**Homotopy fixed point sets of group actions on groups**

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Let $\pi$ be a group acted on by another group $G$. Then $G$ acts on $B\pi$, and we inquire about the homotopy fixed point set of this action.

The space $(B\pi)^{hG}$ is the space of sections of the projection $EG\times_G B\pi\to BG$. We know that

$$EG\times_G B\pi = B(\pi\tilde{\times}G)$$

where $\pi\tilde{\times}G$ is the semi-direct product: $\pi\times G$ as a set, with product

$$(p,\gamma)(p',\gamma')=(p\cdot\gamma p',\gamma\gamma')$$

Write $q:\pi\tilde{\times}G\to G$ for the projection map.

We know that

$$\mathrm{map}(BG,B(\pi\tilde{\times}G))=\coprod_{[\sigma]}BZ_\sigma$$

where $\sigma$ runs over a set of representatives of conjugacy classes of homomorphisms $G\to\pi\tilde{\times}G$ and $Z_\sigma$ denotes the centralizer of $\sigma$,

$$Z_\sigma=\{p\in\pi : p\sigma(\gamma)=\sigma(\gamma)p \text{ for all } \gamma\in G\}$$

The composite $Bq\circ B\sigma$ is homotopic to the identity map on $BG$ if and only if $q\circ\sigma$ is conjugate to the identity map on $G$. This condition is invariant under conjugation, and for a unique $G$-conjugate of $\sigma$ the composite with $q$ is equal to the identity. For such a $\sigma$,

$$\sigma(\gamma)=(\varphi(\gamma),\gamma)$$

for some function $\varphi:\gamma:G\to\pi$. The fact that $\sigma$ is a group homomorphism is equivalent to

$$\varphi(\gamma\gamma')=\varphi(\gamma)\cdot\gamma\varphi(\gamma')$$

—i.e. $\varphi$ is a 1-cocycle on $G$ with values in $\pi$.

For $p\in\pi$, the homomorphism $\gamma\mapsto p\sigma(\gamma)p^{-1}$ corresponds to the 1-cocycle $\gamma\mapsto p\cdot\varphi(\gamma)\cdot\gamma p^{-1}$. This gives an action of $\pi$ on $Z^1(G;\pi)$, with orbit space $H^1(G;\pi)$.

So the set of components of $\mathrm{map}(BG, B(\pi\tilde{\times}G))$ lying over the identity component of $\mathrm{map}(BG, BG)$ is in bijection with $H^1(G; \pi)$.

The component of the mapping space containing $B\sigma$ is $BZ_\sigma$. Suppose that $\sigma$ corresponds to the 1-cocycle $\varphi$. Then $(p, \gamma) \in Z_\sigma$ if and only if

$$(p, \gamma)(\varphi(\gamma'), \gamma') = (\varphi(\gamma'), \gamma')(p, \gamma)$$

for all $\gamma' \in G$. This is demanding that $\gamma \in Z(G)$ and that $p$ is such that

$$\gamma' p = \varphi(\gamma')^{-1} \cdot p \cdot \gamma\varphi(\gamma') \tag{1}$$

for all $\gamma' \in G$. Write $Z(\varphi)$ for this subgroup of $\pi\tilde{\times}G$.

The space of sections is the disjoint union of the fibers of the maps $BZ(\varphi) \to BZ(G)$ as $\varphi$ runs over a set of representatives of $H^1(G; \pi)$.

Notice that for any $\gamma \in Z(G)$, $(\phi(\gamma), \gamma) \in Z(G)$ since

$$\phi(\gamma') \cdot \gamma'\phi(\gamma) = \phi(\gamma'\gamma) = \phi(\gamma\gamma') = \phi(\gamma) \cdot \gamma\phi(\gamma')$$

Let $K(\varphi) = \ker\left(Z(\varphi) \to Z(G)\right)$. Then

$$(B\pi)^{hG} = \coprod_{[\varphi] \in H^1(G;\pi)} BK(\varphi) \tag{2}$$

As a first check suppose that the action is trivial. Then $Z^1(G; \pi) = \mathrm{Hom}(G, \pi)$, the group $\pi$ acts on this by conjugation, and $H^1(G; \pi) = \mathrm{Rep}(G, \pi)$. $Z(\varphi) = Z(G) \times Z_\varphi \subseteq G \times \pi$, $\overline{Z}(\varphi) = Z(G)$, and $K(\varphi) = Z_\varphi$, so (2) reduces to the known formula for maps of classifying spaces.

This is actually more general than it seems: Suppose that $G$ acts on $\pi$ through a homomorphism $f : G \to \pi$: $\gamma p = f(\gamma) p f(\gamma)^{-1}$. Then $(p, \gamma) \mapsto (pf(\gamma), \gamma)$ makes

$$\begin{array}{ccc} \pi\tilde{\times}G & \xrightarrow{\cong} & \pi \times G \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & G & \end{array}$$

commute, so $B\pi^{hG} = \mathrm{map}(BG, B\pi)$ for any homomorphism $f$.

For another example suppose $|G| = 2$, with generator $\gamma$. Write $\gamma p = \overline{p}$. Let $\varphi(\gamma) = \tau \in \pi$. Then $1 = \varphi(\gamma^2) = \tau\overline{\tau}$, so

$$Z^1(G; \pi) = \{\tau \in \pi : \tau\overline{\tau} = 1\}.$$

Conjugation by $p \in \pi$ sends the 1-cocycle corresponding to $\tau$ to the 1-cocycle corresponding to $p\tau\overline{p}^{-1}$, so

$$H^1(G;\pi) = \{\tau \in \pi : \tau\overline{\tau} = 1\}/\tau \sim p\tau\overline{p}^{-1}$$

The group $\pi\tilde{\times}G$ is given by

$$\pi \times \{1\} \coprod \pi \times \{\gamma\}$$

The centralizer of $\tau$ is the subgroup given by

$$Z(\tau) = \{p : \overline{p} = \overline{\tau}p\tau\} \times \{1\} \coprod \{p : \overline{p} = \overline{\tau}p\overline{\tau}\} \times \{\gamma\}$$

It's fun and reassuring to check that this *is* a subgroup!

The kernel group is

$$K(\tau) = \{p : \overline{p} = \overline{\tau}p\tau\} \subseteq \pi$$

Thus:

$$(B\pi)^{hG} = \coprod_{[\tau] \in H^1(G;\pi)} BK(\tau)$$