well, since Ω_1 is left exact. Note that $U(N)$ is an $\mathcal{A}$-Hopf algebra, with $PU(N) = N$. Now suppose that we can give $H^*(B)$ a Hopf algebra structure such that f^* is a Hopf algebra map. Then condition (i) of Theorem 2.3 is automatic. Take

$$Q = N \cap \ker f^* = \ker Pf^*. \tag{2.8}$$

Then we obtain (ii) of Theorem 2.3 as noted; (iii) holds because $U(Q) \to H^*(B_0)$ is monic, being a Hopf algebra map which is monic on primitives; and (iv) holds since $U(Q)$ is the Hopf algebra kernel $\operatorname{Ker} f^* = H^*(B_0) \,\square_{H^*(B)} \mathbb{F}_p$.

The Hopf condition on f^* usually occurs in one of two ways:

(a) $H^*(B) = U(M)$ for some $M \in \mathcal{U}$, and the map f^* induced by $f\colon B \to B_0$ fits into a diagram

$$\begin{tikzcd} U(M) \arrow[d, "\cong"'] & U(N) \arrow[l, "U(\phi)"'] \arrow[d, "\cong"'] \\ H^*(B) & H^*(B_0) \arrow[l, "f^*"'] \end{tikzcd}$$

for an $\mathcal{A}$-module map ϕ.

(b) B and B_0 are homotopy-associative H-spaces such that the Hopf algebra structure on $H^*(B_0) = U(N)$ has N for primitives, and f is an H-map. If $B_0 = K(N)$, the standard H-structure on B_0 will do, and f corresponds to a primitive cohomology class in B.

2.9 Remark. In case (b) above,

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & PB_0 \arrow[d] \\ B \arrow[r, "f"] & B_0 \end{tikzcd}$$

is a Hopf fiber square. We have seen that $R \cong \operatorname{Im} p^*$. We can identify other elements of the situation as follows:

(i) The Hopf algebra kernel $\operatorname{Ker} f^* = H^*(B_0) \,\square_{H^*(B)} \mathbb{F}_p = U(Q)$.

(ii) $\operatorname{Im} f^* = \operatorname{Ker} p^*$.

(iii) $\operatorname{Ker} \partial^*$ has a description in terms of a fundamental sequence. The diagram (2.5) shows that ∂^* factors as

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is an H-map. If $B_0 = K(N)$,
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ntal sequence. The diagram

$$
\begin{tikzcd}
H^*(E) \arrow[rr, "\partial^*"] & & H^*(\Omega B_0) \\
\overline{U}_R(G) \arrow[u, "\cong"] \arrow[r, "q"] & U(\Omega \ker Pf^*) \arrow[r, "U(\Omega k)"] & U(\Omega N) \arrow[u, "\cong"']
\end{tikzcd}
$$

where $k: \ker Pf^* \to N$ is the inclusion. By Lemma 2.2 the map q is epic with Hopf algebra kernel R, but $U(\Omega k)$ is not generally monic; indeed, there is an exact sequence

$$0 \longrightarrow \Omega_1 \operatorname{Im} Pf^* \xrightarrow{\delta} \Omega \ker Pf^* \xrightarrow{\Omega k} \Omega N \longrightarrow \Omega \operatorname{Im} Pf^* \longrightarrow 0.$$

If we form the pull-back diagram

$$
\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, "="'] & K \arrow[r] \arrow[d] & \Omega_1 \operatorname{Im} Pf^* \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & R \arrow[r] & G \arrow[r] & \Omega \ker Pf^* \arrow[r] & 0
\end{tikzcd}
$$

then we see that the top row is a fundamental sequence for

$$\operatorname{Ker} \partial^* \cong \overline{U}_R(K).$$

It will be important for us to compare the fundamental sequence for E against that for ΩE. For this purpose we will write $Q(f)$ for the choice of Q, and agree to take $Q(\Omega f) = \ker P(\Omega f)^*$ as in (2.8). Then the suspension $\omega: \Omega H^*(B_0) \to H^*(\Omega B_0)$ automatically maps $\Omega Q f$ to $Q(\Omega f)$. We will also write $R(f) = H^*(B) \otimes_{H^*(B_0)} \mathbb{F}_p$, etc.

Theorem 2.3 has the

2.10 Addendum. *Suppose $f: B \to B_0$ and $Q(f)$ satisfy the conditions of Theorem 2.3, that B and B_0 are 2-connected, and that $\Omega_1 Q(\Omega f) = 0$. Then there is $\omega: \Omega G(f) \to G(\Omega f)$ such that*

$$
\begin{tikzcd}
 & \Omega R(f) \arrow[r] \arrow[d, "\omega"'] & \Omega G(f) \arrow[r] \arrow[d, "\omega"'] & \Omega^2 Q(f) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R(\Omega f) \arrow[r] & G(\Omega f) \arrow[r] & \Omega Q(\Omega f) \arrow[r] & 0
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
\Omega G(f) \arrow[r] \arrow[d, "\omega"'] & \Omega H^*(E) \arrow[d, "\omega"'] \\
G(\Omega f) \arrow[r] & H^*(\Omega E)
\end{tikzcd}
$$

commute.

We turn to a sketch of proofs. Write E_0 for PB_0. Let E_{0T} be the mapping cylinder of the projection map $p_0\colon E_0 \to B_0$. Then $E_0 \to B_0$ factors canonically as $E_0 \rightarrowtail E_{0T} \xrightarrow{p_0'} B_0$, and p_0' is a homotopy equivalence. Now $E_0 \to E_{0T}$ is a cofibration over B inducing a surjection in cohomology. (This is the general assumption made in [3] by Massey and Peterson; here it is trivial since $E_0 \simeq *$.) Moreover $H^*(E_{0T}) \cong H^*(B_0)$ is a projective $H^*(B_0)$-module. It follows that the following diagram is the beginning of an exact couple yielding the Eilenberg-Moore spectral sequence (cf. Smith [6]):

$$\begin{tikzcd}
H^*(B\times_{B_0}E_0) \arrow[rr, "\delta"] & & H^*(B\times_{B_0}(E_{0T},E_0)) \arrow[dl] \arrow[rr, "\delta"] & & \cdots \arrow[dl] \\
& H^*(B\times_{B_0}E_{0T}) \arrow[ul] & & H^*(B\times_{B_0}B_0\times(E_{0T},E_0)). \arrow[ul] &
\end{tikzcd}$$

Rewriting this, we have

$$\begin{tikzcd}
H^*(E) \arrow[rr, "\delta"] & & H^*(E_T,E) \arrow[dl] \arrow[rr, "\delta"] & & \cdots \arrow[dl] \\
& H^*(B) \arrow[ul, "p^*"] \arrow[d, equal] & & H^*(B)\otimes H^*(E_{0T},E_0) \arrow[ul, "p_1^*"'] \arrow[d, equal] & \\
& E_1^0 & & E_1^{-1}. \arrow[ll, "d_1"'] &
\end{tikzcd} \tag{2.11}$$

Therefore, in the Eilenberg-Moore filtration of $H^*(E)$,

$$F^0 = \ker\delta = \operatorname{Im} p^*,$$
$$F^{-1} = \ker\delta^2 = \delta^{-1}\operatorname{Im} p_1^*.$$

If we identify $H^*(E_{0T},E_0) \cong \bar{H}^*(B_0)$, then $d_1(b\otimes b_0) = b\cdot f^*b_0$. Since $Q\subset \ker f^*$, the map $\Sigma^{-1}Q \to E_1^{-1}$ sending b_0 to $1\otimes b_0$ carries $\Sigma^{-1}Q$ to $\ker(d_1|E_1^{-1})$. Since $E_\infty^{-1}\in\mathcal{U}$, there is a factorization

$$\begin{tikzcd}
\Sigma^{-1}Q \arrow[r, two heads] \arrow[d] & \Omega Q \arrow[d] \\
\ker(d_1|E_1^{-1}) \arrow[r, two heads] & E_\infty^{-1}.
\end{tikzcd} \tag{2.12}$$

Thus we may form the pull-back

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & \Omega Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^0 \arrow[r] & F^{-1} \arrow[r] & E_\infty^{-1} \arrow[r] & 0.
\end{tikzcd} \tag{2.13}$$

The top sequence here is the fundamental sequence. The composite $G \to F^{-1} \rightarrowtail H^*(E)$ is the map i in the statement of Theorem 2.3. The composite $R \to G \to H^*(E)$ is an $\mathcal{A}$-algebra homomorphism, induced by p^*: so we get a map $\bar{i}: \overline{U}_R(G) \to H^*(E)$. To show it is an isomorphism, we filter $\overline{U}_R(G)$. First filter $U(Q)$ by setting $F^0 = \mathsf{F}_p$, $F^{-1} = \mathsf{F}_p \oplus Q$, and $F^{-n} = (F^{-1})^n$; thus $F^{-p} = U(Q)$. Then let $F^{-n}\overline{U}_R(G)$ be the inverse image of $F^{-n}U(Q)$ under the natural map. It is then clear that $\bar{i}$ is filtration preserving when $H^*(E)$ is given the Eilenberg-Moore filtration. We will show that $E_0^*\bar{i}$ is an isomorphism.

Before we carry out this computation, we prove the naturality statements (2.5) and Addendum 2.10. The boundary map $\partial: \Omega B_0 \to E$ extends to a map of squares

$$\begin{array}{ccccccc} \Omega B_0 & \longrightarrow & E_0 & & E & \longrightarrow & E_0 \\ \downarrow & & \downarrow & \longrightarrow & \downarrow & & \downarrow \\ * & \longrightarrow & B_0 & & B & \longrightarrow & B_0, \end{array}$$

so compatibility with ∂ follows by naturality of the Eilenberg-Moore spectral sequence.

As for 2.10, note that the cohomology suspension map ω is induced from a natural transformation $\Sigma\Omega \to \mathrm{id}$. The relevant portion of the exact couple for the square with classifying map f is induced in H^* by

$$\begin{tikzcd} E \arrow[dr] & & (E_T, E) \arrow[dr] & \\ & E_T \arrow[ur] & & B \times (E_{0T}, E_0). \end{tikzcd} \tag{2.14}$$

The corresponding diagram for Ωf is

$$\begin{tikzcd} \Omega E \arrow[dr] & & (\Omega E_T, \Omega E) \arrow[dr] & \\ & \Omega E_T \arrow[ur] & & \Omega B \times (\Omega E_{0T}, \Omega E_0). \end{tikzcd} \tag{2.15}$$

The suspension of (2.15) maps to (2.14), and this leads to the desired naturality.

We now compute

$$E_2^* = \mathrm{Tor}^*_{H^*(B_0)}(H^*(B), \mathsf{F}_p)$$

in the Eilenberg-Moore spectral sequence. Let $A = \mathrm{Im}\, f^*$. Since $H^*(B)$ is a free A-module, the extension spectral sequence

$$\mathrm{Tor}^*_A(H^*(B), \mathrm{Tor}^*_{H^*(B_0)}(A, \mathsf{F}_p)) \Rightarrow \mathrm{Tor}^*_{H^*(B_0)}(H^*(B), \mathsf{F}_p)$$

collapses. Since $H^*(B_0)$ is free over $U(Q)$ and $A \cong H^*(B_0) \otimes_{U(Q)} \mathsf{F}_p$,

$$\mathrm{Tor}^*_{H^*(B_0)}(A, \mathsf{F}_p) \cong \mathrm{Tor}^*_{U(Q)}(\mathsf{F}_p, \mathsf{F}_p).$$

There is thus an $\mathcal{A}$-algebra isomorphism

$$R \otimes \mathrm{Tor}^{*}_{U(Q)}(\mathsf{F}_p, \mathsf{F}_p) \xrightarrow{\cong} \mathrm{Tor}^{*}_{H^{*}(B_0)}(H^{*}(B), \mathsf{F}_p). \tag{2.16}$$

Suppose first $p = 2$. The assumption $\Omega_1 Q = 0$ implies that $U(Q)$ is polynomial with $\mathrm{Tor}^{-1}_{U(Q)}(\mathsf{F}_2, \mathsf{F}_2) \cong \Sigma\Omega Q$ as $\mathcal{A}$-modules, so $\mathrm{Tor}^{*}_{H^{*}(B_0)}(H^{*}(B), \mathsf{F}_p) \cong R \otimes E[\sigma^{-1}\Sigma\Omega Q]$. By multiplicativity, $E_2 = E_\infty$. The map $\bar{i}$ is thus an isomorphism in this case.

Now take $p > 2$. By 1.1, the sub vector space $\Phi^{+}Q$ of ΦQ of elements of degree divisible by $2p$ forms a sub $\mathcal{A}$-module. Since $\Omega_1 Q = 0$ we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \Phi^{+}Q & \longrightarrow & \Phi Q & \longrightarrow & \Phi^{-}Q \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow{\scriptstyle\lambda} & & \\
 & & & & Q & \xrightarrow{=} & Q & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Phi^{-}Q & \xrightarrow{\lambda^{-}} & IQ & \longrightarrow & \Sigma\Omega Q & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

which defines IQ and $\Phi^{-}Q$ in $\mathcal{U}$. Then as an algebra $U(Q)$ is free-commutative and $\mathrm{Tor}^{-1}_{U(Q)}(\mathsf{F}_p, \mathsf{F}_p) = I(Q)$ as $\mathcal{A}$-modules. Therefore $\mathrm{Tor}^{*}_{U(Q)}(\mathsf{F}_p, \mathsf{F}_p)$ is the free algebra with divided powers generated by $I(Q)$. The pth divided power $\gamma_p\colon \mathrm{Tor}^{-1}_{U(Q)}(\mathsf{F}_p, \mathsf{F}_p) \to \mathrm{Tor}^{-p}_{U(Q)}(\mathsf{F}_p, \mathsf{F}_p)$ factors through as an embedding

$$\Sigma^{p-2}\Phi^{-}Q \longrightarrow \mathrm{Tor}^{-p}_{U(Q)}(\mathsf{F}_p, \mathsf{F}_p)$$

sending $\bar{x}$ to $\gamma_p[x]$. Under this correspondence, the differential d_{p-1} satisfies

$$d_{p-1}\gamma_n[x] = \gamma_{n-p}[x] \cdot [\lambda^{-}\bar{x}] \tag{2.17}$$

up to a unit, for $n \geq p$. This follows from the fact that the divided powers in Tor are natural, so we can use them to compute the effect on Tor of the map of squares

$$\begin{array}{ccccccc}
E & \longrightarrow & E_0 & & K_{2n} & \longrightarrow & PK_{2n+1} \\
\downarrow & & \downarrow & \longrightarrow & \downarrow & & \downarrow \\
B & \longrightarrow & B_0 & & * & \longrightarrow & K_{2n+1}
\end{array}$$

to the appropriate universal example; here $|x| = 2n+1$ and $K_m = K(\mathbf{Z}/p, m)$. There, (2.17) is known ([6], Prop. 4.4, page 85).

The effect of d_{p-1} is that E_p^{*} contains no generators in homological degree less than -1, so $E_p^{*} = E_\infty^{*}$; and $E_p^{*} = E^0\overline{U}_R(G)$, so as before we conclude that $\overline{U}_R(G) \cong H^{*}(E)$. ∎

3 Massey-Peterson towers

We review the construction of a tower of fibrations associated to a simply connected space X with $H^*(X) = U(M)$ and a projective resolution

$$M \xleftarrow{\epsilon} P_0 \xleftarrow{d_0} P_1 \xleftarrow{d_1} P_2 \longleftarrow \cdots \tag{3.1}$$

in $\mathcal{U}$. Both $H^*(X)$ and P_s will be assumed to be of finite type, and P_s should be zero in degree $d \leq s+1$.

The tower will have the form

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & E_{s+1} \arrow[d, "p_s"] & \\
X \arrow[ur, "q_{s+1}"] \arrow[r, "q_s"] \arrow[ddr] & E_s \arrow[r, "f_s"] \arrow[d] & K(\Omega^s P_{s+1}) \\
 & \vdots \arrow[d] & \\
K(P_0) = & E_0 \arrow[r, "f_0"] & K(P_1)
\end{tikzcd} \tag{3.2}$$

where p_s is the pull-back under the "k-invariant" f_s of the path-loop fibration; and if $\partial_s: K(\Omega^s P_s) \to E_s$ is the Barratt-Puppe boundary map, then

$$f_s \circ \partial_s \simeq K(\Omega^s d_s). \tag{3.3}$$

Begin by constructing $q_0: X \to K(P_0)$ such that $H^*(q_0) = U(\epsilon)$. Since $\epsilon d_0 = 0$, q_0 lifts to q_1. By Theorem 2.3 we have a diagram

$$\begin{tikzcd}
 & & & \Omega P_2 \arrow[d, "\Omega d_1'"] & \\
0 \arrow[r] & R(f_0) \arrow[r] \arrow[d, "\cong"'] & G(f_0) \arrow[r] \arrow[d, "i"] & \Omega \ker d_0 \arrow[r] \arrow[d] & 0 \\
 & H^*(X) & H^*(E_1) \arrow[l, "q_1^*"'] \arrow[r, "\partial_1^*"] & H^*(K(\Omega P_1)) &
\end{tikzcd} \tag{3.4}$$

in which $R(f_0) = H^*(K(P_0)) \otimes_{H^*(K(P_1))} \mathbb{F}_p \cong U(M) \cong H^*(X)$ and $\overline{U}_{R(f_0)}(G(f_0)) \xrightarrow{\cong} H^*(E_1)$. Thus $q_1^* i$ splits the fundamental sequence for E_1.

It follows that $H^*(E_1) \cong U(M \oplus \Omega \ker d_0)$ as $\mathcal{A}$-algebras, in such a way that the diagram

$$
\begin{array}{ccccc}
H^*(X) & \xleftarrow{q_1^*} & H^*(E_1) & \xrightarrow{\partial_1^*} & H^*(K(\Omega P_0)) \\
\uparrow & & \uparrow & & \uparrow \\
U(M) & \longleftarrow & U(M \oplus \Omega \ker d_0) & \longrightarrow & U(\Omega P_0)
\end{array}
\tag{3.5}
$$

commutes. Here the bottom horizontal arrows are the obvious ones.

Let $\sigma: \Omega \ker d_0 \to M \oplus \Omega \ker d_0$ include the summand, and let $\Omega d_1': \Omega P_2 \to \Omega \ker d_0$ be induced from d_1. Then there exists a unique map $f_1: E_1 \to K(\Omega P_2)$ such that $U(\sigma \Omega d_1') = H^*(f_1)$. It follows that $f_1 q_1 = *$ and that $f_1 \partial_1 = K(\Omega d_1)$.

To carry on we must know $R(f_1) = H^*(E_1) \otimes_{H^*(K(\Omega P_2))} \mathbb{F}_p$. For this we note the exact sequence

$$\Omega^{s-1} P_{s+1} \xrightarrow{d_s'} \ker \Omega^{s-1} d_{s-1} \longrightarrow \Omega_s^{s-1} M \longrightarrow 0$$

defining the derived functor, and remember that $\Omega_s^{s-1} = 0$ (1.6). Since Ω is right exact,

$$\Omega d_s': \Omega^s P_{s+1} \longrightarrow \Omega \ker \Omega^{s-1} d_{s-1} \tag{3.6}$$

is surjective. Thus (taking $s = 1$) tensoring over $H^*(K(\Omega P_2)) = U(\Omega P_2)$ is the same as tensoring over $U(\Omega \ker d_0)$; and that gives us

3.7 *q_1 induces an isomorphism $R(f_1) \cong U(M)$.*

The succeeding inductive steps are identical, and yield

3.8 **Theorem** [4]. *There exists a tower (3.2) in which (3.3) holds and such that*

$$
\begin{array}{ccccc}
H^*(X) & \xleftarrow{q_s^*} & H^*(E_s) & \xrightarrow{\partial_s^*} & H^*(K(\Omega^s P_{s-1})) \\
\uparrow \cong & & \uparrow \cong & & \uparrow \cong \\
U(M) & \longleftarrow & U(M \oplus \Omega^s \ker d_{s-1}) & \longrightarrow & U(\Omega^s P_{s-1})
\end{array}
$$

commutes.

Here we have slipped in the identification

$$\Omega^{s-1} \ker d_{s-1} \xrightarrow{\cong} \ker \Omega^{s-1} d_{s-1},$$

which follows from the exact sequence

$$0 \longrightarrow \Omega_s^{s-1} M \longrightarrow \Omega^{s-1} \ker d_{s-1} \longrightarrow \Omega^{s-1} P_s \xrightarrow{\Omega^{s-1} d_{s-1}} \Omega^{s-1} P_{s-1}$$

(see e.g. [2, page 93]) together with the fact that $\Omega_s^{s-1} M = 0$. A generalization of this identification, Lemma 4.5 below, is a key element to our approach and explains the form of the associated quotients in our main theorem, which we now restate in more precise form.

$$H^*(K(\Omega P_0)) \uparrow U(\Omega P_0) \qquad (3.5)$$

e obvious ones.

nd, and let $\Omega d_1': \Omega P_2 \to \Omega \ker d_0$

up $f_1: E_1 \to K(\Omega P_2)$ such that

$f_1 \partial_1 = K(\Omega d_1)$.

$(\Omega P_2))$ $\mathbb{F}_p$. For this we note the

$\longrightarrow 0$

$^{-1} = 0$ (1.6). Since Ω is right

(3.6)

$K(\Omega P_2)) = U(\Omega P_2)$ is the same

$M)$.

ield

2) in which (3.3) holds and such

$$\xrightarrow{p_s^*} H^*(K(\Omega^s P_{s-1})) \overset{\cong}{\uparrow} \to U(\Omega^s P_{s-1})$$

$$^{-1}P_s \xrightarrow{\Omega^{s-1} d_{s-1}} \Omega^{s-1} P_{s-1}$$

$M = 0$. A generalization of this our approach and explains the , which we now restate in more

3.9 Theorem. *Let $M \in \mathcal{U}$ be of finite type, and let X be a simply connected space with $H^*(X) = U(M)$. Choose a projective resolution $M \longleftarrow P_*$ in $\mathcal{U}$ and construct the associated Massey-Peterson tower (3.2). For any*

$$k \leq \operatorname{conn} M - p^{-2} \operatorname{coconn} M + 1,$$

$$\operatorname{Im}(H^*(\Omega^k E_s) \longrightarrow H^*(\Omega^k X)) = \operatorname{Im}(H^*(\Omega^k E_s) \longrightarrow H^*(\Omega^k E_{s+1})),$$

and is independent of the resolution. If we call this Hopf algebra A_s, then there is a fundamental sequence giving rise to a Hopf algebra extension sequence over $\mathcal{A}$

$$\mathbb{F}_p \longrightarrow A_s \longrightarrow A_{s+1} \longrightarrow U(\Omega\Omega_{s+1}^{s+k} M) \longrightarrow \mathbb{F}_p.$$

4 Looping the tower: proof of the main theorem

Begin again as in Section 3, but assume X is $(k+1)$-connected and P_s is $(s+k)$-connected. We wish to compute $H^*(\Omega^k E_s)$ and the map $\Omega^k p_s^*: H^*(\Omega^k E_s) \to H^*(\Omega^k E_{s+1})$. Start with the fundamental sequence for $\Omega^k E_1$: by 2.10, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^k U(M) & \longrightarrow & \Omega^k G(f_0) & \longrightarrow & \Omega^{k+1} \ker d_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \Omega\tau} & & \\ 0 & \longrightarrow & U(\Omega^k M) & \longrightarrow & G(\Omega^k f_0) & \longrightarrow & \Omega \ker \Omega^k d_0 & \longrightarrow & 0. \end{array} \qquad (4.1)$$

The bottom left term, $R(\Omega^k f_0)$, is as displayed because Ω^k is right exact. Notice that $R(\Omega^k f_0) \to H^*(\Omega^k E_1)$ is a Hopf algebra map and that $PR(\Omega^k f_0) = \Omega^k M$ is zero in degrees greater than $\operatorname{coconn} M - k$.

We next state the inductive assumptions:

4.2 (i)$_s$ The fundamental sequence for $H^*(\Omega^k E_s)$ is given by the bottom row of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega^k R(f_{s-1}) & \to & \Omega^k G(f_{s-1}) & \to & \Omega^{s+k} \ker d_{s-1} & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \Omega\tau} & & \\ 0 & \to & R(\Omega^k f_{s-1}) & \to & G(\Omega^k f_{s-1}) & \to & \Omega \ker \Omega^{s+k-1} d_{s-1} & \to & 0 \end{array}$$

in which $\Omega\tau$ is the natural map.

(ii)$_s$ There is a Hopf algebra fundamental sequence for $R(\Omega^k f_{s-1})$ of the form

$$0 \longrightarrow R(\Omega^k f_{s-2}) \longrightarrow G_{s-1} \longrightarrow \Omega\Omega_{s-1}^{s+k-2} M \longrightarrow 0.$$

(iii)$_s$ $\operatorname{coconn} PR(\Omega^k f_{s-1}) \leq p^{s-1} \operatorname{coconn} M - k$.

Let $\sigma\colon \Omega^s \ker d_{s-1} \to G(f_{s-1})$ be the splitting (as in Section 3). The k-invariant $f_s\colon E_s \to K(\Omega^s P_{s+1})$ is defined as induced by the composite

$$\Omega^s P_{s+1} \overset{\Omega^s d_s'}{\twoheadrightarrow} \Omega^s \ker d_{s-1} \xrightarrow{\sigma} G(f_{s-1}) \longrightarrow H^*(E_s).$$

Since ω is compatible with the cohomology suspension by 2.10, it follows that the k-invariant $\Omega^k f_s$ is induced by the composite

$$\begin{tikzcd} \Omega^{s+k}P_{s+1} \arrow[r, two heads, "\Omega^{s+k}d_s'"] \arrow[drr, "f"'] & \Omega^{s+k}\ker d_{s-1} \arrow[r, tail, "\Omega^k\sigma"] \arrow[dr, "\hat\sigma"] & \Omega^k G(f_{s-1}) \arrow[d, "\omega"] & \\ & & G(\Omega^k f_{s-1}) \arrow[r, tail] & H^*(\Omega^k E_s). \end{tikzcd} \tag{4.3}$$

Write $\hat\sigma = \omega \circ \Omega^k\sigma$. We need to know $H^*(\Omega^k E_s)//\mathrm{Im}\,\Omega^k f_s^*$ and $\ker P(\Omega^k f_s)^*$, and for this it will suffice to find $\ker\hat\sigma$ and $\operatorname{coker}\hat\sigma$.

A diagram chase shows that $\hat\sigma$ is compatible with $\Omega\tau$, and there results a commutative diagram with exact rows which defines an important map "fat theta".

$$\begin{tikzcd} 0 \arrow[r] & \ker\Omega\tau \arrow[r] \arrow[d, "\Theta"] & \Omega^{s+k}\ker d_{s-1} \arrow[r] \arrow[d, "\hat\sigma"] & \mathrm{Im}\,\Omega\tau \arrow[r] \arrow[d, "\Omega\tau"'] & 0 \\ 0 \arrow[r] & R(\Omega^k f_{s-1}) \arrow[r] & G(\Omega^k f_{s-1}) \arrow[r] & \Omega\ker\Omega^{s+k-1}d_{s-1} \arrow[r] & 0. \end{tikzcd} \tag{4.4}$$

We pause to record some information about τ:

4.5 Lemma. *There exists a natural commutative diagram in which each row and column is exact:*

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathrm{Im}\,\Omega^{s+k}d_{s+1} \arrow[r] \arrow[d, "="'] & \ker\Omega^{s+k}d_s \arrow[r] \arrow[d] & \Omega^{s+k}_{s+1}M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Im}\,\Omega^{s+k}d_{s+1} \arrow[r] & \Omega^{s+k}P_{s+1} \arrow[r] \arrow[d, "\Omega\alpha"] & \Omega^{s+k}\ker d_{s-1} \arrow[r] \arrow[d, "\Omega\tau"] & 0 \\ & & \Omega\ker\Omega^{s+k-1}d_{s-1} \arrow[r, "="] \arrow[d] & \Omega\ker\Omega^{s+k-1}d_{s-1} \arrow[d] & \\ & & \Omega\Omega^{s+k-1}_s M \arrow[r, "="] \arrow[d] & \Omega\Omega^{s+k-1}_s M \arrow[d] & \\ & & 0 & 0\,. & \end{tikzcd}$$

Proof. The top row defines $\Omega^{s+k}_{s+1}M$, and the second row follows from exactness of

$$P_{s+2} \xrightarrow{d_{s+1}} P_{s+1} \longrightarrow \ker d_{s-1} \longrightarrow 0$$

and right-exactness of Ω^{s+k}. The rows are thus exact. The map $\ker \Omega^{s+k} d_s \to \Omega^{s+k} P_{s+1}$ is the inclusion. The box to its left is clearly commutative, and this defines the vertical arrow to its right. We now construct the rest of the middle column and show it is exact; a diagram chase shows that this suffices.

The bottom three terms of the middle column are Ω applied to the exact sequence

$$\Omega^{s+k-1} P_{s+1} \xrightarrow{\alpha} \ker \Omega^{s+k-1} d_{s-1} \longrightarrow \Omega_s^{s+k-1} M \longrightarrow 0.$$

It remains to prove exactness at $\Omega^{s+k} P_{s+1}$. For this, consider the short exact sequence

$$\begin{tikzcd} & \Omega^{s+k-1} P_{s+1} \arrow[d, "\alpha"'] \arrow[dr, "\Omega^{s+k-1} d_s"] & & \\ 0 \arrow[r] & \ker \Omega^{s+k-1} d_{s-1} \arrow[r, "i"] & \Omega^{s+k-1} P_s \arrow[r] & \operatorname{Im} \Omega^{s+k-1} d_{s-1} \arrow[r] & 0. \end{tikzcd}$$

Since $\operatorname{Im} \Omega^{s+k-1} d_{s-1}$ is a submodule of the projective $\Omega^{s+k-1} P_{s-1}$, and Ω_1 is left exact, Ωi is monic. Therefore $\ker \Omega\alpha$ and $\ker \Omega^{s+k} d_s$ coincide. ∎

Now by 2.10, $\omega \colon \Omega^k G(f_{s-1}) \to G(\Omega^k f_{s-1})$ is compatible with the cohomology suspension $\Omega^k H^*(E_s) \to H^*(\Omega^k E_s)$, which lands in the coalgebra primitives. Since $R(\Omega^k f_{s-1}) = H^*(\Omega^k E_{s-1})//\operatorname{Im}(\Omega^k f_{s-1}^*) \to H^*(\Omega^k E_s)$ is a Hopf algebra monomorphism, we conclude that Θ takes values in $PR(\Omega^k f_{s-1})$. By Lemmas 4.5 and 1.8, we know that

$$\operatorname{conn} \ker \Omega\tau \geq p^{s+1}(\operatorname{conn} M - k + 1).$$

By inductive assumption 4.2(iii)

$$\operatorname{coconn} PR(\Omega^k f_{s-1}) \leq p^{s-1} \operatorname{coconn} M - k.$$

The assumption

$$k - 1 \leq \operatorname{conn} M - p^{-2} \operatorname{coconn} M$$

of Theorem 3.9 lets us conclude that

$$\Theta = 0.$$

The serpent lemma applied to (4.4), together with Lemma 4.5, then gives an isomorphism

$$\Omega_{s+1}^{s+k} M \xrightarrow{\cong} \ker \hat{\sigma} \tag{4.6}$$

and a short exact sequence

$$0 \longrightarrow R(\Omega^k f_{s-1}) \longrightarrow \operatorname{coker} \hat{\sigma} \longrightarrow \Omega\Omega_s^{s+k-1} M \longrightarrow 0. \tag{4.7}$$

We can now verify the next stage of the inductive assumption 4.2. The third term in the fundamental sequence for $H^*(\Omega^k E_{s+1})$ is the algebraic loops of the kernel of (4.3). To find this kernel, consider the following diagram, whose top half coincides with the diagram in Lemma 4.5. The map f occurs in (4.3).

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Im}\Omega^{s+k}d_{s+1} \arrow[r] \arrow[d, "="] & \ker\Omega^{s+k}d_s \arrow[r] \arrow[d] & \Omega^{s+k}_{s+1}M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Im}\Omega^{s+k}d_{s+1} \arrow[r] & \Omega^{s+k}P_{s+1} \arrow[r] \arrow[d, "f"] & \Omega^{s+k}\ker d_{s-1} \arrow[r] \arrow[d, "\hat{\sigma}"'] & 0 \\
 & & G(\Omega^k f_{s-1}) \arrow[r, "\cong"] \arrow[d] & G(\Omega^k f_{s-1}) \arrow[d] & \\
 & & \operatorname{coker} f \arrow[r, "\cong"] \arrow[d] & \operatorname{coker}\hat{\sigma} \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd} \tag{4.8}$$

Now the rows and the right column are known to be exact, so the middle column is too. We conclude that $4.2(\mathrm{i})_{s+1}$ holds.

To obtain $(\mathrm{ii})_{s+1}$ we establish a fundamental sequence for

$$R(\Omega^k f_s) = H^*(\Omega^k E_s) \otimes_{U(\Omega^{s+k}P_{s+1})} \mathsf{F}_p.$$

By (4.3), the k-invariant f_s^* factors through $U(\hat{\sigma})$ mapping to the second factor in

$$H^*(\Omega^k E_s) = R(\Omega^k f_{s-1}) \otimes_{UR(\Omega^k f_{s-1})} UG(\Omega^k f_{s-1}),$$

so by associativity of tensor product and right exactness of U,

$$R(\Omega^k f_s) \cong R(\Omega^k f_{s-1}) \otimes_{UR(\Omega^k f_{s-1})} U\operatorname{coker}\hat{\sigma}.$$

Thus (4.7) is a fundamental sequence for $R(\Omega^k f_s)$, establishing $(\mathrm{ii})_{s+1}$.

Assumption $4.2(\mathrm{iii})_{s+1}$ follows by applying P to the Hopf algebra short exact sequence

$$\mathsf{F}_p \longrightarrow R(\Omega^k f_{s-1}) \longrightarrow R(\Omega^k f_s) \longrightarrow U(\Omega\Omega_s^{s+k-1}M) \longrightarrow \mathsf{F}_p \tag{4.9}$$

associated to this fundamental sequence, and using Lemma 1.8.

This completes the proof of Theorem 3.9. The unfortunate restrictions on the coconnectivity of M are needed to kill fat theta in the above argument. A reduction in the estimate of the size of the module of primitives in $R(\Omega^k f_s)$ would lead to a better theorem; in homology one wants to guarantee the creation of squares. Examples of D. Kraines seem relevant here. However, some restrictions are necessary, as shown by the very simple tive assumption 4.2. The third term the algebraic loops of the kernel of ; diagram, whose top half coincides urs in (4.3).

$$\begin{tikzcd} 0 \arrow[d] & \\ \Omega^{s+k}_{s+1}M \arrow[d] \arrow[r] & 0 \\ \Omega^{s+k}\ker d_{s-1} \arrow[d, "\hat\sigma"'] \arrow[r] & 0 \\ G(\Omega^k f_{s-1}) \arrow[d] & \\ \operatorname{coker}\hat\sigma \arrow[d] & \\ 0. & \end{tikzcd} \tag{4.8}$$

be exact, so the middle column is

quence for

F_p.

mapping to the second factor in

$UG(\Omega^k f_{s-1})$,

ctness of U,

$\operatorname{coker}\hat\sigma$.

, establishing $(\mathrm{ii})_{s+1}$.

the Hopf algebra short exact se-

$$U(\Omega\Omega^{s+k-1}_s M) \longrightarrow \mathbb{F}_p \tag{4.9}$$

; Lemma 1.8.

e unfortunate restrictions on the the above argument. A reduction s in $R(\Omega^k f_s)$ would lead to a better creation of squares. Examples of ictions are necessary, as shown by

4.10 **Example.** Consider the two-stage Postnikov system

$$\begin{tikzcd} E \arrow[r] \arrow[d] & * \arrow[d] \\ K(\mathbb{Z},2) \arrow[r, "\mathrm{Sq}^2"] & K(\mathbb{Z},4). \end{tikzcd}$$

Let $F'(n) = PH^*K(\mathbb{Z},n)$, so that $H^*K(\mathbb{Z},n) = UF'(n)$. Then the Massey-Peterson theory of Section 2 applies to compute $H^*(E)$, since $(\mathrm{Sq}^2)^* = U(\phi)$ where $\phi\colon F'(4) \to F'(2)$ sends ι_4 to $\mathrm{Sq}^2\iota_2$. The fundamental sequence

$$0 \longrightarrow E[\iota_2] \longrightarrow G \longrightarrow \Omega\ker\phi \longrightarrow 0$$

splits for degree reasons, so $H^*(E) = U(M)$ with

$$M = \langle\iota_2\rangle \oplus \Omega\ker\phi.$$

Consider $H^*(\Omega E)$. The modules $\Omega\Omega^s_s(\Omega\ker\phi)$ are involved in Theorem 3.9. To compute them we have the short exact sequence

$$0 \longrightarrow \ker\phi \longrightarrow F'(4) \longrightarrow \langle \iota_4, \mathrm{Sq}^4\iota_4, \mathrm{Sq}^8\mathrm{Sq}^4\iota_4, \dots\rangle \longrightarrow 0.$$

The right term has trivial Ω_1, so

$$0 \longrightarrow \Omega\ker\phi \longrightarrow F'(3) \longrightarrow \langle\iota_3\rangle \longrightarrow 0$$

is still exact. Since Ω_1 is left exact and $\Omega_1 F'(n) = 0$, $\Omega_1\Omega\ker\phi = 0$. Thus

$$\Omega\Omega^s_s M = \langle\iota_{2s}\rangle$$

for any $s \geq 1$, while

$$U(\Omega\ker\phi) = H^*(S^3\langle 3\rangle).$$

So our estimate of $H^*(\Omega E)$ resembles $H^*(\Omega S^2 \times S^3\langle 3\rangle)$.

But in fact $\Omega E = K(\mathbb{Z},1)\times K(\mathbb{Z},2)$, since $\Omega\mathrm{Sq}^2 \simeq *$. In understanding this example it may help to notice the square of fibrations

$$\begin{tikzcd} & S^3\langle 3\rangle \arrow[r, equal] \arrow[d] & S^3\langle 3\rangle \arrow[d] \\ S^1 \arrow[r] \arrow[d, equal] & S^3 \arrow[r] \arrow[d] & S^2 \arrow[d] \\ S^1 \arrow[r] & K(\mathbb{Z},3) \arrow[r] & E \end{tikzcd}$$

which when looped back gives

$$\begin{tikzcd}
\Omega S^3 \arrow[r] \arrow[d, equal] & \Omega S^2 \arrow[r] \arrow[d] & S^1 \arrow[d, equal] \\
K(\mathbf{Z},2) \arrow[r] \arrow[d] & \Omega E \arrow[r] \arrow[d] & S^1 \\
S^3\langle 3\rangle \arrow[r, equal] & S^3\langle 3\rangle .
\end{tikzcd}$$

The horizontal sequence splits; we have picked up the vertical sequence, which does not.

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