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In September, 1988, I was priveleged to talk here in honor of Albrecht Dold's sixtieth birthday. I talked about the transfer, a central part of homotopy theory to which he contributed so much. Today I am honored again to be here, doubly honored now to celebrate the career of both Albrecht Dold and Dieter Puppe.

The subject of this talk will be Elliptic Cohomology. I will not be able to talk in much detail. But behind the scenes there is an obstruction theory, and the cohomology groups housing the obstructions are direct descendants of the "Homologie nicht-additiver Functoren" which formed the subject of the paper by Professors Dold and Puppe which forms the climax of my contact with German literature.

In this talk I want to outline an approach to Elliptic Cohomology, due mainly to Mike Hopkins. I will touch on work he has done or is doing jointly with Matthew Ando, Neil Strickland, Mark Mahowald, as well as myself. Other contributions to the homotopy theory involved have been made by Charles Rezk, Ethan Devinatz, and Paul Goerss.

To begin, I recall the notion of a *genus*, as explored in Professor Hirzebruch's wonderful book. This is an invariant of manifolds with some specific extra structure (such as a complex structure, or more topologically a complex structure on the normal bundle—"U-manifolds") which is additive, multiplicative, and vanishes on boundaries. In the case of a complex structure, the fundamental example is the "Todd genus." I want to remind you of some of the features of this example.

A genus on U-manifolds is determined (up to torsion) by its values on complex projective spaces, and

$$\mathrm{Td}(\mathbb{C}\mathrm{P}^n) = 1$$

for all n. In fact, $\mathrm{Td}(M)$ is integral for *any* U-manifold, and this encourages one to look for an interpretation as a dimension.

Such an interpretation comes from analysis, at least of M is in actually a *complex* manifold. For then there is the *Dolbeault complex*

$$0 \longrightarrow C^\infty(M) \xrightarrow{\bar\partial} C^\infty(\bar T^* M) \xrightarrow{\bar\partial} C^\infty(\Lambda^2 \bar T^* M) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} C^\infty(\Lambda^n \bar T^* M) \longrightarrow 0.$$

The cohomology groups H^k of this chain complex turn out to be finite-dimensional, and the alternating sum of the dimensions is the *arithmetic genus* of M. Hirzebruch's book is devoted to a proof that the arithmetic genus and the Todd genus coincide. This is a fundamental case of the Atiyah-Singer Index Theorem.

Now suppose that we have a *family* of complex manifolds: a fiber bundle

$$E \xrightarrow{\pi} B,$$

with compatible complex structures on the fibers. Any genus assigns to each fiber a number: locally the same number, of course. But in the case of the Todd genus, the Dolbeault cohomology groups H^k vary smoothly and define *vector bundles* over B. Their alternating sum is more than just a number at each point of the base: it is a *virtual vector bundle* over B, an element of $K(B)$. ϕ thus enriches to an invariant

$$\pi_!(1) = \sum(-1)^k H^k \in K(B). \tag{1}$$

By tensoring the Dolbeault complex with a vector bundle over E, we get a homomorphism $\pi_! : K(E) \to K(B)$, the *push-forward* or *Gysin map.*

Let me start a table. [This table gets filled in bit by bit as the talk progresses.]

$H\mathbb{Q}$	K	$E_{C/R}, C$ etale
$\mu : MU \to H\mathbb{Q}$	$Td : MU \to K$	$\lambda_x : MU \to E_{C/R}$
counting	Dolbeault complex	??
none	complex conjugation	maps in $(\mathcal{M}_{\mathrm{Ell}})_{\mathrm{et}}$
$w_1 = 0$	$w_2 = 0$	$p_1/2 = 0$
$\mu : MSO \to H\mathbb{Q}$	$\hat{A} : MSpin \to KO$	$\varphi_W : MStr \to TMF$
counting	Dirac operator	??

The construction of the Gysin map is completely captured by the "Todd orientation" $Td : MU \to K$; this is a map of ring spectra.

There is an important refinement which uses the symmetry provided by complex conjugation. In fact we should consider Td and $\bar{\mathrm{T}}\mathrm{d}$ on the same footing. Suitably interpeted, the fixed point subspectrum of $\mathbb{Z}/2$ acting on K by complex conjugation is the spectrum KO. There is a corresponding genus, obtained by forming a "difference of square roots": the $\hat{A}$-genus. It takes integral values on *Spin*-manifolds (where $w_1 = w_2 = 0$), and its values are realized analytically by the Dirac operator D. There is as before a push-forward in the presence of a spin structure on the bundle of tangents along the fiber, and this Gysin map is captured by the "Atiyah-Bott-Shapiro" orientation $MSpin \to KO$.

An interesting new feature is that KO_* contains torsion (in dimensions 1 and 2 mod 8). The ABS orientation is thus a proper refinement of the $\hat{A}$ genus. The new torsion-valued invariants also have index theory interpretations (Atiyah and Singer).

Now, I will want to discuss progress in filling in a column to the right of this. To get a running start, though, let us look at what happens in a *simpler* case. The genus I have in mind is quite stupid: It assigns to a U-manifold the value 0 except in dimension 0, where it counts the number of points (with sign). It is realized by an orientation $MU \to H\mathbb{Q}$, representing the natural Thom class. When we evaluate this map on a space X we get the *Steenrod-Thom orientation*

$$MU_n(X) \to H_n(X; \mathbb{Q}).$$

There are no symmetries here, but μ does factor through an orientation $MSO \to H\mathbb{Q}$.

Now suppose we had only the left hand column here. It is quite a jump to the middle! There are a lot of new ideas—index theory, for example, and vector bundles. I believe that the jump in complexity from the middle column to the right one is similar in size. We are just beginning to understand the formal features it presents, and there are a lot of mysteries.

To describe the relevant genera on U-manifolds, I need to remind you of some of the theory of elliptic curves.

One approach to elliptic curves is via the Weierstrass equation

$$C: \quad y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \qquad a_i \in R.$$

The essential point is that this defines a projective plane cubic curve which meets the line at infinity at a single point o (namely $[0, 1, 0]$) in such a way that the line at infinity is the tangent line at that point. I will also want to assume that the curve is nonsingular; this is equivalent to requiring that a certain polynomial in the a_i's, the discriminant Δ, is a unit.

Generically at least any line meets C at three points. Requiring that the sum of those points is o defines a group structure on C.

There are some changes of variable which preserve all this:

$$\begin{aligned} x &= x' + r \\ y &= y' + sx' + t \end{aligned}$$

There is a group-structure on R^3, and this group acts on the set of Weierstrass curves naturally in R. This may be formulated by saying that there is a *groupoid* $\mathcal{W}(R)$ of Weierstrass curves, whose objects are such curves and whose morphisms are coordinate changes. This groupoid is represented by the *Hopf algebroid*

$$\begin{aligned} A &= \mathbb{Z}[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}] \\ \Gamma &= A[r, s, t] \end{aligned}$$

There are structure maps. For example $\eta_L : A \to \Gamma$, representing "source," is the obvious map, while $\eta_R : A \to \Gamma$, representing "target" classifies the effect of the universal change of coordinates.

I am ignoring a scaling change of coordinates, which does not change the projective equation. These have the effect of grading the rings involved: $|a_i| = 2i$, $|\Delta| = 24$, $|r| = 4, |s| = 2, |t| = 6$.

Now, things will work better later if we enlarge this somewhat in the following way. Suppose that C is a Weierstrass curve over R, and that S is a ring extension of R. I consider the ring $S \otimes_R S$. There are two maps, $\eta_L, \eta_R : S \to S \otimes_R S$, and the curve C gets sent to the same curve under the two (since $r \otimes 1 = 1 \otimes r$).

On the other hand, suppose we have a curve C *over* S together with an *isomorphism* $\eta_L C \to \eta_R C$. Does it come from a curve over R? This is the question of *descent*, and it is only reasonable to ask if S is *faithfully flat* over R. Even then, however, the answer is "No." So we repair the defect by defining

$$(\mathcal{M}_{\mathrm{Ell}})_R = \left\{ \begin{matrix} \text{faithfully flat } R \to S, \\ \text{Weierstrass curve } C \text{ over } S, \\ \text{isomorphism of Weierstrass curves} \\ \text{from } \eta_L C \text{ to } \eta_R C \text{ over } S \otimes_R S \end{matrix} : \begin{matrix} \text{compatibility} \\ \text{over } S \otimes_R S \otimes_R S \\ \text{and over } S \end{matrix} \right\}$$

This forms the set of objects in a groupoid, containing and in sense completing $\mathcal{W}(R)$.

Said differently, an object of $(\mathcal{M}_{\mathrm{Ell}})_R$ is a faithfully flat map $R \to S$ together with a Hopf algebroid map from (A, Γ) to $(S, S \otimes_R S)$.

One can also change rings. This gives the "elliptic modular stack" $\mathcal{M}_{\mathrm{Ell}}$ (over affine schemes). Write $(\mathcal{M}_{\mathrm{Ell}})_{\mathrm{flat}}$ for a certain full subcategory of $\mathcal{M}_{\mathrm{Ell}}$. I'll describe the condition in case C/R is Weierstrass, represented by $A \to R$: then

$$A \xrightarrow{\eta_R} \Gamma \cong A \otimes_A \Gamma \longrightarrow R \otimes_A \Gamma$$

is required to be a flat morphism. Roughly speaking this says that R can't be too small.

Theorem 1. There is a functor E from $(\mathcal{M}_{\mathrm{Ell}})_{\mathrm{flat}}$ to the homotopy category of periodic ring spectra, with natural isomorphisms $\pi_0(E_{C/R}) \to R$ and from the formal group of $E_{C/R}$ to the formal completion of C at o.

This is really the Landweber exact functor theorem. A *periodic* ring spectrum is a homotopy-associative and commutative ring spectrum such that $\pi_* R$ is evenly graded and $\pi_2 R$ contains a unit. K-theory is the motivating example. As Dan Quillen explained, to any such theory is associated a *formal group* over $\pi_0 R$. (Actually one must pass to a smaller subcategory than $(\mathcal{M}_{\mathrm{Ell}})_{\mathrm{flat}}$, requiring that the canonical line bundle ω admit a global section in order to obtain a formal group in the traditional sense.)

These periodic ring theories are our analogue of K: there is an abundance of them. The possibility of defining a multiplicity of "elliptic cohomology theories" was first brought up by Jens Franke. The first were constructed by Landweber, Ravenel, and Stong. Each has an associated genus $\lambda_{C/R}$ defined on U-manifolds. The coefficient ring R is in each case torsion free.

If we choose a local parameter x on C near o, we obtain a formal group *law* and an orientation

$$\lambda_x : MU \to E_{C/R}.$$

Next one wants to rigidify this big diagram of spectra, in order to form its "inverse limit." I think it is fair to say that we are still experimenting with this. For some time our approach has been this: In order to cut down the size of the mapping spaces involved, we consider an intermediate step, the category of A_∞ ring spectra. There are several good treatments of this theory now, including an entirely simplicial one due to Jeff Smith, Mark Hovey, and Brooke Shipley. There is a simplicial model category of such objects. This leads to an obstruction theory for the existence of an A_∞ structure, a spectral sequence converging to the homotopy groups of such structures, and a similar story for maps of A_∞ rings spectra. We have been helped here by Charles Rezk, a student of Mike's.

The obstruction groups which arise lie in certain cohomology groups (as defined by Quillen). The groups involved in existence and uniqueness of structure can be made to vanish if we restrict further to the subcategory $(\mathcal{M}_{\mathrm{Ell}})_{\mathrm{et}}$ of *etale* curves. Restricting again to Weierstrass curves, this means that $R \otimes_A \Gamma$ should again be flat over A, and that in addition the relative cotangent complex should vanish. Roughly speaking the added condition means that R can't be too big either. There are fewer of these curves, but enough. Many standard parametrizations of elliptic curves (the Legendre form $y^2 = x(x-1)(x-\lambda)$ over $\mathbb{Z}[1/2, \lambda^{\pm 1}, (1-\lambda)^{-1}]$ for example) are etale. For these purposes Hopkins has written down an etale cover which seems to be new.

There is also a second stage of the obstruction theory, which rigidifies the diagram in the homotopy category of A_∞ ring spectra to one in the honest category. For this we have the work of Dwyer and Kan; the relevant diagram turns out to be "centric."

Theorem 2. There is a commutative diagram

$$\begin{array}{ccc}
(\mathcal{M}_{\mathrm{Ell}})_{\mathrm{et}} & \xrightarrow{E} & \text{Periodic } A_\infty \text{ ring spectra} \\
\downarrow & & \downarrow \\
(\mathcal{M}_{\mathrm{Ell}})_{\mathrm{flat}} & \xrightarrow{E} & \mathrm{Ho}(\text{Periodic ring spectra})
\end{array}$$

Finally we can take the (homotopy) inverse limit, which we call the spectrum of *Topological Modular Forms*, TMF. There is a spectral sequence converging to its homotopy groups, whose E_2 term is the cohomology of the Hopf algebroid (A, Γ). H^0 is the subring of A of polynomials whose value is *independent* of the parametrization of the curve: these were computed by Tate and Deligne, under the name "integral modular forms." They found that

$$H^0 = \mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 = 12^3 \Delta).$$

Topology brings an interpretation of the higher cohomology, and shows that the Tate-Deligne ring is just the first approximation to a more fundamental object.

The higher cohomology is interesting! It is all torsion, killed by 24. There are differentials in the spectral sequence, and they are interesting too! Δ itself is not a permanent cycle, though 24Δ and Δ^{24} are. The result, $\pi_*(TMF)$, is periodic of period 24^2. Rationally, it is the ring of modular forms. But there is something defective about Δ. What?

It would seem that there should be a genus on manifolds whose normal bundle lifts through the next connective cover of $BSpin$: This kills $p_1/2$, and I would like to call the resulting group the *string group*, Str. (Is there a construction of $Str(n)$ analogous to the construction of $Spin(n)$ using Clifford algebras?) As a matter of fact, by thinking about computing the equivariant $\hat{A}$ genus of the free loop space on a manifold by means of the fixed point formula, Ed Witten defined a genus on string manifolds with values in modular forms (or rather their Fourier series). His wonderful formula is:

$$\varphi_W(M) = \langle \hat{A}(M) \cup ch \bigotimes_{n \geq 1} S_{q^n}(T^{\mathbb{C}}M - \dim\ M), [M] \rangle.$$

That this does take values in modular forms was proved by Don Zagier. Hopkins, Ando, and Strickland have done most of the work necessary to construct an orientation $MStr \to TMF$, analogous to the ABS orientation, refining the Witten genus. This will show that the discriminant is not a value of the Witten genus, for example. Gerd Laures has shown that there is a natural orientation from $MStr$ to many elliptic spectra by much simpler methods.

Obviously there are some gaps in the right column of the table, but I think that homotopy theory has done its part!