Fred's thesis

Haynes Miller

November, 2003; May, 2011

Here is a reading of a small part of Fred Cohen's thesis [1], in the special case when $p = 2$ and $n = 1$, so we are speaking of the structure of $H_*(\Omega^2 X; \mathbb{F}_2)$.

We work over the field $\mathbb{F}_2$. An *Lie algebra* is a vector space V together with a symmetric bilinear operation $[-,-]$ which satisfies the Jacobi law

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0\,.$$

A *restriction* on V is a function $\xi : V \to V$ such that

$$[\xi(x), y] = [x,[x,y]] \quad \text{and} \quad \xi(x+y) = \xi(x) + [x,y] + \xi(y)\,.$$

Note that by taking $x = y = 0$ we find $\xi(0) = 0$, and taking $x = y$ we find that $[x,x] = 0$. One should think of $\xi(x)$ as half of $[x,x]$. A *restricted Lie algebra* is a Lie algebra together with a restriction.

The addition formula for ξ extends to

$$\xi\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \xi(x_i) + \sum_{i<j} [x_i, x_j].$$

Often one has a grading on V for which the bracket has some degree n, so $|[x,y]| = |x| + |y| + n$, and then $|\xi(x)| = 2|x| + n$. We'll call n the *degree* of the (restricted) Lie algebra.

A *restricted Poisson algebra* is a commutative algebra together with the structure of a restricted Lie algebra, which satisfies also

$$[x, yz] = y[x,z] + z[x,y] \quad \text{and} \quad \xi(xy) = x^2\xi(y) + x[x,y]y + \xi(x)y^2\,.$$

Note that $[x, y^2] = 0$, so in particular $[x, 1] = 0$, and that $\xi(x^2) = 0$, so in particular $\xi(1) = 0$. Again there may be a degree n; but the commutative product is always assumed to have degree 0. The structure is then called a *restricted Gerstenhaber algebra of degree n*. We will take $n = 1$.

The forgetful functor from restricted Gerstenhaber algebras to restricted Lie algebras has a left adjoint S. I claim that S covers the symmetric algebra functor from vector spaces to commutative algebras. The essential point is that if V is a restricted Lie algebra, then there is a unique restricted Lie algebra structure on the symmetric algebra on V which extends the restricted Lie algebra structure on V and which, with the given commutative algebra structure, forms a restricted Gerstenhaber algebra. Uniqueness is clear from the Gerstenhaber relations. For existence, note first that for any $x \in V$ the linear map $[x, -] : V \to V \to SV$ extends uniquely to a derivation $[x, -] : SV \to SV$. Then for any $s \in SV$, the linear map $[-, s] : V \to SV$ extends uniquely to a derivation $[-, s] : SV \to SV$. This is the bracket on SV. Simple uniqueness arguments must lead to symmetry and the Jacobi law. The restriction takes a little more care.

Let L be the left adjoint to the forgetful functor from restricted Lie algebras to vector spaces, so that SL is left adjoint to the forgetful functor from restricted Gerstenhaber algebras to vector spaces.

Theorem. Let C_2 be the triple on pointed spaces associated to the little squares operad. The homology of a C_2-space is naturally a restricted Gerstenhaber algebra, and the natural map $SL\bar{H}_*(X) \to SL\bar{H}_*(C_2X) \to H_*(C_2X)$ is an isomorphism.

For example take $X = S^n$. Let $\bar{H}_n(S^n) = \langle x_n \rangle$. Then

$$L\langle x_n \rangle = \langle x_n, x_{2n+1}, x_{4n+3}, \ldots \rangle$$

with ξ sending one generator to the next and with trivial bracket; so

$$SL\langle x_n \rangle = \mathbb{F}_2[x_n, x_{2n+1}, x_{4n+3}, \ldots],$$

which is indeed the homology of $\Omega^2 S^{n+2}$ assuming $n > 0$. The bracket is trivial, so the restriction is linear, and we have for example

$$\xi(x_n x_{2n+1}) = x_n^2 x_{4n+3} + x_{2n+1}^3 \,.$$

Lemma. The coproduct of two restricted Gerstenhaber algebras (of given degree) V, W, has as its underlying vector space $V \otimes W$, and structure maps determined by

$$\begin{aligned} (v \otimes w)(x \otimes y) &= vx \otimes wy \,, \\ [v \otimes w, x \otimes y] &= vx \otimes [w, y] + [v, x] \otimes wy \,, \\ \xi(v \otimes w) &= \xi(v) \otimes w^2 + v^2 \otimes \xi(w) \,. \end{aligned}$$

The proof is a straightforward and entertaining calculation. One uses the Poisson relation to check the Jacobi identity in $V \otimes W$. The function ξ has to be extended to decomposable tensors using the addition formula, and then one must check that it is well-defined. For example, using $[v, v] = 0$,

$$\begin{aligned} &\xi(v \otimes (w_1 + w_2)) = \xi(v) \otimes (w_1^2 + w_2^2) + v^2 \otimes \xi(w_1 + w_2) \\ &= \xi(v) \otimes w_1^2 + \xi(v) \otimes w_2^2 + v^2 \otimes \xi(w_1) + v^2 \otimes [w_1, w_2] + v^2 \otimes \xi(w_2) \\ &= \xi(v) \otimes w_1^2 + v^2 \otimes \xi(w_1) + [v, v] \otimes w_1 w_2 + v^2 \otimes [w_1, w_2] + \xi(v) \otimes w_2^2 + v^2 \otimes \xi(w_2) \\ &= \xi(v \otimes w_1) + [v \otimes w_1, v \otimes w_2] + \xi(v \otimes w_2) = \xi(v \otimes w_1 + v \otimes w_2). \end{aligned}$$

This lemma lets us discuss commutative coalgebras in the category of restricted Gerstenhaber algebras: call these Gerstenhaber bialgebras. The diagonal map $\Delta : V \to V \otimes V$ must be a ring-homomorphism and satisfy

$$\begin{aligned} \Delta[v,w] &= \sum (v'w' \otimes [v'',w''] + [v',w'] \otimes v''w'')\,, \\ \Delta\xi(x) &= \sum \left(\xi(x') \otimes x''^2 + x'^2 \otimes \xi(x'')\right) \\ &+ \sum_{i<j} \left(x'_i x'_j \otimes [x''_i, x''_j] + [x'_i, x'_j] \otimes x''_i x''_j\right)\,, \end{aligned}$$

and the augmentation must be a ring-homomorphism and satisfy

$$\epsilon[v,w] = 0 \quad , \quad \epsilon\xi(x) = 0\,.$$

The natural vector space map $V \otimes W \to SLV \otimes SLW$ extends uniquely to a restricted Gerstenhaber algebra map $SL(V \otimes W) \to SLV \otimes SLW$. Thus a commutative coalgebra structure on V determines a Gerstenhaber bialgebra structure on $SL\overline{V}$. The map $SL\overline{H}_*(X) \to H_*(C_2X)$ is an isomorphism of Gerstenhaber bialgebras.

Finally, we have the right action of the Steenrod algebra on homology. On a restricted Gerstenhaber algebra, the action is required to satisfy:

$$\begin{aligned} (xy)\mathrm{Sq}^k &= \sum_{i+j=k} (x\mathrm{Sq}^i)(y\mathrm{Sq}^j)\,, \\ [x,y]\mathrm{Sq}^k &= \sum_{i+j=k} [x\mathrm{Sq}^i, y\mathrm{Sq}^j]\,, \\ (\xi x)\mathrm{Sq}^{2r} &= \xi(x\mathrm{Sq}^r) + \sum_{i=0}^{r-1} [x\mathrm{Sq}^i, x\mathrm{Sq}^{2r-i}]\,, \\ (\xi x)\mathrm{Sq}^{2r+1} &= |x|(x\mathrm{Sq}^r)^2 + \sum_{i=0}^{r} [x\mathrm{Sq}^i, x\mathrm{Sq}^{2r+1-i}]\,. \end{aligned}$$

References

[1] F. R. Cohen, The homology of $\mathcal{C}_{n+1}$-spaces, in The Homology of Iterated Loop Spaces, Springer Lect. Notes in Math. 533 (1976) 207–351.