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HOMOTOPICAL UNIQUENESS OF CLASSIFYING SPACES

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§1. INTRODUCTION

IF G is a connected compact Lie group, then for almost all prime numbers p the mod p cohomology ring of the classifying space BG is a finitely generated polynomial algebra. In 1961, N. Steenrod [24] asked in general for a determination of all spaces X such that $H^*(X, \mathbf{F}_p)$ is a finitely generated polynomial algebra (i.e., such that X *has a polynomial cohomology ring*); at that time, the only examples known were spaces of the form $X = BG$.

There has been a lot of subsequent progress on this problem. On the one hand, the topological constructions of Sullivan [25, p. 4.28], as exploited by Clark–Ewing [8] and Wilkerson [27], have led to the discovery of exotic spaces X with polynomial cohomology rings. On the other hand, the algebraic arguments of Wilkerson [26] and Adams–Wilkerson [1] have shown in some generality that if X is any space with a polynomial cohomology ring then $H^*(X, \mathbf{F}_p)$ must be one of the polynomial algebras listed in [8].

However, there is still a gap here between topology and algebra; in this paper we act to narrow the gap and in some cases to close it. Suppose that p is an odd prime. Let $\mathscr{K}$ be the category of unstable algebras [18] over the mod p Steenrod algebra $\mathscr{A}_p$ and $\mathscr{K}_{\text{poly}}$ the full subcategory of $\mathscr{K}$ consisting of objects which as rings are finitely generated polynomial algebras. If X is a p-complete space with $H^*(X, \mathbf{F}_p) \in \mathscr{K}_{\text{poly}}$ we extend the ideas of [1] by associating to X a finite p-adic linear group W_X generated by pseudoreflections (see 1.1); given any such finite group W such that p does not divide the order of W, we then show (1.2) that there is up to homotopy exactly one p-complete space X with $H^*(X, \mathbf{F}_p) \in \mathscr{K}_{\text{poly}}$ and $W_X = W$. In a variety of particular situations (1.3, 1.4) this gives a bijective correspondence between finite group data and homotopy types of p-complete spaces with polynomial cohomology rings.

As a corollary we observe that if G is a connected compact Lie group and p a prime which does not divide the order of the Weyl group of G then the cohomology ring $H^*(BG, \mathbf{F}_p)$, considered as an object of $\mathscr{K}$, determines the homotopy type of the p-completion of BG (1.7). This generalizes the treatment of $G = SU(2)$ in [12] and is the uniqueness property referred to in the title of the paper.

We will now describe our results in more detail. If R is an object of $\mathscr{K}_{\text{poly}}$, let ρ_R (the rank of R) be the number of polynomial generators in R and v_R the product $\prod_i (|x_i|/2)$ indexed by a set $\{x_i\}$ of polynomial generators for R; recall that the prime p is odd, so that the degree $|x_i|$ of each polynomial generator x_i is even. The integer v_R does not depend upon a choice of polynomial generators for R. If X is a space with $H^*(X, \mathbf{F}_p) \in \mathscr{K}_{\text{poly}}$, we will write ρ_X and v_X for ρ_R and v_R with $R = H^*(X, \mathbf{F}_p)$. If $H^*(X, \mathbf{F}_p) \in \mathscr{K}_{\text{poly}}$ and $X = BG$ for a connected

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compact Lie group G, then ρ_X is the Lie-theoretic rank of G and v_X the order of the Weyl group of G.

Let $\mathbf{Z}_p$ denote the ring of p-adic integers. The Eilenberg–MacLane space $K((\mathbf{Z}_p)^r, 2)$ is the p-completion of the classifying space of the r-torus T^r, and we will denote it $\hat{B}T^r$ or $\hat{B}T$ if r is understood; it is clear that the general linear group $GL(r, \mathbf{Z}_p)$ acts on $\hat{B}T^r$ in a natural way.

Given a commutative domain D, an element $g \in GL(r, D)$ of finite order is said to be a *pseudoreflection* if the $r \times r$ matrix $(g - I_r)$ has rank at most one (here $I_r \in GL(r, D)$ is the identity matrix). If X is a space, say that a subgroup $W \subset GL(r, \mathbf{Z}_p)$ is *adapted* to X if there is a map $\tau: \hat{B}T^r \to X$, equivariant up to homotopy with respect to the natural action of W on $\hat{B}T$ and the trivial action of W on X, such that τ induces an isomorphism

$$H^*(X, \mathbf{F}_p) \cong H^*(\hat{B}T, \mathbf{F}_p)^W.$$

1.1 THEOREM. *Let p be an odd prime and let X be a p-complete space with $H^*(X, \mathbf{F}_p)$ in $\mathscr{K}_{\text{poly}}$. Then there exists up to conjugacy a unique finite subgroup $W_X \subset GL(\rho_X, \mathbf{Z}_p)$ adapted to X. The group W_X is generated by pseudoreflections and $|W_X| = v_X$.*

Remark. Theorem 1.1 does not require the assumption $p \nmid v_X$ and for this reason leads to non-realizability results for many objects of $\mathscr{K}_{\text{poly}}$. The image in $GL(\rho_X, \mathbf{F}_p)$ of the group $W_X \subset GL(\rho_X, \mathbf{Z}_p)$ is the "Galois group" of $H^*(X, \mathbf{F}_p)$ constructed in [1] (see §2). If $H^*(X, \mathbf{F}_p) \in \mathscr{K}_{\text{poly}}$ and X is the p-completion of BG for a connected compact Lie group G, then W_X is the image in $GL(\rho_X, \mathbf{Z}_p)$ of the Lie-theoretic Weyl group $W_G \subset GL(\rho_X, \mathbf{Z})$. In general the group W_X of 1.1 must be a product of the irreducible p-adic pseudoreflection groups enumerated in [8]. This perhaps gives the list in [8] a richer significance.

1.2 THEOREM. *Let p be an odd prime and let W be a finite subgroup of $GL(r, \mathbf{Z}_p)$ generated by pseudoreflections. Suppose that p does not divide the order of W. Then there exists up to homotopy exactly one p-complete space X such that $H^*(X, \mathbf{F}_p) \in \mathscr{K}_{\text{poly}}$ and W is adapted to X.*

It is clear from 1.1 that if $H^*(X, \mathbf{F}_p) \in \mathscr{K}_{\text{poly}}$ then p divides $|W_X|$ iff p divides v_X. This leads to the following corollary.

1.3 COROLLARY. *Let p be an odd prime. Then for any $r \geqslant 1$ there is a bijective correspondence between the following two sets:*

(1) *Homotopy equivalence classes of p-complete spaces X such that $H^*(X, \mathbf{F}_p) \in \mathscr{K}_{\text{poly}}$, $\rho_X = r$ and $p \nmid v_X$.*
(2) *Conjugacy classes of finite subgroups $W \subset GL(r, \mathbf{Z}_p)$ such that W is generated by pseudoreflections and $p \nmid |W|$.*

If $r \leqslant p - 2$ then $GL(r, \mathbf{Z}_p)$ does not contain any non-trivial elements of order p, so Corollary 1.3 takes the following simpler form in this case.

1.4 COROLLARY. *Let p be an odd prime. Then for any r* such that $1 \leqslant r \leqslant p - 2$ *there is a bijective correspondence between the following two sets:*

(1) *Homotopy equivalence classes of p-complete spaces X such that $H^*(X, \mathbf{F}_p) \in \mathscr{K}_{\text{poly}}$ and $\rho_X = r$.*
(2) *Conjugacy classes of finite subgroups $W \subset GL(r, \mathbf{Z}_p)$ such that W is generated by pseudoreflections.*

Moreover, if X is a space as in (1) *then $p \nmid v_X$.*

The following algebraic theorem is both a summary and an extension of some of the results of [8] and [1]; it lists several ways of looking at the groups W which appear in 1.2–1.4. Let $\mathbf{Q}_p$ denote the quotient field of $\mathbf{Z}_p$.

1.5 THEOREM. *Let p be an odd prime. Then for any positive integers r and n with $p \nmid n$ the following sets are in bijective correspondence:*

(1) *Isomorphism classes of objects $R \in \mathscr{K}_{\text{poly}}$ such that $\rho_R = r$ and $v_R = n$.*
(2) *Conjugacy classes of subgroups $W \subset GL(r, \mathbf{F}_p)$ such that W is generated by pseudoreflections and $|W| = n$.*
(3) *Conjugacy classes of finite subgroups $W \subset GL(r, \mathbf{Z}_p)$ such that W is generated by pseudoreflections and $|W| = n$.*
(4) *Conjugacy classes of finite subgroups $W \subset GL(r, \mathbf{C})$ such that W is generated by pseudoreflections, $|W| = n$, and the character field K_W (see §5) can be embedded in $\mathbf{Q}_p$.*

Combining 1.3 with 1.5 gives the following result.

1.6 COROLLARY. *Let p be an odd prime. Then for any positive integers r and n with $p \nmid n$ there is a bijective correspondence between the following two sets:*

(1) *Homotopy equivalence classes of p-complete spaces X such that $H^*(X, \mathbf{F}_p) \in \mathscr{K}_{\text{poly}}$, $\rho_X = r$ and $v_X = n$.*
(2) *Isomorphism classes of objects $R \in \mathscr{K}_{\text{poly}}$ such that $\rho_R = r$ and $v_R = n$.*

Of course, the bijection (1) $\to$ (2) in 1.6 is given by the cohomology functor, so in effect 1.6 states that a space X of the indicated type is determined up to homotopy by $H^*(X, \mathbf{F}_p) \in \mathscr{K}_{\text{poly}}$. This immediately leads to the following homotopical uniqueness theorem for classifying spaces of Lie groups.

1.7 THEOREM. *Let G be a connected compact Lie group with Weyl group W and assume that $p \nmid |W|$. Suppose X is a space with $H^*(X, \mathbf{F}_p)$ isomorphic to $H^*(BG, \mathbf{F}_p)$ as an object of $\mathscr{K}$. Then the p-completion of X is homotopy equivalent to the p-completion of BG.*

Proof. Let r be the rank of G and let $T \subset G$ be a maximal torus. The conjugation action of W on T exhibits W as a finite subgroup of $GL(r, \mathbf{Z})$ generated by (pseudo)reflections. Since $p \nmid |W|$ there is an isomorphism $H^*(BG, \mathbf{F}_p) \cong H^*(BT, \mathbf{F}_p)^W$, where the action of W on $H^*(BT, \mathbf{F}_p)$ can be interpreted as the algebra action induced under the composite $W \to GL(r, \mathbf{Z}) \to GL(r, \mathbf{F}_p)$ by the natural action of $GL(r, \mathbf{F}_p)$ on $H^2(BT, \mathbf{F}_p) = \text{Hom}((\mathbf{F}_p)^r, \mathbf{F}_p)$. The second part of [1, 1.2] (cf. (2)$\leftrightarrow$(1) in 1.5) now shows that $H^*(BG, \mathbf{F}_p) \in \mathscr{K}_{\text{poly}}$ and $p \nmid v_{BG}$, so the theorem follows immediately from 1.6.

We will give one concrete example of the calculations possible with 1.5 and 1.6. Let $\mathbf{F}_p[x_{24}, x_{48}]$ denote a graded polynomial algebra over $\mathbf{F}_p$ on generators of the indicated dimensions.

1.8 THEOREM. *If $p > 3$ then the number of isomorphism classes of objects of $\mathscr{K}$ with underlying ring $\mathbf{F}_p[x_{24}, x_{48}]$ is 1 if $p \equiv 1 \bmod 48$, 3 if $p \equiv 25 \bmod 48$, 2 if $p \equiv 13, 17 \bmod 48$ and zero otherwise. This number is also the number of homotopy types of p-complete spaces X such that $H^*(X, \mathbf{F}_p) \cong \mathbf{F}_p[x_{24}, x_{48}]$.*

This follows from a check of the group tables in [8]. The following more qualitative result is proved by combining 1.5, 1.6, the spectral sequence argument of [3, 13.1], and the observation that a finite group has only a finite number of conjugacy classes of subgroups.

1.9 THEOREM. *Let p be an odd prime and let Y be a one-connected p-complete space such that $H^*(Y, \mathbf{F}_p)$ is an exterior algebra on generators of degree $2d_i - 1$, $i = 1, \ldots, n$. Suppose that $p \nmid d_i$, $i = 1, \ldots, n$. Then up to homotopy there are at most finitely many connected spaces X such that ΩX is equivalent to Y.*

Further work. The results in this paper deal mostly with odd primes. For several reasons the results do not extend directly to cover the case $p = 2$; one of these reasons is that there are more elements of Hopf invariant one mod 2 than there are mod p for odd p (cf. proof of 4.1). Removing the restriction $p \nmid |W|$ in Theorem 1.2 will probably require a new technique. Nevertheless, one example in which $p = 2$ *and* $|W| = 2$ (i.e. $X = BSU(2)$) has been treated successfully in [12]; there is some hope that the techniques of that paper, as extended by [16] and [17], may lead to further progress in this area.

Organization of the paper. Our main technique is to take a p-complete space X with $H^*(X, \mathbf{F}_p) \in \mathscr{K}_{\text{poly}}$ and find a homotopical "maximal torus" for X; this is essentially a map $t: BT \to X$ with certain homological properties (see 2.6). We describe the theory of homotopical maximal tori and of their associated "Weyl groups" in §2, where we also detail how this theory leads to proofs of 1.1 and 1.2.

If X is the p-completion of BG for a compact Lie group G, then a homotopical maximal torus for X can be constructed from the map $BT \to BG$ induced by the inclusion $T \subset G$ of an ordinary maximal torus (cf. proof of 1.7). If p is an odd prime, though, it is easy to see that T is in fact the centralizer in G of the subgroup $V \subset T$ generated by elements of order p, and so [14] the map $BT \to BG$ can also be obtained (up to p-completion) by restricting the domain of the basepoint evaluation map $\mathrm{Map}(BV, BG) \to BG$ to an appropriate component. It turns out that a corresponding function space construction gives a homotopical maximal torus for any p-complete X with $H^*X \in \mathscr{K}_{\text{poly}}$; this is proved in §4 with the help of some properties of Lannes' functor "T^V" which are established in §3. Section 5 contains the proof of 1.5.

1.10 *Notation and terminology.* The letter p will stand for a fixed rational prime, which except in §3 is assumed to be odd. For a space X, H^*X will denote the object $H^*(X, \mathbf{F}_p)$ of $\mathscr{K}$ and $\hat{X}$ the p-completion of X in the sense of [6]. For a topological group or group-like topological monoid G, EG will denote some functorial contractible G-CW complex on which G acts freely, $BG = EG/G$ the classifying space of G, and $\hat{BG}$ the p-completion of BG; if X is some space on which G acts, then $B(G, X)$ will denote $EG \times_G X$ and $\hat{B}(G, X)$ the p-completion of $B(G, X)$. If $f: X \to Y$ is a map of spaces, we will let $\mathrm{Map}(X, Y)_f$ denote the path component of the space of maps from X to Y containing f.

An object of $\mathscr{K}$ is of finite type if in each dimension it is of finite rank as an $\mathbf{F}_p$ vector space. An object $R \in \mathscr{K}$ which is a domain is said to have *transcendence degree* r if its (graded) field of fractions $\mathrm{Frac}(R)$ has transcendence degree r over $\mathbf{F}_p$. We will let $\mathscr{K}_{\text{ftd}}$ (resp. $\mathscr{K}^r_{\text{ftd}}$) denote the full subcategory of $\mathscr{K}$ consisting of domains of finite transcendence degree (resp. transcendence degree r). Note that $\mathscr{K}_{\text{ftd}} \supset \mathscr{K}_{\text{poly}}$; many of our methods give information about objects of $\mathscr{K}_{\text{ftd}}$ as well as about objects of $\mathscr{K}_{\text{poly}}$. Let $\mathscr{K}^r_{\text{poly}}$ denote $\mathscr{K}_{\text{poly}} \cap \mathscr{K}^r_{\text{ftd}}$.

§2. MAXIMAL TORI

In this section we will describe the properties of homotopical maximal tori and then show how these properties lead to 1.1 and 1.2. The proofs of the theorems in this section will appear in §4.

It is convenient to begin by recalling some material from [1] and [28] which deals with properties of an algebraic analogue of a maximal torus for rings $R \in \mathscr{K}^r_{\text{ftd}}$.

2.1 THEOREM. [1] *For any object R of $\mathscr{K}^r_{\text{ftd}}$ there exists a $\mathscr{K}$-map $t\colon R \to H^*\hat{B}T^r$ which is an embedding (i.e. monomorphism). If t_1 and t_2 are two such embeddings, then there is an automorphism w of $H^*\hat{B}T^r$ such that $w \cdot t_1 = t_2$.*

According to 2.1, considering general objects of $\mathscr{K}^r_{\text{ftd}}$ is equivalent to considering suitable $\mathscr{A}_p$ subalgebras of $H^*\hat{B}T^r$.

The group $GL(r, \mathbf{F}_p)$ can be identified with the group of automorphisms of $H^*\hat{B}T^r$ (in the category $\mathscr{K}$). Given an $\mathscr{A}_p$ subalgebra R of $H^*\hat{B}T^r$, let W_R denote the subgroup of $GL(r, \mathbf{F}_p)$ consisting of automorphisms of $H^*\hat{B}T$ which leave R elementwise fixed. The geometric multiplication on $\hat{B}T$ gives a Hopf algebra structure on $H^*\hat{B}T$ which is preserved by the action of $GL(r, \mathbf{F}_p)$, and we will let R_{sep} denote the smallest Hopf subalgebra of $H^*\hat{B}T$ which is closed under the action of $\mathscr{A}_p$ and contains R. The action of W_R on $H^*\hat{B}T$ clearly restricts to an action of W_R on R_{sep}.

Remark. The ring R_{sep} is in some sense the separable closure of R in $H^*\hat{B}T$ and in generated by iterated pth powers of two-dimensional classes (cf. proof of 4.1).

2.2 THEOREM. [28] *Let $R \in \mathscr{K}^r_{\text{ftd}}$ be an $\mathscr{A}_p$-subalgebra of $H^*\hat{B}T^r$. Assume that as an algebra R is noetherian and integrally closed. Then the natural map*

$$R \to (R_{\text{sep}})^{W_R}$$

is an isomorphism.

Theorem 2.2 of course applies if R is a polynomial algebra, but even more is true in this case.

2.3 THEOREM. [1] [28] *Let $R \in \mathscr{K}^r_{\text{poly}}$ be an $\mathscr{A}_p$ subalgebra of $H^*\hat{B}T^r$. Assume that $p \nmid v_R$. Then $R_{\text{sep}} = H^*\hat{B}T^r$, the group $W_R \subset GL(r, \mathbf{F}_p)$ is generated by pseudoreflections, and $|W_R| = v_R$.*

Finally, there is an algebraic existence theorem.

2.4 THEOREM. [1] [28] *Let W be a subgroup of $GL(r, \mathbf{F}_p)$ and R the ring $(H^*\hat{B}T^r)^W$. Then $R \in \mathscr{K}^r_{\text{ftd}}$, R is noetherian and integrally closed, $R_{\text{sep}} = H^*\hat{B}T^r$, and the natural map $W \to W_R$ is an isomorphism. Moreover, if W is generated by pseudoreflections and $p \nmid |W|$ then $R \in \mathscr{K}_{\text{poly}}$ and $v_R = |W|$.*

What we will now do is to give geometric strengthenings of the above theorems under the assumption that the ring R in question is the cohomology ring of a space. The basic philosophy behind our approach is due to Rector [22].

Let $\hat{B}\mathscr{T}^r$ stand for a CW-complex with some chosen homotopy equivalence to $\hat{B}T^r$. We introduce this notation because below we would like to take a map $f\colon \hat{B}T^r \to X$ and consider for instance self-maps of $\hat{B}T^r$ over X; to do this is a homotopy invariant way it is necessary to replace f by an equivalent Serre fibration $t\colon \hat{B}\mathscr{T}^r \to X$.

2.5 THEOREM. *For any p-complete space X with $H^*X \in \mathscr{K}^r_{\text{ftd}}$ there exists a space $\hat{B}\mathscr{T}^r$ and a Serre fibration $t\colon \hat{B}\mathscr{T}^r \to X$ such that the induced cohomology homomorphism t^* is an embedding. If $t_i\colon \hat{B}\mathscr{T}^r_i \to X$, $(i = 1,2)$ are two such maps, then there is a homotopy equivalence $w\colon \hat{B}\mathscr{T}^r_1 \to \hat{B}\mathscr{T}^r_2$ such that $t_1 = w \cdot t_2$.*

2.6 *Definition.* A *space X with a maximal r-torus* is a pair (X, t) in which X is a p-complete space with $H^*X \in \mathscr{K}^r_{\mathrm{ftd}}$ and $t: \hat{B}\mathscr{T}^r \to X$ is a Serre fibration such that t^* is an embedding.

The above definition of "maximal r-torus" is not the most useful general one but it is convenient for the purposes of this paper. According to 2.5, considering a general p-complete X with $H^*X \in \mathscr{K}^r_{\mathrm{ftd}}$ is more or less equivalent to considering a space X with a maximal r-torus.

2.7 THEOREM. *Let X be a space with a maximal r-torus $t: \hat{B}\mathscr{T}^r \to X$ and let R be the algebra $t^*(H^*X) \subset H^*\hat{B}T^r$. Then $R_{\mathrm{sep}} = H^*\hat{B}T^r$ and hence $R = (H^*\hat{B}T^r)^{W_R}$.*

In the above theorem, the geometric assumption that R is the cohomology of a space has allowed us to sidestep requirements (cf. 2.3) of the type $p \nmid \nu_R$. This theorem is false for $p = 2$ (for instance, let X be the 2-completion of $BSU(2)$).

2.8 *Definition.* Let X be a space with a maximal r-torus $t: \hat{B}\mathscr{T}^r \to X$. The *Weyl space* $\mathscr{W}_X = \mathscr{W}_X(t)$ of X is the space of self-equivalences $w: \hat{B}\mathscr{T} \to \hat{B}\mathscr{T}$ such that $t \cdot w = t$. The *Weyl group* $W_X = W_X(t)$ of X is the component group $\pi_0 \mathscr{W}_X$.

Remark. Composition of maps gives the Weyl space $\mathscr{W}_X$ the structure of a group-like topological monoid, and actions of $\mathscr{W}_X$ can be used in forming suitable Borel constructions (1.10).

Given X as in 2.8, let R stand for the subalgebra $t^*(H^*X)$ of $H^*\hat{B}T^r$. The group of homotopy classes of self homotopy equivalences of $\hat{B}T^r$ is $GL(r, \mathbf{Z}_p)$, so that there is a commutative diagram

$$\begin{array}{ccc} W_X & \to & W_R \\ \downarrow & & \downarrow \\ GL(r, \mathbf{Z}_p) & \to & GL(r, \mathbf{F}_p) \end{array}$$

in which the horizontal arrows are constructed by taking induced cohomology homomorphisms. In this diagram, it is not *a priori* clear that W_X is finite or that the left-hand vertical arrow is a monomorphism, although by definition the right-hand vertical arrow is a monomorphism.

2.9 THEOREM. *Let X be a space with a maximal r-torus $t: \hat{B}\mathscr{T}^r \to X$ and let R be the algebra $t^*(H^*X) \subset H^*\hat{B}T^r$. Then the Weyl space $\mathscr{W}_X$ is homotopically discrete (i.e., each component of $\mathscr{W}_X$ is contractible), and the natural map $W_X \to W_R$ is an isomorphism.*

If $\hat{B}\mathscr{T}^r \to X$ is a space with a maximal r-torus, there is an induced map $B(\mathscr{W}_X, \hat{B}\mathscr{T}^r) \to X$ and up to homotopy a map $\tau: \hat{B}(\mathscr{W}_X, \hat{B}\mathscr{T}^r) \to X$ (see 1.10 for the notation here). Note that the homomorphism $W_X \to GL(r, \mathbf{Z}_p)$ gives by naturality an action of W_X on the Eilenberg–MacLane space $K((\mathbf{Z}_p)^r, 2) = \hat{B}T^r$.

2.10 THEOREM. *Suppose that X is a space with a maximal r-torus and that as an algebra H^*X is noetherian and integrally closed. Assume that $p \nmid |W_X|$. Then the above map.*

$$\tau: \hat{B}(\mathscr{W}_X, \hat{B}\mathscr{T}^r) \to X$$

is a homotopy equivalence. Moreover, the space $\hat{B}(\mathscr{W}_X, \hat{B}\mathscr{T}^r)$ is homotopy equivalent to $\hat{B}(W_X, \hat{B}T^r)$.

As in 2.3, there are additional algebraic restrictions in the case of a space with a polynomial cohomology ring. Note (2.9) that under the hypotheses of the following theorem the group W_X can be identified with a subgroup of $GL(r, \mathbf{Z}_p)$, well-defined up to conjugacy.

2.11 THEOREM. *Suppose that X is a space with a maximal r-torus and that H^*X is noetherian and integrally closed. Then the natural maps*

$$H^*X \to H^*(\hat{B}T^r)^{W_X}$$

$$H^*(X, \mathbf{Z}_p) \to H^*(\hat{B}T^r, \mathbf{Z}_p)^{W_X}$$

$$\mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^*(X, \mathbf{Z}_p) \to (\mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^*(\hat{B}T^r, \mathbf{Z}_p))^{W_X}$$

are isomorphisms. If $H^*X \in \mathscr{K}_{\text{poly}}$, then $W_X \subset GL(r, \mathbf{Z}_p)$ is generated by pseudoreflections.

Finally, there is a geometric existence theorem.

2.12 THEOREM. *Let W be a finite subgroup of $GL(r, \mathbf{Z}_p)$ and let X be the space $\hat{B}(W, \hat{B}T^r)$ formed with respect to the natural action of W on $\hat{B}T^r = K((\mathbf{Z}_p)^r, 2)$. Assume that $p \nmid |W|$. Then $H^*X = (H^*\hat{B}T^r)^W \in \mathscr{K}^r_{\text{ftd}}$, H^*X is noetherian and integrally closed, and, given a maximal r-torus $t: \hat{B}\mathscr{T}^r \to X$ for X (2.5, 2.6), the group $W_X(t)$ is conjugate as a subgroup of $GL(r, \mathbf{Z}_p)$ to W. Moreover, if W is generated by pseudoreflections then $H^*X \in \mathscr{K}_{\text{poly}}$ and $\nu_X = |W|$.*

We will finish this section by giving proofs for the homotopy theoretic theorems of the introduction.

Proof of 1.1. Let $r = \rho_X$ (so that $H^*X \in \mathscr{K}^r_{\text{poly}}$) and let $t: \hat{B}\mathscr{T}^r \to X$ be a maximal r-torus for X (2.5). By 2.9, 2.7 and 2.2, the group $W_X(t)$ defined in 2.8 is adapted to X. By 2.11 the group $W_X(t)$ is generated by pseudoreflections; the equality $\nu_X = |W_X(t)|$ follows from 2.4. Suppose that $W \subset GL(r, \mathbf{Z}_p)$ is some other subgroup which is adapted to X. Let S be the set of homotopy classes of maps $f: \hat{B}T^r \to X$ such that f^* is an embedding. It follows from 2.5 that $GL(r, \mathbf{Z}_p)$ acts transitively on S and that the isotropy subgroup of any element of S is conjugate to the image of the natural map $\pi_0 \mathscr{W}_X(t) \to GL(r, \mathbf{Z}_p)$. As a consequence, W is conjugate to a subgroup of $W_X(t)$ and so by (2.9) the reduction mod p map $W \to GL(r, \mathbf{F}_p)$ is a monomorphism. By 2.4 and 2.1 the image of W in $GL(r, \mathbf{F}_p)$ is a conjugate to the image of $W_X(t)$, which by the above implies that $|W_X(t)| = |W|$ and finishes the proof.

Proof of 1.2. This is a direct consequence of 2.4 and 2.2.

§3. THE FUNCTOR T^V

In this section V will denote an $\mathbf{F}_p$-vector space of dimension r. The purpose of this section is to describe some properties of Lannes' functor T^V [20].

Recall [20, §2] that T^V is left adjoint to the functor $\mathscr{K} \to \mathscr{K}$ which sends $R \in \mathscr{K}$ to $H^*(BV) \otimes R$. The functor T^V is exact [20, 2.4.1] and preserves tensor products [20, 2.4.3].

If $f: R \to H^*BV$ is a map of $\mathscr{K}$, then corresponding to f under adjointness is a map $T^V(R) \to \mathbf{F}_p$ or in other words a ring homomorphism $\tilde{f}: T^V(R)^0 \to \mathbf{F}_p$. We will let $T^V_f(R)$ denote the tensor product $T^V(R) \otimes_{T^V(R)^0} \mathbf{F}_p$, where the action of $T^V(R)^0$ on $\mathbf{F}_p$ is given by $\tilde{f}$. If $f: R \to S$ and $g: S \to H^*BV$ are maps of $\mathscr{K}$ with $g \cdot f = h$ then f induces a natural map $T^V_h R \to T^V_g S$. We will denote this map $T^V_g(f)$; using the notation $T^V_h(f)$ would leave some

ambiguity as to the range of the map, since there might be several maps $g: S \to H^*BV$ with $g \cdot f = h$. The inclusion $\{0\} \subset V$ induces for any $R \in \mathscr{K}$ a natural map $\varepsilon: R = T^{\{0\}}R \to T^V R$ which for $f: R \to H^*BV$ passes to a map $\varepsilon_f: R \to T^V_f R$.

For any space X, the evaluation map $BV \times \mathrm{Map}(BV, X) \to X$ induces a cohomology homomorphism

$$H^*X \to H^*(BV) \otimes H^*\mathrm{Map}(BV, X)$$

which has as adjoint a natural map

$$\lambda: T^V(H^*X) \to H^*\mathrm{Map}(BV, X).$$

For each map $f: BV \to X$, λ induces a map $\lambda_f: T^V_f(H^*X) \to H^*\mathrm{Map}(BV, X)_f$, which fits into a commutative diagram

$$\begin{array}{ccc} H^*X & \xrightarrow{=} & H^*X \\ {\scriptstyle \varepsilon_f}\downarrow & & \downarrow \\ T^V_f(H^*X) & \xrightarrow{\lambda_f} & H^*\mathrm{Map}(BV, X)_f \end{array}$$

in which the right vertical map is induced by evaluation at the basepoint of BV. (Here and subsequently we write $T^V_f(H^*X)$, etc., instead of $T^V_{f^*}(H^*X)$.)

Lannes has proved the following two theorems.

3.1 Theorem. [20, 3.1.4] *If X is a simply-connected p-complete space such that H^*X is of finite type, then the natural map*

$$[BV, X] \to \mathrm{Hom}_{\mathscr{K}}(H^*X, H^*BV)$$

is a bijection.

3.2 Theorem. [20, 3.2.2] *Let X be a simply-connected p-complete space such that H^*X is of finite type and let $f: BV \to X$ be a map. Assume that $T^V_f(H^*X)$ is of finite type and vanishes in degree 1. Then the natural map*

$$\lambda_f: T^V_f(H^*X) \to H^*\mathrm{Map}(BV, X)_f$$

is an isomorphism.

We will need to establish some algebraic properties of T^V. Recall that the topological multplication on BV gives H^*BV the structure of a Hopf algebra with comultiplication $\Delta: H^*BV \to H^*BV \otimes H^*BV$. For any $g: H^*BV \to H^*BV$, the map $T^V(H^*BV) \to H^*BV$ which is adjoint to $(g \otimes 1) \cdot \Delta$ induces a map $\mu_g: T^V_g(H^*BV) \to H^*BV$. For any map $f: BV \to BV$, μ_f fits into a commutative diagram

$$\begin{array}{ccc} T^V_f(H^*BV) & \xrightarrow{\mu_f} & H^*BV \\ {\scriptstyle \lambda_f}\downarrow & & \downarrow = \\ H^*\mathrm{Map}(BV, BV)_f & \xrightarrow[\tau]{} & H^*BV \end{array}$$

in which τ is induced by the map $BV \to \mathrm{Map}(BV, BV)_f$ which sends a point $x \in BV$ to the map obtained by composing f with (right) multiplication by x.

3.3 Lemma. *For any map $g: H^*BV \to H^*BV$ the maps $\varepsilon_g: H^*BV \to T^V_g(H^*BV)$ and $\mu_g: T^V_g(H^*BV) \to H^*BV$ are isomorphisms. If f is an endomorphism of H^*BV then $\mu_g \cdot T^V_g(f) = f \cdot \mu_{gf}$.*

Proof. The first two statements can be proved by using the elementary fact that $\lambda: T^V(H^*BV) \to H^*\mathrm{Map}(BV, BV)$ is an isomorphism [20, 3.4.4] and then calculating with function space cohomology; for instance, the statement about ε_g follows from the calculation that for any $h: BV \to BV$ the map $\mathrm{Map}(BV, BV)_h \to BV$ given by evaluation at the basepoint is an equivalence. The final assertion is proved by showing that both maps involved correspond under adjointness to $((gf) \otimes f) \cdot \Delta$.

The following lemma will be used in §4. It is proved by using the fact (cf. [14] or [19]) that $\lambda: T^V(H^*\hat{B}T^s) \to H^*\mathrm{Map}(BV, \hat{B}T^s)$ is an isomorphism.

3.4 LEMMA. *For any $s \geq 0$ and map $f: H^*\hat{B}T^s \to H^*BV$, the map $\varepsilon_f: H^*\hat{B}T^s \to T^V_f(H^*\hat{B}T^s)$ is an isomorphism.*

3.5 PROPOSITION. *Let $f: R \to H^*BV$ be a monomorphism and let ι denote the identity map of H^*BV. Then the map $T^V_\iota(f): T^V_f R \to T^V_\iota(H^*BV) \cong H^*BV$ is a monomorphism.*

Proof. Denote the tensor product $T^V_f(R) \otimes_{T^V(R)} T^V(H^*BV)$ by S. The algebra $T^V_f R$ is flat as a module over $T^V R$ (this is a consequence of the associativity of the tensor product and the fact that every module over the p-boolean ring $(T^V R)^0$ is flat [20, 1.8]). It follows from the exactness property of T^V [20, 2.4.1] that the natural map $T^V_f R \to S$ is a monomorphism. It is easy to calculate (cf. 3.3) that S is the product $\prod_g T^V_g(H^*BV)$ indexed by $g: H^*BV \to H^*BV$ such that $g \cdot f = f$; under this identification the monomorphism $T^V_f R \to S$ is $\prod_g T^V_g(f)$. To complete the proof, then, it is enough to show that if $g: H^*BV \to H^*BV$ is a map with $g \cdot f = f$, then the kernel of $T^V_g(f)$ contains the kernel of $T^V_\iota(f)$.

Let u denote the composite map $\mu_\iota \cdot T^V_\iota(f)$ and v the corresponding composite $\mu_g \cdot T^V_g(f)$. By 3.3, we will be done if we can show that the kernel of v contains the kernel of u. The map $g: H^*BV \to H^*BV$ is $(B\gamma)^*$ for a unique homomorphism $\gamma: V \to V$. Write V as a direct sum $V_1 \oplus V_2$ where V_1 is the subgroup killed by some power of γ and V_2 is the intersection of the images of all powers of γ. Note that the restriction of γ to V_2 gives an automorphism of V_2. Let $\varepsilon: V \to V_2$ be the projection map, and choose N large enough that the restriction of γ^N to V_1 is zero. Let $e: H^*BV \to H^*BV$ be the map $(B\varepsilon)^*$. Since $e \cdot g^N = g^N$, it follows from 3.3 that $\mu_g \cdot T^V_g(g^N) = e \cdot \mu_g \cdot T^V_g(g^N)$. Consequently, there are equalities

$$\begin{aligned} v &= \mu_g \cdot T^V_g(f) \\ &= \mu_g \cdot T^V_g(g^N) \cdot T^V_{g^N - \iota}(f) \\ &= e \cdot \mu_g \cdot T^V_g(g^N) \cdot T^V_{g^N - \iota}(f) \\ &= e \cdot \mu_g \cdot T^V_g(f) \end{aligned}$$

where the first equality is the definition of v, the second and fourth follow from $g \cdot f = f$, and the third is the above remark. Let γ^{-1} denote the self-map of V which is zero on V_1 and the inverse of $\gamma|_{V_2}$ on V_2; let g^{-1} denote $(B\gamma^{-1})^*$. The identity $\gamma \cdot \gamma^{-1} \cdot \varepsilon = \varepsilon$ gives $e = e \cdot g^{-1} \cdot g$. It follows from 3.3 that $e \cdot \mu_g = e \cdot g^{-1} \cdot \mu_\iota \cdot T^V_\iota(g)$. Combining this with the above gives that v agrees with the map $g^{-1} \cdot \mu_\iota \cdot T^V_\iota(g) \cdot T^V_g(f) = g^{-1} \cdot \mu_\iota \cdot T^V_\iota(f)$, where the equality follows from $g \cdot f = f$. In other words, $v = g^{-1} \cdot u$, which proves the desired result.

3.6 PROPOSITION. *Suppose that $R \in \mathscr{K}$ is a subalgebra of H^*BV. Let $f: R \to H^*BV$ be the inclusion map and let ι be the identity map of H^*BV. Then $(\varepsilon_\iota)^{-1} \cdot T^V_\iota(f)$ gives an isomorphism from $T^V_f R$ to the smallest Hopf subalgebra of H^*BV which is closed under the action of $\mathscr{A}_p$ and contains R.*

This proposition does not require the assumption that R is noetherian.

Proof of 3.6. Adjoint to the identity map of $T^V R$ is a map $R \to H^*(BV) \otimes T^V R$. Compose this with the diagonal map of H^*BV to get a map

$$R \to H^*(BV) \otimes H^*(BV) \otimes T^V R$$

and reverse the process of taking adjoints; this gives a map

$$\Delta : T^V R \to H^*(BV) \otimes T^V R$$

which makes $T^V R$ into a comodule over H^*BV. (This comodule structure is an algebraic counterpart to the action of BV on $\mathrm{Map}(BV, X)$ induced by the left translation action of BV on itself). The comodule structure passes to a quotient comodule structure $\Delta_f : T_f^V R \to H^*(BV) \otimes T_f^V R$ which fits into a commutative diagram

$$\begin{array}{ccccc} R & \xrightarrow{\varepsilon_f} & T_f^V R & \xrightarrow{\Delta_f} & H^*(BV) \otimes T_f^V R \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle 1 \otimes g} \\ H^*BV & \xrightarrow{=} & H^*BV & \xrightarrow{\Delta} & H^*(BV) \otimes H^*(BV) \end{array}$$

where we have used ε_i (3.3) to identify $T_i^V(H^*BV)$ with H^*BV and $g = (\varepsilon_i)^{-1} \cdot T_i^V(f)$. Since f is a monomorphism the map g is a monomorphism (3.5). It is easy to see that Δ_i is the Hopf algebra comultiplication map on H^*BV. It now follows from the fact that the comultiplication on H^*BV is cocommutative that $\Delta_f(T_f^V R)$ is contained in $T_f^V R \otimes T_f^V R$ and thus that $T_f^V R$ is a Hopf subalgebra of H^*BV. On the other hand, if S is a Hopf subalgebra of H^*BV closed under the action of $\mathscr{A}_p$ and $h : S \to H^*BV$ is the inclusion, then $\varepsilon_h : S \to T_h^V S$ is an isomorphism [13, 4.8]. The proposition follows.

§4. MAXIMAL TORI, CONTINUED

In this section we will give proofs of the theorems from §2. We will continue to use the notation of §2 and §3.

4.1 LEMMA. *Let $R \in \mathscr{K}'_{\mathrm{fd}}$ be an $\mathscr{A}_p$ subalgebra of $H^*\hat{B}T^r$. Then there exists a space X with $H^*X \cong R_{\mathrm{scp}}$ iff $R_{\mathrm{scp}} = H^*\hat{B}T^r$.*

Proof. Define a filtration of $H^2\hat{B}T^r$

$$F_0 \subset F_1 \subset \cdots \subset H^2\hat{B}T$$

by declaring that $x \in H^2\hat{B}T$ belongs to F_s iff the element x^{p^s} of the field (1.10) $\mathrm{Frac}(H^*\hat{B}T)$ is separable [28] over $\mathrm{Frac}(R)$. According to [28] $R_{\mathrm{scp}} \supset R$ is the subalgebra of $H^*\hat{B}T$ generated by $\sum_{s \geq 0}(F_s)^{p^s}$. Suppose that there exists such a space X of the indicated type, so that in particular $H^2X \cong F_0$. Let s be the $\mathbf{F}_p$ rank of F_0. Since $H^3X = 0$ the group $H^2(X, \mathbf{Z}_p)$ has no torsion, so there exists a map $f : X \to \hat{B}T^s$ such that $H^2(f)$ is an isomorphism. Let Y be the homotopy fibre of f. It is clear that H^*Y is isomorphic to the (polynomial) subalgebra of $H^*\hat{B}T^r$ generated by $\sum_{s \geq 1}(F_s)^{p^s}$, and thus that H^*Y is non-zero only in degrees divisible by $2p$. An argument [15] involving the non-existence of higher elements of Hopf invariant one mod p implies that this fibre Y is contractible.

Proof 2.5. Let V be the group ${}_pT^r$ generated by the elements of order p in the torus T^r. The map $BV \to \hat{B}T^r$ induces a monomorphism on cohomology which we will use to identify

$H^*\hat{B}T^r$ with the subalgebra of H^*BV generated by the images under the Bockstein of one-dimensional classes. By 2.1 and 3.1 there is a map $f: BV \to X$ such that f^* embeds H^*X into $H^*\hat{B}T^r \subset H^*BV$. Let $R \subset H^*\hat{B}T^r$ denote $f^*(H^*X)$.It follows from 3.6 and the fact that $H^*\hat{B}T^r$ is a Hopf subalgebra of H^*BV that $T_f^V(H^*X)$ is isomorphic to R_{sep}. On the other hand, by 3.2 there is an isomorphism $\lambda_f: T_f^V(H^*X) \to H^*\mathrm{Map}(BV, X)_f$ and so $T_f^V(H^*X) \cong R_{sep}$ is realizable as the cohomology of a space. Lemma 4.1 now shows that $R_{sep} = H^*\hat{B}T^r$ and thus that $H^*\mathrm{Map}(BV, X)_f \cong H^*\hat{B}T^r$. There is a map $c: \mathrm{Map}(BV, X)_f \to \hat{B}T^r$ which induces this cohomology isomorphism, and, since the space $\mathrm{Map}(BV, X)_f$ is $H_*(-, \mathbf{F}_p)$-local [5, 12.9], the map c is an equivalence. Let ι denote the identity map of BV. There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}(BV, BV)_\iota & \xrightarrow{f\cdot(-)} & \mathrm{Map}(BV, X)_f \\ e_\iota \downarrow & & \downarrow e_f \\ BV & \xrightarrow{f} & X \end{array}$$

in which the vertical maps are given by basepoint evaluation and the left-hand vertical map is an equivalence. It follows that the map e_f induces an embedding $(e_f)^*: H^*X \to H^*\mathrm{Map}(BV, X)_f$ and that the desired map $t: \hat{B}\mathscr{T}^r \to X$ can be obtained by replacing e_f by a weakly equivalent Serre fibration.

Now suppose that $g: Y \to X$ is a Serre fibration with $Y \cong \hat{B}T^r$ and g^* an embedding. By the fact [20] that H^*BV is an injective object of $\mathscr{K}$, the map $f^*: H^*X \to H^*BV$ extends over g^* to a map $\chi: H^*Y \to H^*BV$ (i.e. $\chi \cdot g^* = f^*$). If $h: BV \to Y$ is a map with $h^* = \chi$, then the composite $g \cdot h$ is homotopic to f(3.1). Since g is a fibration it is possible by the covering homotopy property to adjust h so that $g \cdot h = f$. There is then a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}(BV, Y)_h & \xrightarrow{g\cdot(-)} & \mathrm{Map}(BV, X)_f \\ e_h \downarrow & & \downarrow e_f \\ Y & \xrightarrow{g} & X \end{array}$$

in which the vertical arrows are given by basepoint evaluation. The map e_h is an equivalence by a simple direct argument or by a combination of 3.2 and 3.4. To complete the proof it is enough by standard homotopy theory to show that the upper horizontal arrow in this diagram is an equivalence, or, since the spaces involved are $H_*(-, \mathbf{F}_p)$-local [5, 12.9], that the corresponding cohomology map $T_h^V(g)$ (see 3.2) is an isomorphism. The map $T_\iota^V(f): T_f^V(H^*X) \to T_\iota^V(H^*BV)$ is a monomorphism by 3.5; since $T_\iota^V(f) = T_\iota^V(h) \cdot T_h^V(g)$, this implies that $T_h^V(g)$ is a monomorphism. Since $T_h^V(H^*Y) \cong H^*\hat{B}T^r$ by 3.4 and $T_f^V(H^*X) \cong H^*\hat{B}T^r$ by the argument above, dimension counting in fact shows that $T_h^V(g)$ must be an isomorphism.

Proof of 2.7. Let $V = (\mathbf{F}_p)^r$ and choose a map $g: BV \to \hat{B}\mathscr{T}^r$ such that g^* is a monomorphism. Let $f = t \cdot g$. It is possible to argue as above in the proof of 2.5 that R_{sep} is realizable as the cohomology of the space $\mathrm{Map}(BV, X)_f$. The desired conclusion follows from 4.1.

4.2 LEMMA. *Let X be a space with a maximal r-torus $t: \hat{B}\mathscr{T}^r \to X$ and V the vector space $(\mathbf{F}_p)^r$. Suppose that $f: BV \to \hat{B}\mathscr{T}^r$ is a map such that f^* is an embedding, and let $g = t \cdot f$. Then the natural map*

$$\mathrm{Map}(BV, \hat{B}\mathscr{T}^r)_f \xrightarrow{t\cdot(-)} \mathrm{Map}(BV, X)_g$$

is an equivalence.

The proof of this is contained in the proof above of 2.5. We now head towards a proof of 2.9. First we have an algebraic lemma, whose proof comes down to a straightforward calculation.

4.3 LEMMA. *Suppose that R is an object of $\mathscr{K}^r_{\mathrm{ftd}}$ and that $t: R \to H^*\hat{B}T^r$ is an embedding. Let V be the vector space $(\mathbf{F}_p)^r$, and identify $H^*\hat{B}T^r$ with the subalgebra of H^*BV generated by the images under the Bockstein of one-dimensional classes. Let G be the group of $\mathscr{K}$ automorphisms g of H^*BV such that $g \cdot t = t$. Then each element of G carries $H^*\hat{B}T^r$ to itself, and the subsequent induced homomorphism $G \to W_R$ is an isomorphism.*

Proof of 2.9. Let $V = {}_pT^r$ as above. The main idea of this proof is to find a relationship between maps $BV \to X$ and maps $\hat{B}T^r \to X$ by using the fibration sequence $BV \to \hat{B}T^r \to \hat{B}T^r$. We begin with some general considerations.

Suppose that $q: E \to B$ is a (Serre) fibration of connected CW-complexes with fibre F, and that Y is a space. By elementary homotopy theory there is a space $\mathrm{Map}(E/B, Y)$ and a fibration $\dot{q}: \mathrm{Map}(E/B, Y) \to B$ such that the space of sections $\Gamma(\dot{q})$ is naturally weakly equivalent to $\mathrm{Map}(E, Y)$. The fibre of $\dot{q}$ is $\mathrm{Map}(F, Y)$.

Suppose in addition that $f: E \to X$ and $g: Y \to X$ are maps, with g a fibration. Let $\mathrm{Map}_X(E, Y)$ denote the space of all maps $h: E \to Y$ such that $f = g \cdot h$. The map $\mathrm{Map}(E, Y) \to \mathrm{Map}(E, X)$ given by composition with g is a fibration with fibre over $f \in \mathrm{Map}(E, X)$ equal to $\mathrm{Map}_X(E, Y)$. It is clear that there is a space $\mathrm{Map}_X(E/B, Y)$ and a fibration $\dot{q}_X: \mathrm{Map}_X(E/B, Y) \to B$ such that $\Gamma(\dot{q}_X) \cong \mathrm{Map}_X(E, Y)$; the map $\dot{q}_X$ can be obtained by constructing a (homotopy) fibre square

$$\begin{array}{ccc} \mathrm{Map}_X(E/B, Y) & \to & \mathrm{Map}(E/B, Y) \\ {\scriptstyle \dot{q}_X}\downarrow & & \downarrow \\ B & \to & \mathrm{Map}(E/B, X) \end{array}$$

in which the lower horizontal arrow corresponds to f. It follows from this fibre square that the (homotopy) fibre of $\dot{q}_X$ is $\mathrm{Map}_X(F, Y)$.

Suppose that X is a space with a maximal torus $t: \hat{B}\mathscr{T}^r \to X$. Let Y be the space $\hat{B}\mathscr{T}^r$ and $g: Y \to X$ the map t. Let $q: E \to B$ be a fibration equivalent to the map $K((\mathbf{Z}_p)^r, 2) \to K((\mathbf{Z}_p)^r, 2)$ induced by the homomorphism $(\mathbf{Z}_p)^r \to (\mathbf{Z}_p)^r$ given by multiplication by p, and let $f: E \to X$ be the composite of t with an equivalence $E \cong \hat{B}\mathscr{T}^r$. Up to homotopy we will identify the fibre of q with BV, where V is the vector space $(\mathbf{F}_p)^r = {}_p\mathbf{T}^r$. By the above considerations, $\mathrm{Map}_X(Y, Y) \cong \mathrm{Map}_X(E, Y)$ is equivalent to the space of sections of a fibration $\dot{q}_X$ over B with fibre $\mathrm{Map}_X(BV, Y)$.

We will use a superscript " + " (e.g. "$\mathrm{Map}^+(X, Y)$") to denote the subspace of a mapping space consisting of maps with induce a monomorphism on cohomology. It is clear that there are equivalences $\mathscr{W}_X \cong \mathrm{Map}^+_X(Y, Y) \cong \mathrm{Map}^+_X(E, Y)$; in effect, a map $Y \to Y$ is a homotopy equivalence iff it induces a monomorphism on cohomology. A map $h: E \to Y$ induces a monomorphism on cohomology iff the composite of h with the map $BV \to E$ induces a monomorphism $H^*Y \to H^*BV$. It follows that $\mathrm{Map}^+_X(E, Y)$ is equivalent to the space of sections of a fibration $\dot{q}^+_X: \mathrm{Map}^+_X(E/B, Y) \to B$ with fibre $\mathrm{Map}^+_X(BV, Y)$.

The space $\mathrm{Map}^+_X(BV, Y)$ is the fibre of the map $\mathrm{Map}^+(BV, Y) \to \mathrm{Map}^+(BV, X)$ given by composition with g over the point represented by the composite $h: BV \to E \to X$. Thus by 4.2 the space $\mathrm{Map}^+_X(BV, Y)$ is homotopically discrete and the component set $\pi_0\mathrm{Map}^+_X(BV, Y)$ is isomorphic in a natural way to the set of homotopy classes of maps

$a: BV \to Y$ such that a^* is a monomorphism and $q \cdot a \sim h$. This last set is isomorphic (3.1) to the set of monomorphisms $a': H^*Y \to H^*BV$ such that $a' \cdot g^* = h^*$. It now follows easily that the group W_R, considered (4.3) as a group of automorphisms of H^*BV or equivalently as a group of homotopy classes of self-equivalences of BV, acts simply transitively on $\pi_0 \mathrm{Map}_X^+(BV, Y)$.

The fibration $\dot{q}_X^+$ is necessarily fibre-homotopically trivial, since it is a fibration with a homotopically discrete fibre over a simply connected base. The basepoint evaluation map $\Gamma(\dot{q}_X^+) \to \mathrm{Map}_X^+(BV, Y)$ is thus an equivalence. But $\Gamma(\dot{q}_X^+) \sim \mathrm{Map}_X^+(E, Y)$, so, by naturality, the restriction map $\mathrm{Map}_X^+(E, Y) \to \mathrm{Map}_X^+(BV, Y)$ is an equivalence. This shows that $\mathrm{Map}_X^+(E, Y) \cong \mathscr{W}_X$ is homotopically discrete; the desired statement about W_X ($\cong \pi_0 \mathscr{W}_X$) follows in a straightforward way from the above identification of $\pi_0 \mathrm{Map}_X^+(BV, Y)$.

Proof of 2.10. Let $R \subset H^*\hat{B}\mathscr{T}'$ be the image of H^*X. By 2.2 the natural restriction map gives an isomorphism $H^*X \cong (H^*\hat{B}\mathscr{T}')^{W_R}$, while by a twisted coefficient Serre spectral sequence argument the corresponding restriction map gives an isomorphism $H^*B(\mathscr{W}_X, \hat{B}\mathscr{T}') \cong H^*(\hat{B}\mathscr{T}')^{W_R}$. Since W_R is naturally isomorphic to W_X, it follows immediately that the map $B(\mathscr{W}_X, \hat{B}\mathscr{T}') \to X$ induces an isomorphism on cohomology and thus that the completed map τ is an equivalence.

The spaces $B(\mathscr{W}_X, \hat{B}\mathscr{T}')$ and $B(W_X, \hat{B}T')$ are each two-stage Postnikov systems with fundamental group W_X and second homotopy group $(\mathbf{Z}_p)^r$; by construction both spaces have the same action of the fundamental group on the second homotopy group. The difference between the two spaces can therefore be measured by a k-invariant in $H^3(BW_X, (\mathbf{Z}_p)^r)$; since this group vanishes ($p \nmid |W_X|$) the two spaces (and hence their completions) must be equivalent.

The following lemmas will be used in proving 2.11.

4.4 Lemma. *Let W be a finite group and $f: A \to B$ a map of modules over the group ring $\mathbf{Z}_p[W]$. Suppose that A and B are finitely generated free modules over $\mathbf{Z}_p$, that W acts trivially on A, and that f induces an isomorphism $A/pA \cong (B/pB)^W$. Then f induces isomorphisms $A \cong B^W$ and $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} A \cong (\mathbf{Q}_p \otimes_{\mathbf{Z}_p} B)^W$.*

Proof. Let C denote B^W. Pick $x \in B$ and suppose that $px \in C$. For each $w \in W$ the element $px - wpx = p(x - wx)$ vanishes; since B is p-torsion free, this implies that $x = wx$ and thus that $x \in C$. Consequently, the maps $C/pC \to B/pB$ and $C/pC \to (B/pB)^W$ are monomorphisms and hence (since the composite $A/pA \to C/pC \to (B/pB)^W$ is an isomorphism) the map $A/pA \to C/pC$ is an isomorphism. The fact that $A \to C$ is an isomorphism now follows from the fact that A and C are both finitely generated free modules over $\mathbf{Z}_p$. For the second isomorphism, note that since $\mathbf{Q}_p$ is flat as a module over $\mathbf{Z}_p$ (i.e. $\mathbf{Q}_p \otimes_{\mathbf{Z}_p}$ is an exact functor) and the fixed point functor $(-)^W$ can be expressed naturally as a kernel, the natural map $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} (B^W) \to (\mathbf{Q}_p \otimes_{\mathbf{Z}_p} B)^W$ is an isomorphism.

4.5 Lemma. *If $H^*X \in \mathscr{K}_{\mathrm{poly}}$ then the ring $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^*(X, \mathbf{Z}_p)$ is a polynomial algebra over $\mathbf{Q}_p$ on a finite number of generators.*

Proof. We can assume that X is p-complete and in particular simply connected. The Eilenberg–Moore spectral sequence (or Serre spectral sequence) shows that $H^*\Omega X$ is finite and hence that $H^*(\Omega X, \mathbf{Z}_p)$ is finitely generated over $\mathbf{Z}_p$. This implies that $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^*(\Omega X, \mathbf{Z}_p)$ is a finite connected commutative Hopf algebra over $\mathbf{Q}_p$ and thus isomorphic as an algebra to an exterior algebra over $\mathbf{Q}_p$ (see [21, Appendix and 5.15]).

(Another way to obtain this last conclusion is to use obstruction theory and the fact that k-invariants of ΩX are torsion to produce a map

$$\Omega X \to \prod_i K(\pi_i \Omega X/\text{torsion}, i)$$

which induces an isomorphism on $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^*(-, \mathbf{Z}_p)$. The fact that $H^*(\Omega X, \mathbf{Z}_p)$ is finitely generated implies that except for a finite number of odd integers i the spaces in the above product are contractible.) By the Samelson–Leray theorem [21, 7.20], the dual of the Hopf algebra $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^*(\Omega X, \mathbf{Z}_p)$ is also an exterior algebra over $\mathbf{Q}_p$. The spectral sequence argument of Borel [3, 13.1] (or an easy argument using the bar construction spectral sequence) now shows that $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^*(X, \mathbf{Z}_p)$ is the desired polynomial algebra.

Proof of 2.11. If Y is a space such that H^*Y is of finite type and vanishes in odd dimensions, then for each i the group $H^i(Y, \mathbf{Z}_p)$ is a finitely generated free module over $\mathbf{Z}_p$ and there is a natural isomorphism $H^iY \cong \mathbf{F}_p \otimes_{\mathbf{Z}_p} H^i(Y, \mathbf{Z}_p)$. This remark applies to the spaces X and $\hat{B}\mathscr{T}^r$, where $t: B\mathscr{T}^r \to X$ is the maximal torus for X. The three isomorphisms listed in the theorem follows from 2.2 (cf. proof of 2.10) and 4.4. By 4.5, then, the ring of invariants $(\mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^*(B\mathscr{T}^r, \mathbf{Z}_p))^{W_X}$ is a polynomial algebra. Under the monomorphism (2.9) $W_X \to GL(r, \mathbf{Z}_p) \to GL(r, \mathbf{Q}_p)$ the action of W_X on $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^*(\hat{B}\mathscr{T}^r, \mathbf{Z}_p)$ can be identified with the natural induced action of W_X on the symmetric algebra of the dual of $(\mathbf{Q}_p)^r$, so the desired result is a consequence of 5.1.

Proof of 2.12. It is clear (cf. proof of 2.10) that the natural map $H^*X \to (H^*\hat{B}T^r)^W$ is an isomorphism, so it follows immediately from 2.4 that $H^*X \in \mathscr{K}^r_{\text{ftd}}$, that H^*X is noetherian and integrally closed, and that $H^*X \in \mathscr{K}_{\text{poly}}$ and $\nu_X = |W|$ if W is generated by pseudoreflections. Let $f: \hat{B}\mathscr{T}^r \to X$ be the maximal r-torus for X obtained by replacing the evident map $\hat{B}T^r \to X$ with a Serre fibration, and let $R \subset H^*\hat{B}\mathscr{T}^r$ denote $f^*(H^*X)$. By construction $W \subset W_X(f)$, by 2.4 the natural map $W \to W_R$ is an isomorphism, and by 2.9 the natural map $W_X(f) \to W_R$ is an isomorphism. It follows that $W = W_X(f)$. The desired conjugacy statement is a consequence of 2.5.

§5. PSEUDOREFLECTION GROUPS

In this section we will prove Theorem 1.5.

If D is a commutative domain, a subgroup of $GL(n, D)$ is said to be a *pseudoreflection* (p.r.) subgroup if it is generated by pseudoreflections (see §1). The main property of p.r. subgroups that makes them interesting for our purposes is the following one.

5.1 THEOREM. [10] [7] [4, V, Sect 6, Exer. 8(a)] *Suppose that F is a field and that W is a finite subgroup of $GL(n, F)$. Let W act on the polynomial algebra $R = F[t_1, \ldots, t_n]$ in the natural way. If the ring of invariants R^W is a polynomial algebra, then W is a p.r. subgroup. If the characteristic of F is zero or prime to $|W|$, then the converse holds as well.*

5.2 LEMMA. *If $W \subset GL(n, \mathbf{F}_p)$ is a subgroup of order prime to p, then W lifts to a subgroup $\bar{W} \subset GL(n, \mathbf{Z}_p)$ and the conjugacy class of $\bar{W}$ depends only on that of W. If $\bar{W} \subset GL(n, \mathbf{Z}_p)$ is a subgroup of finite order prime to p and $\bar{W}$ reduces mod p to a p. r. subgroup, then $\bar{W}$ is a p.r. subgroup.*

5.3 *Remark.* It is possible to have a finite p.r. subgroup W of $GL(n, \mathbf{F}_p)$ with the following paradoxical combination of properties:

(1) $\mathbf{F}_p[t_1, \ldots, t_n]^W$ is a polynomial ring,
(2) W lifts to a subgroup $\bar{W}$ of $GL(n, \mathbf{Z}_p)$, but
(3) W does not lift to a p.r. subgroup of $GL(n, \mathbf{Z}_p)$.

Of course, by 5.2 the prime p must divide the order of W, and by 5.1 the ring of invariants $(\mathbf{Z}_p[t_1, \ldots, t_n])^W$ must fail to be a polynomial ring. For example, let p be 3, let $W \subset GL(2, \mathbf{F}_p)$ be the cyclic group of strictly upper-triangular matrices, and let $\bar{W} \subset GL(2, \mathbf{Z}_p)$ be the group generated by the matrix $\begin{pmatrix} -\frac{1}{3} & -\frac{1}{2} \end{pmatrix}$. It is easy to check that $(\mathbf{F}_p[t_1, t_2])^W$ is the polynomial algebra on t_1 and $t_2^3 - t_1^2 t_2$. No generator of W can lift to a pseudoreflection, since $\mathbf{Q}_p$ does not contain a cube root of unity.

Proof of 5.2. The fact that a group W of the indicated type lifts to a subgroup of $GL(n, \mathbf{Z}_p)$ in a way which is unique up to conjugacy is proven in [1, pp. 141–142]. To finish the proof we show in fact that if $A \in GL(n, \mathbf{Z}_p)$ is an element of finite order prime to p which reduces mod p to a pseudoreflection then A itself is a pseudoreflection. Pick such an A, let s be the order of A, and let N denote $s^{-1}(\sum_{i=0}^{s-1} A^i)$. Choose linearly independent elements $v_1, \ldots, v_{n-1}$ in $(\mathbf{F}_p)^n$ which are fixed by A, and let $\bar{v}_i \in (\mathbf{Z}_p)^n$ be a lift of v_i. Then for each $i \leqslant n-1$ the element $N\bar{v}_i$ is fixed by A and reduces mod p to v_i. Since the v_i are linearly independent, it follows easily that the set $\{N\bar{v}_1, \ldots, N\bar{v}_{n-1}\}$ generates a rank $(n-1)$ submodule of $(\mathbf{Z}_p)^n$; by construction the elements of this submodule are in the kernel of $(A - I_n)$ and so the existence of this submodule shows that A is a pseudoreflection.

If F is a field of characteristic zero and $\chi: W \to GL(n, F)$ is a representation of the finite group W, then the *character field* K_χ is by definition the extension field of $\mathbf{Q}$ generated by the values of the character of χ (i.e. by the algebraic numbers trace $(\chi(w))$, $w \in W$). If W is a finite subgroup of $GL(n, F)$, we will let K_W denote the character field of the inclusion representation $W \subset GL(n, F)$. Note that $K_W \subset F$. For any $w \in W$ the algebraic number trace (w) is the sum of the eigenvalues of w and therefore a sum of roots of unity; this implies that K_W lies in a cyclotomic, hence abelian, extension of $\mathbf{Q}$ and thus that K_W itself is a Galois extension of $\mathbf{Q}$.

5.4 PROPOSITION. [8] [2] *If F is a field of characteristic 0 and $W \subset GL(n, F)$ is a p.r. subgroup, then W is conjugate in $GL(n, F)$ to a subgroup of $GL(n, K_W)$.*

Remark. This proposition amounts to the statement that the Schur index of a reflection representation is one [2].

5.5 PROPOSITION. *If $W \subset GL(n, \mathbf{C})$ is a p.r. subgroup and σ is a field automorphism of $\mathbf{C}$, then the group $\sigma(W)$ is conjugate as a subgroup of $GL(n, \mathbf{C})$ to W.*

Remark. If $i: W \to GL(n, \mathbf{C})$ is the inclusion of a p.r. subgroup it is *not* necessarily true in the situation of 5.5 that the representation $\sigma \cdot i$ of W is conjugate to the representation i; for instance, the dihedral group D_{10} of order 10 has two distinct p. r. representations over $\mathbf{Q}(\sqrt{5})$ which are interchanged by the non-identity automorphism of $\mathbf{Q}(\sqrt{5})$. According to 5.5, these two representations must also be interchanged by an outer automorphism of D_{10}.

Proof of 5.5. We may assume that the action of W on $\mathbf{C}^n$ gives an irreducible representation of W. The action of $\sigma(W)$ on $\mathbf{C}^n$ is then also irreducible. It is clear that W and $\sigma(W)$ are irreducible p.r. groups of the same rank (i.e. both W and $\sigma(W)$ lie in $GL(n, \mathbf{C})$) and the same

order. Since the character field K_W is Galois and hence carried to itself by σ, $K_{\sigma(W)} = K_W$. Moreover, W is *primitive* [9] as a p.r. group iff $\sigma(W)$ is primitive. The fact that W and $\sigma(W)$ have all of these properties in common now implies, by inspection of the tables in [8], that W and $\sigma(W)$ are conjugate.

5.6 PROPOSITION. *Let W be a finite group of order prime to p. Then the inclusion $GL(n, \mathbf{Z}_p) \to GL(n, \mathbf{Q}_p)$ induces a bijection from conjugacy classes of homomorphisms $W \to GL(n, \mathbf{Z}_p)$ to conjugacy classes of homomorphisms $W \to GL(n, \mathbf{Q}_p)$.*

Proof. The argument in [11, proof of 73.5] shows that every homomorphism $W \to GL(n, \mathbf{Q}_p)$ is derived up to conjugacy from a homomorphism $W \to GL(n, \mathbf{Z}_p)$. The proposition now follows from [11, 76.17].

5.7 PROPOSITION. *Let W be a finite group, F a field of characteristic 0, M and N finitely generated $F[W]$ modules, and L a field containing F. Then $L \otimes_F M$ is isomorphic to $L \otimes_F N$ as a module over $L[W]$ iff M is isomorphic to N as a module over $F[W]$.*

Proof. A finite-dimensional representation of a finite group over a field of characteristic 0 is determined up to isomorphism by the character which it affords [11, 30.14]. This character is not changed by extension of scalars.

Proof of 1.5. The bijection (1)↔(2) is obtained from 2.4. The bijection (2)↔(3) is obtained from 5.2.

It remains to give the bijection (3)↔(4). Suppose that W is a p.r. subgroup of $GL(n, \mathbf{Z}_p)$. The character field $K_W \subset \mathbf{Q}_p$ is a finite extension of $\mathbf{Q}$, and by 5.4 the group W is conjugate in $GL(n, \mathbf{Q}_p)$ to a subgroup W' of $GL(n, K_W)$. By 5.7 the subgroup W' is well-defined up to conjugacy in $GL(n, K_W)$. Pick an embedding $j: K_W \to \mathbf{C}$ and observe that $j(W') \subset GL(n, \mathbf{C})$ is a p.r. subgroup. Since K_W is a Galois extension of $\mathbf{Q}$ any two embeddings $K_W \to \mathbf{C}$ differ by an automorphism of $\mathbf{C}$; in this way 5.5 shows that the conjugacy class of the subgroup $j(W') \subset GL(n, \mathbf{C})$ does not depend on the choice of j. This gives the desired function from the set of conjugacy classes of p.r. subgroups of $GL(n, \mathbf{Q}_p)$ to the set of conjugacy classes of p.r. subgroups of $GL(n, \mathbf{C})$ with character field embeddable in $\mathbf{Q}_p$.

To construct an inverse to this function, reverse the procedure. Let $W \subset GL(n, \mathbf{C})$ be a p.r. group with character field K_W, and assume that K_W can be embedded in $\mathbf{Q}_p$. By 5.4 the group W is conjugate to a subgroup W' of $GL(n, K_W)$; by 5.7 there is only one such W' up to conjugacy. Choose an embedding $k: K_W \to \mathbf{Q}_p$, and observe that $k(W')$ is a p. r. subgroup of $GL(n, \mathbf{Q}_p)$. Since K_W is a Galois extension of $\mathbf{Q}$ any two such embeddings differ by a field automorphism of K_W and thus by 5.5 the conjugacy class of the subgroup $k(W') \subset GL(n, \mathbf{Q}_p)$ does not depend on the choice of k. Finally, by 5.6 there is up to conjugacy a unique subgroup $W'' \subset GL(n, \mathbf{Z}_p)$ such that W'' is conjugate in $GL(n, \mathbf{Q}_p)$ to $k(W')$.

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