Ext above the b_{10}^n line

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January, 2009

Following [1], let P denote the dual reduced powers mod p (for p odd) and $Q = \mathbb{F}_p[q_0, \ldots]$ the comodule algebra so that A is the semi-tensor product of P and Q. Q has an additional grading by weight in the q_i's, which is respected by the P coaction. Then

$$H^{***}(A) = H^{**}(P; Q)$$

and

$$H^{***}(A; E[\tau_0, \ldots, \tau_{n-1}]) = H^{**}(P; Q/(q_0, \ldots, q_{n-1}))$$

The element q_n is primitive in $Q/(q_0, \ldots, q_{n-1})$, and in [1] I computed the localization:

$$q_n^{-1} H^{**}(P; Q/(q_0, \ldots, q_{n-1})) = H^{**}(P/(\xi_1^{p^n}, \xi_2^{p^n}, \ldots))[q_n^{\pm 1}]$$

In particular,

$$q_0^{-1} H^{**}(A) = \mathbb{F}_p[q_0^{\pm 1}]$$

and

$$q_1^{-1} H^{**}(A; E[\tau_0]) = \mathbb{F}_p[q_1^{\pm 1}] \otimes E[h_{10}, h_{20}, \ldots] \otimes \mathbb{F}_p[b_{10}, b_{20}, \ldots]$$

where the $(s, t-s)$ gradings are given by

$$|h_{i0}| = (1, 2(p^i - 1) - 1)\,, \quad |b_{i0}| = (2, 2p(p^i - 1) - 2)$$

The localization map is an isomorphism in a range which is particularly interesting in the case $n = 1$: for q-weight k, this is a line of slope $1/(p^2 - p - 1)$, with $(s - t)$-intercept given by a constant plus $2(p-1)k$. When $p = 3$, $|q_1| = (1, 4)$ and $b_{10} = (2, 10)$, so this result computes 20% of $H^{**}(A; E[\tau_0])$. When $p = 5$, $|q_1| = (1, 8)$ and $|b_{10}| = (2, 38)$, so it computes 11/19 of the cohomology. In general it computes $(p^2 - 3p + 1)/(p^2 - p - 1)$, a fraction which tends to 1 as $p \to \infty$. It does significantly better, actually, when you consider the q-grading.

Since $q_1^{p^k}$ is primitive modulo $q_0^{p^k}$, one may define

$$q_1^{-1} H^{**}(A)$$

and attempt to compute it using the Bockstein spectral sequence

$$E_1^{****} = H^{**}(P; Q/(q_0))[q_0] \Rightarrow H^{**}(A)$$

By commutativity, $q_1^{p^k}$ survives to E_k.

Christian Nassau's computation indicates the following pattern of Bockstein differentials. As usual we'll leave the power of q_0 undenoted. We will use the notation

$$p^{[n]} = \frac{p^n - 1}{p - 1}$$

so that $p^{[0]} = 0, p^{[1]} = 1, p^{[2]} = p + 1, \ldots$.

$$\begin{array}{rclrcl}
d_1 q_1 & = & h_{10} & d_{p-1}(q_1^{-1} h_{10}) & = & q_1^{-p} b_{10} \\
d_{p^{[2]}} q_1^p & = & q_1^{-1} h_{20} & d_{p^2-1}(q_1^{-p^{[2]}} h_{20}) & = & q_1^{1-p^{[3]}} b_{20} \\
d_{p^{[3]}} q_1^{p^2} & = & q_1^{-p^{[2]}} h_{30} & d_{p^3-1}(q_1^{-p^{[3]}} h_{30}) & = & q_1^{1-p^{[4]}} b_{30} \\
 & \vdots & & & \vdots & \\
d_{p^{[k]}} q_1^{p^{k-1}} & = & q_1^{-p^{[k-1]}} h_{k,0} & d_{p^k-1}(q_1^{-p^{[k]}} h_{k,0}) & = & q_1^{1-p^{[k+1]}} b_{k,0} \\
 & \vdots & & & \vdots &
\end{array}$$

Since $q_1^{p^k}$ survives to a cycle in E_{p^k-1}, we may multiply the right hand calculation by $q_1^{p^k}$ if we like.

For example when $p = 5$,

$$\begin{array}{rclrcl}
d_1 q_1 & = & h_{10} & d_4(q_1^4 h_{10}) & = & b_{10} \\
d_6 q_1^5 & = & q_1^{-1} h_{20} & d_{24}(q_1^{19} h_{20}) & = & q_1^{-5} b_{20} \\
d_{31} q_1^{25} & = & q_1^{-6} h_{30} & d_{124}(q_1^{94} h_{30}) & = & q_1^{-30} b_{30} \\
 & \vdots & & & \vdots &
\end{array}$$

One may also inquire about the smallest q_1-multiple of an element in the image of the localization map. For example the subalgebra generated by h_{10} and b_{10} is in the image already. When $p = 5$, it seems that $q_1 h_{20}$ is in the image but none of the elements of the form $b_{10}^k h_{20}$ or $b_{10}^k h_{10} h_{20}$ are. It seems that $q_1^6 b_{20}$ is the first q_1 multiple in the image, and $q_1^6 h_{30}$.

References

[1] H. R. Miller, A localization theorem in homological algebra, Math. Proc. Camb. Phil. Soc 84 (1978) 73–84.