ISRAEL JOURNAL OF MATHEMATICS, Vol. 66, Nos. 1-3, 1989

FIBREWISE COMPLETION AND UNSTABLE ADAMS SPECTRAL SEQUENCES†

BY
WILLIAM DWYER,[a] HAYNES MILLER[b] AND JOSEPH NEISENDORFER[c]
[a]*Department of Mathematics, University of Notre Dame, Notre Dame, Indiana, USA*;
[b]*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA*;
and [c]*Department of Mathematics, University of Rochester, Rochester, New York, USA*

ABSTRACT
We describe a tower of spaces whose inverse limit is a "fiberwise completion" of a fibration $E \to B$, and study the resulting spectral sequence converging to the homotopy groups of the space of lifts of a map $X \to B$. This is used to give a proof of the "generalized Sullivan conjecture".

0. Introduction

In this paper we construct a relative unstable mod p Adams spectral sequence which is adapted to studying the homotopy groups of the space of sections of a fibration $E \to B$. We calculate the E_2-term of the spectral sequence and then analyze the calculation in some detail in the special case in which B is the space BZ/p. This leads to a new proof of the generalized Sullivan conjecture, a proof which is closely related to the proof of Lannes [4] and to an unpublished proof by the second author but which is streamlined at a critical point by use of the relative spectral sequence. The way in which this spectral sequence is used depends heavily on ideas of Lannes.

Let R be the field F_p of p elements. The mod p unstable Adams spectral sequence of [2] is obtained by working with a cosimplicial construction of the *R-completion* $R_\infty X$ of a space X. It is natural to expect that a relative version of this spectral sequence might be obtained from some sort of cosimplicial construction of the *fibrewise R-completion* of a fibration $E \to B$. Bousfield and

†All three authors were supported in part by the National Science Foundation.
Received October 1, 1988

Kan do give a cosimplicial construction $\dot{R}_\infty E$ of the fibrewise R-completion but the resulting spectral sequence has an E_2-term which cannot be described in terms of the homology groups of E and of B. We modify their approach and give a cosimplicial recipe for a slightly different space $BR_\infty E$ which up to homotopy fits into a fibre square

$$\begin{array}{ccc} BR_\infty E & \rightarrow & R_\infty E \\ \downarrow & & \downarrow \\ B & \rightarrow & R_\infty B. \end{array}$$

This recipe leads, for each space Y over B, to a spectral sequence which

(1) has a homologically identifiable E_2-term, and

(2) converges to the homotopy of the space $\Gamma(Y, BR_\infty E)$ of maps from Y to $BR_\infty E$ over B.

Of course, one usually wants to know about $\Gamma(Y, E)$ and not about $\Gamma(Y, BR_\infty E)$. We do not understand in general the properties of the natural map $\Gamma(Y, E) \rightarrow \Gamma(Y, BR_\infty E)$ but some information is available. For one thing the fibre lemma of [2] shows that under some circumstances, for instance if the fundamental group of B is a finite p-group, the space $BR_\infty E$ is weakly equivalent over B to $\dot{R}_\infty E$. Suppose now that π is a finite p-group, X is a simply connected space with a π-action, and $E \rightarrow B$ is the Borel construction $E\pi \times_\pi X \rightarrow B\pi$. The results of [3] then imply that the map $\Gamma(B\pi, E) \rightarrow \Gamma(B\pi, BR_\infty(E))$ is a mod p homotopy isomorphism, that is, that the fibres are simple spaces with homotopy groups which are uniquely p-divisible.

We would like to express our gratitude to A. K. Bousfield for his many letters to us describing unstable Adams spectral sequences, bicosimplicial spaces, the role of derivations, and convergence criteria. We also owe thanks to J.-P. Meyer for pointing out that an invalid proof of Proposition 7.1 was contained in a previous version of this paper.

Notation and Terminology. Throughout the paper p will be a fixed prime and R will denote the field F_p. All homology and cohomology is taken with coefficients in R. We will work in the category of simplicial sets and adopt the convention that "space" means "simplicial set". Cosimplicial objects will be denoted $A^\bullet, B^\bullet, \ldots$ and their augmentations if any by $A^{-1} \rightarrow A^\bullet$ or $X \rightarrow A^\bullet$. The symbol $c^\bullet X$ will stand for the constant cosimplicial object with $(c^\bullet X)^s = X$ for all $s \geqq 0$ and all coface operators d^i and codegeneracy operators s^i given by identity maps. An augmentation $X \rightarrow A^\bullet$ is equivalent to a cosimplicial map $c^\bullet X \rightarrow A^\bullet$.

Our basic reference is the book of Bousfield and Kan [2]. Many of the ideas there are also expounded in [5].

1. The construction of $BR_\infty E$

The purpose of this section is to give the construction of $BR_\infty E$. This depends on two standard definitions.

DEFINITION 1.1. Let **C** be a category and T a triple [2] in **C** with structure maps $\eta : 1 \to T$ and $\mu : T^2 \to T$. Given an object X of **C**, the *canonical resolution* of X with respect to T is the augmented cosimplicial object $X \to T^\bullet X$ defined by:

$$(T^\bullet X)^s = T^{s+1}X,$$

$$d^i = T^i\eta(T^{s-i}X) : (T^\bullet X)^{s-1} \to (T^\bullet X)^s,$$

$$s^i = T^i\mu(T^{s-i}X) : (T^\bullet X)^{s+1} \to (T^\bullet X)^s,$$

where $i = 0, \dots, s$.

The augmentation is $d^0 = \eta : X \to (T^\bullet X)^0$.

Let $\Delta^\bullet$ be the cosimplicial space with $\Delta^s = \Delta[s]$, the standard s-simplex, and with d^i and s^i the standard coface and codegeneracy maps. Given cosimplicial spaces $A^\bullet$ and $B^\bullet$, let $\hom(A^\bullet, B^\bullet)$ be the cosimplicial mapping space [2], in which the n-simplices $\hom(A^\bullet, B^\bullet)_n$ are the cosimplicial morphisms $\Delta[n] \times A^\bullet \to B^\bullet$. Here $(\Delta[n] \times A^\bullet)^s = \Delta[n] \times A^s$, $d^i = 1 \times d^i$, and $s^i = 1 \times s^i$. The face operators $d_i : \hom(A^\bullet, B^\bullet)_n \to \hom(A^\bullet, B^\bullet)_{n-1}$ and the degeneracy operators $s_i : \hom(A^\bullet, B^\bullet)_n \to \hom(A^\bullet, B^\bullet)_{n+1}$ are given by $d_i(f) = f(d^i \times 1)$ and $s_i(f) = f(s^i \times 1)$.

DEFINITION 1.2 The *total complex* tot $Y^\bullet$ of a cosimplicial space $Y^\bullet$ is the mapping space $\hom(\Delta^\bullet, Y^\bullet)$

It is not difficult to check that tot $c^\bullet X = X$. Hence, if $X \to Y^\bullet$ is an augmented cosimplicial space, applying tot to $c^\bullet X \to Y^\bullet$ gives a map $X \to$ tot $Y^\bullet$.

For any space X, Bousfield and Kan construct a space RX by letting $(RX)_n$ be the set of finite formal sums $\Sigma\, r[x]$ where $r \in R$ and $x \in X_n$. This gives rise to a triple in which the structure maps $\eta_X : X \to RX$ and $\mu_X : R^2X \to RX$ are given by the formulas $\eta_X(x) = [x]$ and $\mu_X(\Sigma\, s[\Sigma\, r[x]]) = \Sigma\, sr[x]$. The R-completion $R_\infty X$ is then defined to be tot $R^\bullet X$.

REMARK. Bousfield and Kan actually work with a slightly different triple [2, I, §2], which we will denote $\bar{R}$; our RX is what they denote $R \otimes X$. We will show in Proposition 6.6 that tot $\bar{R}^{\bullet}X$ is homotopy equivalent to tot $R^{\bullet}X$.

Let B be a fixed space. Given a space E over B with map $\phi : E \to B$, let $BR(E)$ be the product $B \times RE$ and regard $BR(E)$ as a space over B by projecting on the first factor. There are maps $\eta : E \to BR(E)$ and $\mu : (BR)^2(E) \to BR(E)$ given by the formulas $\eta = (\phi, \eta_E) : E \to B \times RE$ and $\mu = 1 \times (\mu_E R(\pi_2))$: $B \times R(B \times RE) \to B \times RE$ where $\pi_2 : B \times RE \to RE$ is projection on the second factor. These maps provide the functor BR with the structure of a triple.

DEFINITION 1.3. Let E be a space over B. The *relative R-completion* $BR_\infty(E)$ of E is the space tot $BR^{\bullet}(E)$.

Since there is a map $BR^{\bullet}(E) \to c^{\bullet}B$, $BR_\infty(E)$ is a space over B. The evident augmentation $E \to BR^{\bullet}E$ induces a map $E \to BR_\infty(E)$ which is a map over B.

THEOREM 1.4. *Let B be a fibrant space (i.e. Kan complex) and $E \to B$ a fibration. Then up to homotopy the space $BR_\infty(E)$ fits into a homotopy fibre square*

$$\begin{array}{ccc} BR_\infty(E) & \to & R_\infty E \\ \downarrow & & \downarrow \\ B & \to & R_\infty B \end{array}$$

This will be proved in Section 7.

2. Derived functor of derivations

This section discusses some algebra which is needed for the description of the E_2-term of the relative unstable Adams spectral sequence.

Let **CA** denote the category of graded commutative unstable coalgebras with unit over the mod p Steenrod algebra [5]. If B is an object in **CA**, **CA**/B will stand for the category of objects of **CA** over B.

Let S^t denote the mod p homology of the t-sphere considered as an object of **CA**, and let X be an object in **CA**/B for some B. The projection $S^t \otimes X \to X$ makes $S^t \otimes X$ into an object in **CA**/B in such a way that this projection is a map in the category. Similarly, choosing augmentations $F_p \to S^t$ (these are unique unless $t = 0$) gives maps $X \to S^t \otimes X$.

DEFINITION 2.1. Let B be an object of **CA** and $\psi : X \to E$ a map in **CA**/B. A map $\varphi : S^t \otimes X \to E$ in $\mathbf{CA}/B$ is called a *derivation of degree* t with respect to ψ if the diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & E \\ \downarrow & \nearrow_{\varphi} & \\ S^t \otimes X & & \end{array}$$

commutes. The set of all derivations of degree t with respect to ψ is denoted $\mathrm{Der}^t_{\mathbf{CA}/B}(X, E)_\psi$.

The set $\mathrm{Der}^0_{\mathbf{CA}/B}(X, E)_\psi$ is just $\mathrm{Hom}_{\mathbf{CA}/B}(X, E)$ and, for $t \geqq 1$, $\mathrm{Der}^t_{\mathbf{CA}/B}(X, E)_\psi$ is an F_p vector space of what would usually be called coalgebra derivations over the Steenrod algebra.

Let $\mathbf{F}_p$ be the category of non-negatively graded F_p modules. The forgetful functor $J : \mathbf{CA} \to \mathbf{F}_p$ has a right adjoint $G : \mathbf{F}_p \to \mathbf{CA}$. We also write G for the resulting triple $G = GJ : \mathbf{CA} \to \mathbf{CA}$ with triple structure $\eta_E : E \to GE$ and $\mu_E : G^2E \to GE$. Now define a triple $B - G : \mathbf{CA}/B \to \mathbf{CA}/B$ as follows: if E is in $\mathbf{CA}/B$ with map $\phi : E \to B$, let $(B - G)(E) = B \otimes G(E)$, $\eta = (\phi \otimes \eta_E)\Delta : E \to E \otimes E \to B \otimes G(E)$, and $\mu = (1 \otimes \mu_E)(1 \otimes G(\pi_2)) : B \otimes G(B \otimes G(E)) \to B \otimes G(G(E)) \to B \otimes G(E)$ where $\pi_2 : B \otimes G(E) \to G(E)$ is the projection given by the unit $B \to F_P$.

Definition 2.2. Let $\psi : X \to E$ be a map in $\mathbf{CA}/B$, so that the augmentation $E \to (B - G)^\bullet(E)$ gives maps $X \to (B - G)^{n+1}(E)$, also denoted ψ. The *right derived functors* of $\mathrm{Der}^t_{\mathbf{CA}/B}(X, E)_\psi$ are defined, for all $s \geqq 0$ if $t \geqq 1$ and for $s = 0$ if $t = 0$, by $\mathrm{Ext}^{s,t}_{\mathbf{CA}/B}(X, E)_\psi = \pi^s(\mathrm{Der}^t_{\mathbf{CA}/B}(X, (B - G)^\bullet(E))_\psi)$.

Remark. Here we have used the following notation: if $A^\bullet$ is a cosimplicial abelian group, $\pi^s A^\bullet$ is the cohomology group $H^s(A^\bullet, d)$ where $d = \Sigma(-1)^i d^i$; if $A^\bullet$ is a cosimplicial set, $\pi^0 A^\bullet$ is the equalizer of d^0 and $d^1 : A^0 \to A^1$.

Remark 2.3. It is easy to see that $\mathrm{Ext}^{0,t}_{\mathbf{CA}/B}(X, E)_\psi$ is naturally isomorphic to $\mathrm{Der}^t_{\mathbf{CA}/B}(X, E)_\psi$. In particular, $\mathrm{Ext}^{0,0}_{\mathbf{CA}/B}(X, E)_\psi$ is essentially independent of ψ and is isomorphic to the set $\mathrm{Hom}_{\mathbf{CA}/B}(X, E)$.

The preceding considerations dualize to algebras as follows. Let $\mathbf{A}$ be the category of unstable algebras over the mod p Steenrod algebra. If B^* is an object in $\mathbf{A}$, let $\mathbf{A} \backslash B^*$ be the category of objects under B^*, that is, of objects E^* in $\mathbf{A}$ together with maps $\phi : B^* \to E^*$. The forgetful functor $J^* : \mathbf{A} \to \mathbf{F}_p$ has a left adjoint G^* and we also write G^* for the cotriple $G^* = G^*J^*$. There is a cotriple $(B^* - G^*)$ on $\mathbf{A} \backslash B^*$ defined by $(B^* - G^*)(E^*) = B^* \otimes G^*(E^*)$. If

$\psi : E^* \to X^*$ is a map in $\mathbf{A}\backslash B^*$, we define $\mathrm{Der}_{\mathbf{A}\backslash B^*}(E^*, X^*)_\psi$ and $\mathrm{Ext}^{s,t}_{\mathbf{A}\backslash B^*}(E^*, X^*)_\psi$ by a procedure dual to the one for coalgebras above. Then, if X, E, B are finite type objects in **CA** with dual objects X^*, E^*, B^* in **A**, we have

$$\mathrm{Ext}^{s,t}_{\mathbf{CA}/B}(X, E)_\psi = \mathrm{Ext}^{s,t}_{\mathbf{A}\backslash B^*}(E^*, X^*)_{\psi^*}.$$

If $\varepsilon : B^* \to F_p$ is an augmented object in **A** and E^* is an object of $\mathbf{A}\backslash B^*$, let $E^*_\varepsilon = F_p \otimes_{B^0} E^*$ be the sum of the "components" of E^* which cover the component $B^*_\varepsilon = F_p \otimes_{B^0} B^*$ of B^* corresponding to ε. If $\varphi : E^* \to F_p$ is an augmentation extending ε, let $\bar{\varphi} : E^*_\varepsilon \to F_p$ be the induced quotient augmentation.

LEMMA 2.4. *If B^* is an object of* **A** *augmented by $\varepsilon : B^* \to F_p$, and E^* is an object of $\mathbf{A}\backslash B^*$ with an augmentation $\varphi : E^* \to F_p$ extending ε, then* $\mathrm{Ext}^{s,t}_{\mathbf{A}\backslash B^*}(E^*, F_p)_\varphi$ *is naturally isomorphic to* $\mathrm{Ext}^{s,t}_{\mathbf{A}\backslash B^*_\varepsilon}(E^*_\varepsilon, F_p)_{\bar{\varphi}}$.

The proof of this is essentially the same as the proof [4] in the special case of $\mathbf{A} = \mathbf{A}\backslash F_p$.

Let $\pi = Z/pZ$ and $H^* = H^*(B\pi, F_p)$. Recall that Lannes [4] has given a functor $T : \mathbf{A} \to \mathbf{A}$ which is left adjoint to the functor given by tensor product with H^*:

$$\mathrm{Hom}_{\mathbf{A}}(B^*, H^* \otimes C^*) = \mathrm{Hom}_{\mathbf{A}}(TB^*, C^*).$$

LEMMA 2.5 (Lannes [4]). *The functor T has the following properties.*

(a) *T preserves free objects.*
(b) *T preserves tensor products: $T(A^* \otimes B^*) = TA^* \otimes TB^*$.*
(c) *T is exact, in particular, T preserves simplicial resolutions.*
(d) *If A^* is finite dimensional, the natural map $A^* \to H^* \otimes A^*$ induces via adjointness an isomorphism $TA^* \to A^*$.*

These properties lead easily to a proof of the following lemma.

LEMMA 2.6. *The functor T preserves derivations and derived functors of derivations, that is, if B^* is an object of* **A**, *E^* and X^* are objects of $\mathbf{A}\backslash B^*$, and $\psi : E^* \to H^* \otimes X^*$ is a map of $\mathbf{A}\backslash B^*$, then there are natural isomorphisms*

$$\mathrm{Ext}^{s,t}_{\mathbf{A}\backslash B^*}(E^*, H^* \otimes X^*)_\psi \simeq \mathrm{Ext}^{s,t}_{\mathbf{A}\backslash TB^*}(TE^*, X^*)_{\psi^*}$$

where $\psi^{\#} : TE^* \to X^*$ is the morphism corresponding by adjointness to $\psi : E^* \to H^* \otimes X^*$.

3. The relative mod p unstable Adams spectral sequence

In this section we describe the spectral sequence which is the subject of this paper, compute the E_2 term, and use the computation to give a proof of the generalized Sullivan conjecture.

Fix a space B. For any two spaces Y and E over B, let $\Gamma(Y, E)$ denote the space of maps over B from Y to E. The spaces $\Gamma(Y, BR^n(E))$ for $n \geqq 0$ combine to form a cosimplicial space, denoted $\Gamma(Y, BR^{\bullet}(E))$, which is augmented by $\Gamma(Y, E)$. It is easy to see that tot $\Gamma(Y, BR^{\bullet}(E))$ is isomorphic to $\Gamma(Y, BR_{\infty}(E))$.

We will need one technical observation.

LEMMA 3.1. *For any two spaces Y and E over B, $\Gamma(Y, BR^{\bullet}(E))$ is a fibrant termwise simple cosimplicial space.*

REMARK. The notion of *fibrant* cosimplicial space is discussed in [2] and in Section 4. A cosimplicial space $X^{\bullet}$ is *termwise simple* if each X^n is a simple space (i.e., if each component of X^n has an abelian fundamental group which acts trivially on the component's higher homotopy groups).

PROOF OF 3.1. For any $n \geqq 0$, $BR^n(E)$ is an abelian group object over B. It is straightforward to deduce that $\Gamma(Y, BR^{\bullet}(E))$ is a grouplike cosimplicial space in the sense of [2]. The conclusion follows easily.

Specifying a particular map $\psi: Y \to E$ over B provides compatible basepoints for $\Gamma(Y, E)$, $\Gamma(Y, BR^n(E))$ and $\Gamma(Y, BR_{\infty}(E))$. We will denote the resulting pointed spaces by, for instance, $\Gamma(Y, E, \psi)$.

DEFINITION 3.2. Let $\psi: Y \to E$ be a map of spaces over B. The *relative unstable* mod *p Adams spectral sequence* $E_r^{s,t}(Y, E, \psi)$ is the homotopy spectral sequence in the sense of [1] of the pointed fibrant cosimplicial space $\Gamma(Y, BR^{\bullet}(E), \psi)$.

REMARK. The terms $E_r^{s,t}(Y, E, \psi)$ are defined for $t \geqq s \geqq 0$ and in certain other cases; in particular, in view of Lemma 3.1 and [1, 2.5], $E_2^{s,t}$ is defined for $(s, t) = (0, 0)$ or for $s \geqq 0$ and $t \geqq 1$. There are differentials $d_r: E_r^{s,t} \to E_r^{s+r,t+r-1}$ and the abutment of the spectral sequence is $\pi_{t-s}\Gamma(Y, BR_{\infty}(E))$. A great deal of information about the convergence of this spectral sequence appears in [1].

LEMMA 3.3. *In the situation of Definition 3.2 there are natural isomorphisms*

$$E_2^{s,t}(Y, E, \psi) \cong \mathrm{Ext}^{s,t}_{\mathrm{CA}/H_*B}(H_*Y, H_*E)_{H_*\psi}$$

for $(s, t) = (0, 0)$ *or* $s \geqq 0$ *and* $t \geqq 1$.

PROOF. Let Σ^t denote the topological t-sphere. A map $\Sigma^t \to \Gamma(Y, E)$ has an associated map $\Sigma^t \times Y \to E$; applying the homology functor to this associated map produces a morphism from $\pi_t\Gamma(Y, E, \psi)$ to $\mathrm{Der}^t_{\mathrm{CA}/H_*B}(H_*Y, H_*E)_{H_*\psi}$ which is an isomorphism when $E = BR(E')$. Since the topological triple BR and the algebraic triple $H_*(B) - G$ are related by a natural isomorphism $\theta: H_*(BR(E)) \to (H_*B - G)(H_*E)$ which respects the triple structures, one concludes that there are natural isomorphisms

$$\pi_t\Gamma(Y, BR^s(E), \psi) \to \mathrm{Der}^t_{\mathrm{CA}/H_*B}(H_*Y, (H_*(B) - G)^s H_*E)_{H_*\psi}.$$

The lemma now follows from the definitions of Section 2 and from the general fact [1, 2.5] that for a pointed, fibrant, termwise simple cosimplicial space $(X^\bullet, x)$, $E_2^{s,t}(X^\bullet, x)$ is isomorphic to $\pi^s\pi_t(X^\bullet, x)$.

Given Lemma 3.1, the following is a special case of [1, 6.3].

LEMMA 3.4. *Let* Y *be a space over* B *and* $\phi: E_1 \to E_2$ *a map between two spaces over* B. *Suppose that for each* ψ *in* $\mathrm{Hom}_{\mathrm{CA}/H_*B}(H_*Y, H_*E_1)$, $H_*\phi$ *induces isomorphisms*

$$\mathrm{Ext}^{s,t}_{\mathrm{CA}/H_*B}(H_*Y, H_*E_1)_\psi \simeq \mathrm{Ext}^{s,t}_{\mathrm{CA}/H_*B}(H_*Y, H_*E_2)_{(H_*\phi)\psi}$$

for $s \geqq 0$ *and* $t \geqq 1$. *Then* ϕ *induces a homotopy equivalence*

$$\Gamma(B, BR_\infty(E_1)) \to \Gamma(B, BR_\infty(E_2)).$$

REMARK. By Remark 2.3 the hypothesis of Lemma 3.4 includes the assumption that $H_*\phi$ induces an isomorphism $\mathrm{Hom}_{\mathrm{CA}/H_*B}(H_*Y, H_*E_1) \simeq \mathrm{Hom}_{\mathrm{CA}/H_*B}(H_*Y, H_*E_2)$.

Now we are ready to work on the generalized Sullivan conjecture. In what follows we will tacitly replace CW-complexes by their singular complexes whenever it is necessary to apply functors like R_∞, BR_∞, etc.

PROPOSITION 3.5 (Generalized Sullivan conjecture). *Let* π *be* Z/p, X *a finite CW-complex with a cellular* π*-action*, X^π *the fixed-point set in* X *of the action of* π, *and* $E_\pi(X) = E\pi \times_\pi X \to B\pi$, *the Borel construction on* X. *Then there is a natural homotopy equivalence*

$$R_\infty(X^\pi) \to \Gamma(B\pi, E_\pi(R_\infty X)).$$

REMARK. The space $\Gamma(B\pi, E_\pi(R_\infty X))$ is just the space of sections of the fibration $E_\pi(R_\infty X) \to B\pi$. This space of sections is sometimes called the *homotopy fixed-point set* of the action of π on $R_\infty X$.

For the remainder of this section, H^* will denote $H^*(B\pi)$.

LEMMA 3.6 (Lannes [4]). *The algebra $T(H^*)$ is naturally isomorphic to the product*

$$\prod_{\varphi \in \hom(\pi,\pi)} H^*.$$

Under this isomorphism, the identity map $1 : H^* \to H^*$ corresponds via adjointness to the unique algebra homomorphism from the product to F_p which factors through projection onto the $\varphi = 1$ component. More generally, let π and X be as in (3.5) *and let E denote $E_\pi(X)$. Then $T(H^*E)$ is naturally isomorphic to the product*

$$\left(\prod_{\varphi \neq 0} H^* \otimes H^*(X^\pi)\right) \prod H^*E$$

(*here φ runs through the non-zero maps $\pi \to \pi$*). *Under this isomorphism the map $T\psi^* : TH^* \to TH^*E$ induced by the projection map $\psi : E \to B\pi$ corresponds via adjointness to the map between the products given by the natural inclusion $H^* \to H^* \otimes H^*X^\pi$ on the $\varphi \neq 0$ components and to $\psi^* : H^* \to H^*E$ on the $\varphi = 0$ component.*

In the following statement we will use the notation of Lemma 2.5.

LEMMA 3.7. *Let π and X be as in* (3.5), *and let $\varepsilon : T(H^*) \to F_p$ be the map which corresponds by adjointness to the identity map $1 : H^* \to H^*$. Then the map $E_\pi(X^\pi) \to E_\pi(X)$ induces an isomorphism*

$$(TH^*(E_\pi(X)))_\varepsilon \to (TH^*(E_\pi(X^\pi)))_\varepsilon.$$

PROOF. This is an immediate consequence of (3.6).

PROOF OF 3.5. Let B denote $B\pi$, so that $E_\pi(X)$ and $E_\pi(X^\pi)$ are spaces over B. Theorem 1.4 and the fibre lemma of [2] imply that $E_\pi(R_\infty X) \simeq BR_\infty(E_\pi(X))$ and that there is a similar equivalence with X replaced by X^π. In view of the fact that X^π is finite-dimensional, the Sullivan conjecture (as proven in [5]) states that $\Gamma(B, E_\pi(R_\infty(X^\pi)))$ is equivalent to $R_\infty(X^\pi)$. Hence to prove (3.5) it is enough to show that the natural map $\varphi : E_\pi(X^\pi) \to E_\pi(X)$ induces a homotopy equivalence $\Gamma(B, BR_\infty(E_\pi(X^\pi))) \to \Gamma(B, BR_\infty(E_\pi(X)))$. Let U^* denote $H^*(E_\pi(X^\pi))$ and let V^* denote $H^*(E_\pi(X))$. By the cohomological version of (3.4), the desired result will follow if we can show that for each $\psi: U^* \to H^*$ in $\mathrm{A}\backslash H^*$ the map $H^*(\varphi)$ induces isomorphisms $\mathrm{Ext}^{s,t}_{\mathrm{A}\backslash H^*}(U^*, H^*)_\psi \to \mathrm{Ext}^{s,t}_{\mathrm{A}\backslash H^*}(V^*, H^*)_{\psi H^*(\varphi)}$. By Lemmas 2.5 and 2.6, this is equivalent to showing that for each map $\rho: T(U^*) \to F_p$ in $\mathrm{A}\backslash T(H^*)$ the map $TH^*(\varphi)$ induces isomorphisms

$$\mathrm{Ext}^{s,t}_{\mathrm{A}\backslash T(H^*)_\varepsilon}(T(U^*)_\varepsilon, F_p)_\rho \to \mathrm{Ext}^{s,t}_{\mathrm{A}\backslash T(H^*)_\varepsilon}(T(V^*)_\varepsilon, F_p)_{\rho TH^*(\varphi)},$$

where $\varepsilon: T(H^*) \to F_p$ is the map which corresponds via adjointness to $1: H^* \to H^*$. The proof is completed by appealing to Lemma 3.7, which in fact asserts that $T(U^*)_\varepsilon \simeq T(V^*)_\varepsilon$.

4. Fibrations of cosimplicial spaces

The next three sections are devoted to technical material which is needed for the proof of Theorem 1.4 in Section 7.

Let $X^\bullet$ be a cosimplicial space.

Definition 4.1 [2, X, 4.5]. The *matching space* $M^n X^\bullet$ for $n \geqq 0$ is the subset of the $n+1$ fold product $X^n \times \cdots \times X^n$ consisting of those $(x^0, \ldots, x^n)$ such that $s^i x^j = s^{j-1} x^i$ whenever $0 \leqq i \leqq j < n$.

By convention, $M^{-1}X^\bullet$ is a one-point space $*$. For $n \geqq -1$ there are natural maps

$$(*)_n: X^{n+1} \to M^n X^\bullet$$

given by $x \mapsto (s^0 x, \ldots, s^n x)$ if $n \geqq 0$.

Definition 4.2. A map $f: X^\bullet \to Y^\bullet$ is called a *fibration* if, for each $n \geqq -1$, the natural map

$$(**)_n: X^{n+1} \to Y^{n+1} \times_{M^n Y^\bullet} M^n X^\bullet$$

is a fibration. A cosimplicial space $X^\bullet$ is called *fibrant* if $X^\bullet \to c^\bullet(*)$ is a fibration, equivalently, if the maps $(*)_n$ are fibrations for all $n \geqq -1$.

Remark. The definitions of fibration and of fibrant object do not involve the coface operators.

It is easy to see that pullbacks of fibrations are fibrations; hence fibres of fibrations are fibrant and products of fibrant cosimplicial spaces are fibrant.

Proposition 4.3 [2]. *If $B^\bullet$ is a pointed cosimplicial space and $E^\bullet \to B^\bullet$ is a* *fibration with fibre* $F^\bullet$, *then* tot $E^\bullet \to$ tot $B^\bullet$ *is a fibration with fibre* tot $F^\bullet$. *In particular, if* $X^\bullet$ *is fibrant then* tot $X^\bullet$ *is a fibrant space* (*i.e., a Kan complex*).

We now list some examples.

(1) If B is a fibrant space, then the constant cosimplicial space $c^\bullet B$ is fibrant. The maps $(*)_n$ are $B \to *$ if $n = -1$ and isomorphisms if $n \geqq 0$.

(2) Any isomorphism $X^\bullet \to Y^\bullet$ is a fibration.

(3) If $Y^\bullet$ is fibrant, then the projection $X^\bullet \times Y^\bullet \to X^\bullet$ is a fibration.

(4) A cosimplicial space $X^\bullet$ is called *grouplike* if, for all $n \geqq 0$, $(X^\bullet)^n$ is a simplicial group and the coface and codegeneracy operators except possibly for d^0 are all homomorphisms. Bousfield and Kan show that grouplike objects are fibrant or more generally that any surjective homomorphism of grouplike objects is a fibration. (The canonical resolution $R^\bullet Y$ of a space Y is grouplike.)

(5) Let $p : E \to B$ be a space over B with E and B fibrant. Consider the triple $(B \times -)(E) = B \times E$ where $B \times E$ is regarded as a space over B by projecting on the first factors and where the triple structure $\eta(E) : E \to B \times E$ and $\mu(E) : B \times B \times E \to B \times E$ is given by $\eta(E) = (p, 1)$ and $\mu(E) =$ projection on the first and third factors. The canonical resolution $E \to (B \times -)^\bullet(E)$ is called the *Rector complex*. The natural map $(B \times -)^\bullet(E) \to c^\bullet B$ is a fibration: the map $(**)_n$ is $B \times E \to B$ if $n = -1$, and an isomorphism if $n \geqq 0$.

(6) Let $p : E \to B$ be a space over B and consider the resolution $E \to BR^\bullet(E)$ introduced in Section 1. We claim that $q : BR^\bullet(E) \to c^\bullet B$ is a fibration. Pick a basepoint in B and let $F^\bullet$ be the fibre of q. If we forget the coface operators, then $BR^\bullet(E) \to c^\bullet B$ is the projection $c^\bullet B \times F^\bullet \to c^\bullet B$. Since $F^\bullet$ is grouplike, $F^\bullet$ is fibrant. Hence, $BR^\bullet(E) \to c^\bullet B$ is a fibration. In particular, if B is fibrant, then so is $BR^\bullet(E)$.

DEFINITION 4.4 [2]. A map $f: X^\bullet \to Y^\bullet$ is a *weak equivalence* if $f^n : X^n \to Y^n$ is a weak equivalence for all $n \geqq 0$.

PROPOSITION 4.5 [2]. *If* $f: X^\bullet \to Y^\bullet$ *is a weak equivalence with* $X^\bullet$ *and* $Y^\bullet$ *fibrant, then* tot f: tot $X^\bullet \to$ tot $Y^\bullet$ *is a homotopy equivalence.*

In light of the fact that the notion of a homotopy fibre square and the notion of the total complex of a fibrant cosimplicial space can both be interpreted in terms of homotopy inverse limits [2, XI, 4.1(iv); XI, 4.4], the following is a consequence of the homotopy inverse limit interchange principle [2, XI, 4.3].

Proposition 4.6. *Suppose that*

$$\begin{array}{ccc} A^\bullet & \longrightarrow & B^\bullet \\ \downarrow & & \downarrow \\ C^\bullet & \longrightarrow & D^\bullet \end{array}$$

is a square of fibrant cosimplicial spaces such that for each $n \geqq 0$ *the induced square of spaces*

$$\begin{array}{ccc} A^n & \longrightarrow & B^n \\ \downarrow & & \downarrow \\ C^n & \longrightarrow & D^n \end{array}$$

is a homotopy fibre sqaure. Then the square

$$\begin{array}{ccc} \operatorname{tot} A^\bullet & \longrightarrow & \operatorname{tot} B^\bullet \\ \downarrow & & \downarrow \\ \operatorname{tot} C^\bullet & \longrightarrow & \operatorname{tot} D^\bullet \end{array}$$

is a homotopy fibre square.

5. Fibrations of bicosimplicial spaces

If $A^{\bullet\bullet}$ is a bicosimplicial object, denote the horizontal operators by $d_h^i : A^{n-1,m} \to A^{n,m}$ and $s_h^i : A^{n+1,m} \to A^{n,m}$ and the vertical operators by $d_v^i : A^{n,m-1} \to A^{n,m}$ and $s_v^i : A^{n,m+1} \to A^{n,m}$.

If $X^\bullet$ and $Y^\bullet$ are two cosimplicial spaces, the *external product* $X^\bullet \mathring{\times} X^\bullet$ is defined to be the bicosimplicial space with $(X^\bullet \mathring{\times} Y^\bullet)^{n,m} = X^n \times Y^m$ and $d_h^i = d^i \times 1$, $s_h^i = s^i \times 1$, $d_v^i = 1 \times d^i$, $s_v^i = 1 \times s^i$.

If $A^{\bullet\bullet}$ and $B^{\bullet\bullet}$ are two bicosimplicial spaces, the *bicosimplicial mapping space* $\hom(A^{\bullet\bullet}, B^{\bullet\bullet})$ is defined to be the space in which the n-simplices are bicosimplicial maps $\Delta^n \times A^{\bullet\bullet} \to B^{\bullet\bullet}$ and in which the face and degeneracy operators are given by the formulas $d_i(f) = f(d^i \times 1)$ and $s_i(f) = f(s^i \times 1)$.

The following lemma is straightforward.

Lemma 5.1. *If* $X^\bullet$ *and* $Y^\bullet$ *are cosimplicial spaces and* $C^{\bullet\bullet}$ *is a bicosimplicial space, then* $\hom(X^\bullet \mathring{\times} Y^\bullet, C^{\bullet\bullet}) = \hom(X^\bullet, \hom(Y^\bullet, C^{\bullet\bullet}))$, *where* $\hom(Y^\bullet, C^{\bullet\bullet})_n$ *is the cosimplicial space with* $\hom(Y^\bullet, C^{\bullet\bullet})^n = \hom(Y^\bullet, C^{n\bullet})$.

For any bicosimplicial space $A^{\bullet\bullet}$, define tot $A^{\bullet\bullet}$ to be the space $\hom(\Delta^\bullet \times \Delta^\bullet, A^{\bullet\bullet})$. Also, define horizontal and vertical tot functors by letting $\mathrm{tot}_h A^{\bullet\bullet}$ be the cosimplicial space with $(\mathrm{tot}_h A^{\bullet\bullet})^m = \mathrm{tot}\, A^{\bullet m}$ and $\mathrm{tot}_v A^{\bullet\bullet}$ the cosimplicial space with $(\mathrm{tot}_v A^{\bullet\bullet})^n = \mathrm{tot}\, A^{n\bullet}$. Lemma 5.1 has the following corollary.

Corollary 5.2. *If $A^{\bullet\bullet}$ is a bicosimplicial space, then there are natural isomorphisms* $\mathrm{tot}\, A^{\bullet\bullet} \simeq \mathrm{tot}\,\mathrm{tot}_h A^{\bullet\bullet} \simeq \mathrm{tot}\,\mathrm{tot}_v A^{\bullet\bullet}$.

Remark. One also has $\mathrm{tot}\, A^{\bullet\bullet} = \mathrm{tot}\,(\mathrm{diag}\, A^{\bullet\bullet})$ where $\mathrm{diag}\, A^{\bullet\bullet}$ is the cosimplicial space with $(\mathrm{diag}\, A^{\bullet\bullet})^n = A^{nn}$ and with coface and codegeneracy operators, $d^i = d^i_h d^i_v = d^i_v d^i_h$, $s^i = s^i_h s^i_v = s^i_v s^i_h$.

If $A^{\bullet\bullet}$ is a bicosimplicial space, we define horizontal and vertical coisimplicial matching spaces by $(M^n_h A^{\bullet\bullet})^m = M^n A^{\bullet m}$ and $(M^m_v A^{\bullet\bullet})^n = M^m A^{n\bullet}$.

Definition 5.3. A map $g : A^{\bullet\bullet} \to B^{\bullet\bullet}$ of bicosimplicial spaces is a *fibration* if $A^{n+1,\bullet} \to B^{n+1,\bullet} \times_{M^n_h B^{\bullet\bullet}} M^n_h A^{\bullet\bullet}$ is a fibration of cosimplicial spaces for all $n \geqq -1$.

Remark. The above definition is due to Bousfield. He points out that this notion of fibration extends to a closed model category structure on the category of bicosimplicial spaces in which a map $g : A^{\bullet\bullet} \to B^{\bullet\bullet}$ is a weak equivalence if and only if $g^{nm} : A^{nm} \to B^{nm}$ is a homotopy equivalence for all $n, m \geqq 0$. In addition, he observes the surprising fact that the condition of (5.3) is equivalent to the condition that $A^{\bullet,m+1} \to B^{\bullet,m+1} \times_{M^m_v B^{\bullet\bullet}} M^m_v A^{\bullet\bullet}$ be a fibration for all $n \geqq -1$.

As before, $A^{\bullet\bullet}$ is called *fibrant* if $A^{\bullet\bullet} \to c^{\bullet\bullet}(*)$ is a fibration, where $c^{\bullet\bullet}(*)$ is the bicosimplicial space which is a constant point.

The functors tot_h and tot_v preserve matching spaces, that is,

$$M^m(\mathrm{tot}_h A^{\bullet\bullet}) \simeq \mathrm{tot}(M^m_v A^{\bullet\bullet})$$

and there is a similar formula for tot_v. Furthermore, since tot preserves fibre products, the following lemma is a consequence of (4.3).

Lemma 5.4. *If $g : A^{\bullet\bullet} \to B^{\bullet\bullet}$ is a fibration, then* $\mathrm{tot}_h\, g$, $\mathrm{tot}_v\, g$, *and* $\mathrm{tot}\, g$ *are all fibrations. In particular, if $A^{\bullet\bullet}$ is fibrant, then* $\mathrm{tot}_h A^{\bullet\bullet}$, $\mathrm{tot}_v A^{\bullet\bullet}$, *and* $\mathrm{tot}\, A^{\bullet\bullet}$ *are all fibrant.*

This lemma and (4.5) imply:

Proposition 5.5. *If $g : A^{\bullet\bullet} \to B^{\bullet\bullet}$ is a map of bicosimplicial spaces with $A^{\bullet\bullet}$*

and $B^{\bullet\bullet}$ fibrant and if either, $\mathrm{tot}_h\, g$ *or* $\mathrm{tot}_v\, g$ *is a weak equivalence, then* $\mathrm{tot}\, g$ *is a homotopy equivalence.*

A bicosimplicial space $A^{\bullet\bullet}$ can have either a *vertical augmentation* $X^\bullet \to A^{0\bullet}$ or a *horizontal augmentation* $X^\bullet \to A^{\bullet 0}$. The above proposition specializes to

COROLLARY 5.6. *Let* $X^\bullet \to A^{\bullet\bullet}$ *be a vertical (resp. horizontal) augmentation with* $X^\bullet$ *and* $A^{\bullet\bullet}$ *fibrant and assume that* $X^\bullet \to \mathrm{tot}_h\, A^{\bullet\bullet}$ *(resp.* $X^\bullet \to \mathrm{tot}_v\, A^{\bullet\bullet}$*) is a weak equivalence of cosimplicial spaces. Then* $\mathrm{tot}\, X^\bullet \to \mathrm{tot}\, A^{\bullet\bullet}$ *is a homotopy equivalence.*

The cosimplicial examples of Section 4 have analogues for bicosimplicial spaces. In particular, pullbacks of fibrations are fibrations, if B is a fibrant space then the constant bicosimplicial space $c^{\bullet\bullet}B$ is fibrant, isomorphisms are fibrations, and if $B^{\bullet\bullet}$ is fibrant then the projection $A^{\bullet\bullet} \times B^{\bullet\bullet} \to A^{\bullet\bullet}$ is a fibration.

DEFINITION 5.7. A bicosimplicial space $A^{\bullet\bullet}$ is *grouplike* if each fixed row $A^{\bullet m}$ and each fixed column $A^{n\bullet}$ is a grouplike cosimplicial space.

PROPOSITION 5.8. *A surjective homomorphism* $A^{\bullet\bullet} \to B^{\bullet\bullet}$ *of grouplike bicosimplicial spaces is a fibration. In particular, a grouplike bicosimplicial space is fibrant.*

PROOF. As in [2, p. 276], $A^{n+1\bullet} \to B^{n+1\bullet} \times_{M_h^n B^{\bullet\bullet}} M_h^n A^{\bullet\bullet}$ is a surjective homomorphism of grouplike cosimplicial space, therefore, a fibration.

6. Contractions

Let $X^\bullet$ be a cosimplicial object augmented by $d^0: X^{-1} \to X^0$.

DEFINITION 6.1. The augmented cosimplicial object $X^{-1} \to X^\bullet$ is said to admit a *left contraction* if for $n \geqq -1$ there are maps $s^{-1}: X^{n+1} \to X^n$ which satisfy the evident extensions of the usual cosimplicial identities, that is:

$$\begin{aligned} s^{-1}d^j &= 1 && \text{if } j = 0, \\ &= d^{j-1}s^{-1} && \text{if } j > 0 \\ s^{-1}s^j &= s^{j-1}s^{-1} && \text{if } j > -1. \end{aligned}$$

The augmented cosimplicial object is said to admit a *right contraction* if for $n \geqq -1$ there are maps $s^{n+1}: X^{n+1} \to X^n$ which satisfy corresponding extensions of the cosimplicial identities.

LEMMA 6.2. *If* $X^{-1} \to X^{\bullet}$ *is an augmented cosimplicial space which admits a contraction then the natural* map $d: X^{-1} \to \operatorname{tot} X^{\bullet}$ *is a homotopy equivalence.*

SKETCH OF PROOF. A left contraction, say, gives a map $S: X^{\bullet} \to c^{\bullet}X^{-1}$ defined by $S^n = s^{-1} \cdots s^{-1}: X^n \to X^{-1}$. If $s = \operatorname{tot} S: \operatorname{tot} X^{\bullet} \to X^{-1}$, it is easy to see that the composite $sd: X^{-1} \to X^{-1}$ is the identity. The proof is completed by showing that the contraction provides an explicit homotopy from the composite ds to the identity map of tot $X^{\bullet}$. If X is a fibrant space and $X^{\bullet}$ is a fibrant cosimplicial space, it is also possible to derive this lemma from the fact that the existence of a contraction implies that the homotopy spectral sequence of $X^{\bullet}$ collapses, cf. [2, II, 2.7].

There are a few standard examples of contractions. Let T be a triple and $X \to T^{\bullet}X$ the canonical resolution.

(1) A left contraction for $TX \to T(T^{\bullet}X)$ is defined for $n \geqq -1$ by $s^{-1} = \mu(T^{n+1}x): T(T^{n+2}X) \to T(T^{n+1}X)$.

(2) A right contraction for $TX \to T^{\bullet}(TX)$ is defined for $n \geqq -1$ by $s^{n+1} = T^{n+1}\mu(X): T^{n+2}(TX) \to T^{n+1}(TX)$.

(3) Let E be a space over B with E and B fibrant. The Rector complex $E \to (B \times -)^{\bullet}(E)$ of example 5 in Section 4 admits a left contraction: define it for $n \geqq -1$ by letting

$$s^{-1}: (B \times -)^{n+2}(E) \to (B \times -)^{n+1}(E)$$

be projection on the last $n + 2$ factors.

PROPOSITION 6.3. *Let T be a triple on the category of spaces and M a functor from spaces to spaces. Suppose that X is a space with the property that the map $\eta \circ M: M(X) \to TM(X)$ has a left inverse. Then the map* $MX \to \operatorname{tot} T^{\bullet}(MX)$ *is a homotopy equivalence.*

PROPOSITION 6.4. *Let T and M be as in Proposition* 6.3, *and suppose that for any space X the map $M(\eta): M(X) \to MT(X)$ has a left inverse which is natural in X. Then for any space X the map* $MX \to \operatorname{tot}(MT^{\bullet}X)$ *is a homotopy equivalence.*

PROOF OF 6.3 AND 6.4. In the case of 6.3 the assumptions imply that the map $MX \to \operatorname{tot} T^{\bullet}(MX)$ is a retract of the map $TM(X) \to \operatorname{tot} T^{\bullet}TM(X)$. The desired conclusion follows from example 1 above, Lemma 6.2, and the fact that any retract of a homotopy equivalence is also a homotopy equivalence. A similar argument works in the case of 6.4.

Proposition 6.5. *Let S and T be two triples on the category of spaces and $\alpha : S \to T$ a natural transformation of triples. Suppose that for any space X*

(1) *the map $\eta_S : T(X) \to ST(X)$ has a left inverse,*

(2) *the map $S(\eta_T) : S(X) \to ST(X)$ has a left inverse which is natural in X, and*

(3) *the objects $S^{\bullet}X$ and $T^{\bullet}X$ are grouplike cosimplicial spaces.*

Then for any X the map $\alpha_\infty : S_\infty X \to T_\infty X$ is a homotopy equivalence.

Proof. Consider the bicosimplicial spaces $S^{\bullet}T^{\bullet}(X)$ and $T^{\bullet}T^{\bullet}(X)$ with, e.g., $S^{\bullet}T^{\bullet}(X)^{mn} = S^{m+1}T^{n+1}(X)$. There is a commutative diagram

$$\begin{array}{ccc} S^{\bullet}(X) & \xrightarrow{\alpha^{\bullet}} & T^{\bullet}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ S^{\bullet}T^{\bullet}(X) & \xrightarrow{\alpha^{\bullet\bullet}} & T^{\bullet}T^{\bullet}(X) \end{array}$$

in which $\alpha^{\bullet}$ and $\alpha^{\bullet\bullet}$ are induced by α and f and g are the evident horizontal augmentations. There are three things to observe about this diagram.

(1) The map $\operatorname{tot}(f) : \operatorname{tot} S^{\bullet}(X) \to \operatorname{tot} S^{\bullet}T^{\bullet}(X)$ is a homotopy equivalence. Indeed, assumption (2) implies that for any $m \geqq 1$ the map $S^m(\eta_T) : S^m(X) \to S^mT(X)$ has a left inverse which is natural in X. It follows from Proposition 6.4 that the map $S^{\bullet}(X) \to \operatorname{tot}_v S^{\bullet}T^{\bullet}(X)$ induced by f is a weak equivalence of cosimplicial spaces and then from assumption (3), Proposition 5.8, and Proposition 5.6 that $\operatorname{tot}(f)$ is a homotopy equivalence.

(2) The map $\operatorname{tot}(g) : \operatorname{tot} T^{\bullet}(X) \to \operatorname{tot} T^{\bullet}T^{\bullet}(X)$ is a homotopy equivalence. This is proved exactly as above.

(3) The map $\operatorname{tot}(\alpha^{\bullet\bullet}) : \operatorname{tot} S^{\bullet}T^{\bullet}(X) \to \operatorname{tot} T^{\bullet}T^{\bullet}(X)$ is a homotopy equivalence. Assumption (1) implies that for any $n \geqq 1$ the map $T^n(\eta_S) : T^nX \to S(T^nX)$ has a left inverse and it is evident that for any $n \geqq 1$ the map $T^n(\eta_T) : T^nX \to T(T^nX)$ has a left inverse. It follows from Proposition 6.3 that the map $T^{\bullet}(X) \to \operatorname{tot}_h S^{\bullet}T^{\bullet}(X)$ is a weak equivalence of cosimplicial spaces, and then from assumption (3), Proposition 5.8, and Proposition 5.6 that $\operatorname{tot}(\alpha^{\bullet\bullet})$ is a homotopy equivalence.

These facts together immediately imply that the map $\alpha_\infty = \operatorname{tot}(\alpha^{\bullet})$ is a homotopy equivalence.

Let $\bar{R}$ be the triple of [2, I, 2.1] and let $\bar{R}_\infty X$ denote $\operatorname{tot} \bar{R}^{\bullet}X$. Let $\alpha : \bar{R} \to R$ be the evident natural transformation of triples. The following corollary shows that the R-completion $R_\infty X$ which we use in this paper is the same up to homotopy as the R-completion of [2].

COROLLARY 6.6. *For any space X the natural transformation α induces a homotopy equivalence $\alpha_\infty : \bar{R}_\infty X \to R_\infty X$.*

PROOF. It is easy to handle directly the case in which X is the empty space. The remaining cases are implied by the analogue of Proposition 6.5 for triples on the category of *pointed* spaces (note that both R and $\bar{R}$ lift to the category of pointed spaces). The affine structure underlying RX gives a left inverse to $\eta_{\bar{R}} : R(X) \to \bar{R}R(X)$. The basepoint determines a splitting σ of α [2, I, 2.2] and hence a left inverse to $\bar{R}(\eta_R) : \bar{R}(X) \to \bar{R}R(X)$, given by the composite $\mu \cdot \bar{R}\sigma$. Finally, both $\bar{R}^\bullet X$ and $R^\bullet X$ are evidently grouplike.

7. The homotopy pullback property

This section contains the proof of Theorem 1.4. The proof consists of two stages. First we show that in the case of a product bundle $B \times F \to B$ there is a homotopy equivalence $BR_\infty(B \times F) \to B \times R_\infty(F)$. We then apply this fact to state and prove a slightly more precise form of Theorem 1.4.

Observe first that the natural map $BR(B \times F) = B \times R(B \times F) \to B \times RF$ is compatible with the triple structures of BR and R and consequently defines a map $\theta : BR^\bullet(B \times F) \to B \times R^\bullet(F)$ and a map $\theta_\infty : BR_\infty(B \times F) \to B \times R_\infty(F)$ over B.

PROPOSITION 7.1. *For any spaces B and F the map $\theta_\infty : BR_\infty(B \times F) \to B \times R_\infty(F)$ is a homotopy equivalence.*

PROOF. A proof may be constructed along the lines of the argument used by Bousfield–Kan [2, pages 34–39] to prove that the natural map $\bar{R}_\infty(X \times Y) \to \bar{R}_\infty X \times \bar{R}_\infty Y$ is a homotopy equivalence. This proof, which is based on the categorical acyclic models theorem of Barr and Beck, uses as acyclic models the functor $T(B, F) = B \times RF$ defined on the category of pairs of spaces. This was our original proof. We are grateful to J.-P. Meyer for pointing out that a subsequent proof of ours is invalid and for providing an alternative proof which may be found in [6].

The map $BR(E) = B \times R(E) \to R(E)$ is compatible with the triple structures and thus defines a map $\psi : BR^\bullet(E) \to R^\bullet(E)$ and a map $\psi_\infty : BR_\infty(E) \to R_\infty(E)$. Proposition 7.1 implies that the natural map $B \to BR_\infty(B)$ is a homotopy equivalence and it is clear that the composite of this map with

$\psi_\infty : BR_\infty(B) \to R_\infty(B)$ is the standard map from B to its R-completion. In order to prove Theorem 1.4, then, it is sufficient to prove the following result.

PROPOSITION 7.2. *Let B be a fibrant space and $q : E \to B$ a fibration. Then the square*

$$\begin{array}{ccc} BR_\infty(E) & \xrightarrow{\psi_\infty} & R_\infty E \\ {\scriptstyle BR_\infty(q)}\downarrow & & \downarrow{\scriptstyle R_\infty q} \\ BR_\infty(B) & \xrightarrow{\psi_\infty} & R_\infty(B) \end{array}$$

is a homotopy fibre square.

The *Rector complex* $E \to (B \times -)^\bullet(E)$ of a fibration $E \to B$ was described in Section 4. This is essentially a resolution of $E \to B$ by product fibrations.

LEMMA 7.3. *If $E \to B$ is a fibration, then the natural maps $BR_\infty(E) \to$ $\mathrm{tot}(BR_\infty(B \times -)^\bullet(E))$ and $R_\infty E \to \mathrm{tot}(R_\infty(B \times -)^\bullet(E))$ are homotopy equivalences.*

PROOF. The two maps are handled in a similar way; we will treat only the first one. Consider the bicosimplicial space $A^{\bullet\bullet}$ which in bidegree (m, n) contains the space $BR^{m+1}(B \times -)^{n+1}(E)$. By one of the examples in Section 6, the Rector complex $E \to (B \times -)^\bullet(E)$ admits a left contraction; by functoriality this contraction is inherited by each of the derivative complexes $BR^m(E) \to BR^m(B \times -)^\bullet(E)$. It follows from Lemma 6.2, that the augmentation $BR^\bullet(E) \to \mathrm{tot}_v A^{\bullet\bullet}$ is a weak equivalence of cosimplicial spaces, and therefore, since the cosimplicial spaces involved are fibrant, that the induced map $BR_\infty(E) \to \mathrm{tot}\, A^{\bullet\bullet}$ is a homotopy equivalence. However, the target of this homotopy equivalence can also be interpreted as $\mathrm{tot}\,\mathrm{tot}_h(A^{\bullet\bullet}) =$ $\mathrm{tot}\, BR_\infty(B \times -)^\bullet(E)$.

LEMMA 7.4. *The conclusion of Proposition 7.2 holds if $E \to B$ is a product fibration.*

PROOF. By Proposition 7.1 and the product lemma of [2, I, 7.2], the square of Proposition 7.2 is homotopy equivalent in the case of a product fibration $B \times F \to B$ to the square

$$\begin{array}{ccc} B \times R_\infty F & \longrightarrow & R_\infty B \times R_\infty F \\ \downarrow & & \downarrow \\ B & \longrightarrow & R_\infty B \end{array}$$

This is evidently a homotopy fibre square.

PROOF OF 7.2. By Lemma 7.4, it is possible to apply Proposition 4.6 to the square

$$\begin{array}{ccc} BR_\infty(B \times -)^\bullet(E) & \longrightarrow & R_\infty(B \times -)^\bullet(E) \\ \downarrow & & \downarrow \\ c^\bullet BR_\infty(B) & \longrightarrow & c^\bullet R_\infty(B) \end{array}$$

By Lemma 7.3, this gives the desired result.

REFERENCES

1. A. K. Bousfield, *Homotopy spectral sequences and obstructions*.

2. A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin, 1972.

3. W. G. Dwyer and A. Zabrodsky, *Maps between classifying spaces*, in *Algebraic Topology, Barcelona 1986*, Lecture Notes in Mathematics, Vol. 1298, Springer-Verlag, Berlin, 1987.

4. J. Lannes, *Sur la cohomologie modulo p des p-groupes Abéliens élémentaires*, in *Homotopy Theory, Proc. Durham Symp. 1985* (E. Rees and J. D. S. Jones, eds.), Cambridge Univ. Press, Cambridge, 1987.

5. H. R. Miller, *The Sullivan conjecture on maps from classifying spaces*, Ann. of Math. **120** (1984), 39–87.

6. J.-P. Meyer, *Cosimplicial homotopies*, Proc. Am. Math. Soc., to appear.