Speculations around the immersion conjecture

Haynes Miller

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The idea is to construct a space BO/I_n over $BO(n-\alpha(n))$ inductively as a "homotopy stratified space." The cores of the strata should be classifying spaces of braid groups. Disk bundles over them should be attached to lower strata by means of maps from the sphere bundle. These vector bundles should be such that when added to the pullback of the canonical $(n-\alpha(n))$-plane bundle ξ you get the bundle associated to the canonical representation of the braid group (at least stably). At the level of the Thom space of ξ, the filtration should split and recover the known splitting of MO/I_n into suspensions of Brown-Gitler spectra. Presumably the filtration is given by the dimensions of the basis vectors in $\pi_*(MO)$ indexing that splitting.

It's not clear exactly what is needed for this to serve as a solution to the Immersion Conjecture.

The bottom stratum will be BBr_n. Ralph Cohen's theory of "quasi-normal" bundles (codifying earlier work of Brown and Peterson) assures us that there is a map $BBr_n \to BO/I_n$ such that the restriction of ξ to BBr_n is at least stably the canonical bundle.

The canonical bundle over BBr_n has geometric dimension (exactly?) $n-\alpha(n)$. The Thom spectrum is the Brown-Gitler spectrum $B(n)$, where we are indexing them in such a way that $B(n)$ has top cell in dimension $n-\alpha(n)$. With this indexing, $B(2k) = B(2k+1)$.

To get my bearings I would like to understand the homology of all these graded spaces:

$$\coprod_{n\geq 0} BBr_n \to \coprod_{n\geq 0} BO/I_n \to \coprod_{n\geq 0} BO(n-\alpha(n)) \to \coprod_{n\geq 0} BO(n)\,.$$

The first and last have E_2 models, and the composite is an E_2 map.

The free E_2-algebra on Y is given by

$$\coprod_{n\geq 0} C_2(n) \times Y^n/\sim$$

where the equivalence relation has two parts: equivariance and a base-point identification. If $Y = X_+$, the result is

$$\coprod_{n\geq 0} C_2(n) \times_{\Sigma_n} X^n\,,$$

a disjoint union. If for example $X = *$, we get

$$\coprod_{n\geq 0} C_2(n)/\Sigma_n \simeq \coprod_{n\geq 0} BBr_n\,.$$

This is the E_2 structure on the coproduct of classifying spaces of braid groups. It is of course a "graded" E_2 space, in the sense that

$$C_2(2)\times(C_2(m)\times_{\Sigma_m} X^m)\times(C_2(n)\times_{\Sigma_n} X^n)\to C_{m+n}\times_{\Sigma_{m+n}} X^{m+n}\,.$$

There is a map

$$\overline{MO}_n(K_{n-q})\to \mathrm{Hom}(H^q(BO),\mathbb{F}_2)=H_q(BO)$$

that sends $[x : M\to K_{n-q}]$ to $(w\mapsto\langle x\cdot\nu_M^*(w),[M]\rangle)$.

It's the same as the map

$$\alpha:\overline{MO}_n(K_{n-q})\to MO_n(\Sigma^{n-q}H)\cong H_q(MO)\cong H_q(BO)$$

induced by the stabilization map $\Sigma^\infty K_{n-q}\to\Sigma^{n-q}H$. The anti-automorphism comes in when we swap H and MO.

$H_*(BO/I_n)$ is the image of this map, so

$$H_q(BO/I_n)\cong QMO_n(K_{n-q})\cong (MO_*\otimes QH_*(K_{n-q}))_n\,.$$

We have a right action by Sq^i is induced by $\chi\mathrm{Sq}^i : K_{n-q}\to K_{n-q+i}$, but this seems to be the wrong action. In any case, note that the top nonzero dimension is

$$H_n(BO/I_n)\cong MO_n\ :$$

the homotopy of MO is spread out along the main diagonal, as the top homology of the spaces BO/I_n.

The coproduct of configuration spaces is the free E_2 space on S^0, and its homology is generated by $x_0\in H_0(BBr_1)$ under the action of the Dyer-Lashof operation Q_1: with

$$x_n = Q_1 x_{n-1}\in H_{2^n-1}(BBr_{2^n})\,,$$

$$H_*(\coprod_{n\geq 0} BBr_n)=\mathbb{F}_2[x_0,x_1,\ldots]\,.$$

At the other end, let a_i generate $H_i(BO(1))$ for $i \geq 0$. Then

$$H_*(\coprod_{n\geq 0} BO(n)) = \mathbb{F}_2[a_0, a_1, \ldots]$$

Stewart Priddy (QJM 26 (1975)) computes the Dyer-Lashof operations on this E_∞ space; we find

$$Q_1 a_i = a_{2i+1}a_0 + a_{2i}a_1 + \cdots + a_{i+1}a_i \,.$$

The fact that it is E_∞ implies that the degree 1 Browder bracket is trivial and hence Q_1 is linear. The homology of the other graded spaces embeds into this, so the brackets are zero there too, and Q_1 is linear.

The map

$$\coprod_{n\geq 0} BBr_n \to \coprod_{n\geq 0} BO(n)$$

can be thought of as the E_2 extension of the map from S^0 sending the basepoint to $BO(0)$ and the non-basepoint to $BO(1)$, and this lets us compute the effect in homology:

$$x_0 \mapsto a_0 \,, \quad x_1 \mapsto a_1 a_0 \,, \quad x_2 \mapsto a_3 a_0^3 + a_2 a_1 a_0^2 + a_1^3 a_0 \,, \ldots \,.$$

In 1988 Vince Giambalvo told me that there are elements

$$u_q \in H_q(BO/I_n) \,, q \geq 0 \,,$$

where $n = q$ unless $q = 2^i - 1$, in which case $n = q+1 = 2^i$, with the property that

$$H_*(\coprod_{n\geq 0} BO/I_n) = \mathbb{F}_2[u_0, u_1, \ldots] \,.$$

Pick any set of algebra generators $y_q \in MO_q, q \neq 2^i - 1$. They define (with $n = q$)

$$y_q \otimes 1 \in MO_q \otimes QH_0(K_0) \subseteq H_q(BO/I_n) \,.$$

(Remember, $K_0 = \mathbb{Z}/2$ so $QH_0(K_0) = Q\mathbb{F}_2[\mathbb{Z}/2] = \mathbb{F}_2$, with generator we write as 1.) This is u_q.

For $q = 2^i - 1$, we have generators

$$x_i \in H_{2^i}(K_1) \,.$$

With $n = q+1 = 2^i$, let $u_q \in H_q(BO/I_{q+1})$ correspond to

$$1 \otimes x_i \in MO_0 \otimes QH_n(K_1) \subseteq H_q(BO/I_n)\,.$$

This includes the case $q = 0$, when $x_0 \in H_1(K_1)$ gives rise to $u_0 \in H_0(BO/I_1) = \mathbb{F}_2$.

The collection of $H_*(BO/I_*)$ forms a bigraded ring, since α is multiplicative:

$$\begin{array}{ccc} \overline{MO}_m(K_{m-p}) \otimes \overline{MO}_n(K_{n-q}) & \longrightarrow & H_p(BO) \otimes H_q(BO) \\ \downarrow & & \downarrow \\ \overline{MO}_{m+n}(K_{m+n-p-q}) & \longrightarrow & H_{p+q}(BO) \end{array}$$

The generators u_q for $q \neq 2^i - 1$ are obviously nonunique. But the other ones involve choices too. We have unique generators $x_i \in H_{2^i}(K_1)$. But to get into $QMO_n(K_1)$ you need to pick a ring map $H \to MO$. We do have the map induced from $\Omega^2 S^3 \to BO$. (Note that the resulting map on Thom spectra gives another proof of Thom's theorem!) Said differently, we can use the map

$$\coprod_{n \geq 0} BBr_n \to \coprod_{n \geq 0} BO/I_n$$

to specify the generators u_{2^i-1}.

The graded ring $H_*(\coprod_{n\geq 0} BO(n-\alpha(n)))$ can be described too, but it's much messier as a ring. In fact

$$QH_*\left(\coprod_{n\geq 0} BO(n-\alpha(n))\right) = \bigoplus_{i\geq 0} \mathrm{Sym}_{2^i-1}(\overline{H}_*(K_1))[2^i]$$

where the bracket indicates that this module contributes to the 2^ith term in the graded space, namely $BO(2^i - \alpha(2^i)) = BO(2^i - 1)$. This is definitely not a Hopf algebra, but then it doesn't have to be. It's a "graded Hopf algebra," a monoid object in graded commutative coalgebras. That is, we are given commutative coalgebras $A(n), n \geq 0$, and coalgebra maps $k \to A(0)$ and $A(m) \otimes A(n) \to A(m+n)$ that are unital and associative. Is there a structure theorem for such things, extending the Borel structure theorem?

Here's a guess: $\coprod_{n\geq 0} BO/I_n$ is only an E_1-space, but the swiss cheese operad of Sasha Voronov and Justin Thomas acts on the pair

$$\left(\coprod_{n\geq 0} BBr_n, \coprod_{n\geq 0} BO/I_n\right) :$$

$\coprod_{n\geq 0} BO/I_n$ is a module over $\coprod_{n\geq 0} BBr_n$ in a highly structured way. In fact it has a skeletal filtration as such in which the cells are indexed by generators of MO_*, and each cell contributes a free $H_*(\coprod_{n\geq 0} BBr_n)$-module in homology.

We know the homological structure of E_n-algebras. Questions: What is the homological structure of a Swiss cheese pair? Thomas shows that the space of E_n-maps from B into $THC(A)$ is the space of Swiss cheese algebra structures on the pair (B, A). Is this also the space of E_{n-1}-maps from $THH(B)$ into A? Is Vigleik's thesis relevant to this?

Mike's point: The usual two-fold loop map $\Omega^2 S^3 \to BO$ is the fiber of a loop map $BO \to Y$ where $H_*(Y) \cong \pi_*(MO)$. You might hope that one can construct a filtration of BO lifting the skeleton filtration of Y that intersects with $BO/I_n \subseteq BO$ in the desired filtration of BO/I_n.

This map also presumably gives a multiplicative splitting of the Hurewicz map,

$$H_*(MO) \cong H_*(BO) \to H_*(Y) \cong \pi_*(MO)\,,$$

and so a choice of isomorphism $H_*(MO) \to A_* \otimes \pi_*(MO)$.

We know that $H_*(Y)$ is the coequalizer (in A_*-comodule commutative algebras) of maps $H_*(\Omega^2 S^3) \rightrightarrows H_*(BO)$ where one is induced by the standard double loop map and the other factors through $\mathbb{F}_2$. $H_*(\Omega^2 S^3) = \mathbb{F}_2[x_1, x_2, \ldots]$ where $x_i = Q_1 x_{i-1}$. The effect of Q_1 on $H_*(BO)$ follows from Priddy's work but is quite complicated since the generators of $H_*(BO)$ are of the form $a_i * [-1]$. So writing down the Steenrod action on $H_*(Y)$ explicitly seems hard.