"Chromatic" homotopy theory
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Homotopy theory deals with spaces of large but finite dimension. *Chromatic homotopy theory* is an organizing principle which is highly developed in the *stable* situation.

1. The Spanier-Whitehead category. We'll work with the category of finite polyhedra (or finite CW complexes) and homotopy classes of continuous maps between them. We will always fix a basepoint in all spaces, and assume that maps and homotopies preserve them. Write $\mathcal{F}$ for this homotopy category, and write $[K, L]$ for the set of pointed homotopy classes of pointed maps. It's just a pointed set, and very difficult to compute in even quite simple cases. The idea of *stable homotopy theory* is to try to simplify this problem by a certain type of localization.

There is a shift operator on $\mathcal{F}$. Embed K into the *cone* on K,

$$C(K) = \frac{[0,1] \times K}{1 \times K \cup [0,1] \times *}$$

as the subspace $0 \times K$, and then collapse this subspace to a point. This is the *suspension* ΣK of K. The construction may be iterated. There is a natural isomorphism

$$\overline{H}_{q+n}(\Sigma^n K) = \overline{H}_q(K)$$

By functoriality, there are maps

$$[K, L] \to [\Sigma K, \Sigma L] \to [\Sigma^2 K, \Sigma^2 L] \to \cdots$$

and these maps are eventually isomorphisms. Elements of the direct limit are called *stable maps* from K to L. Maps from a suspension form a group, maps from a double suspension form an abelian group, and the suspension maps are homomorphisms when they can be: so the set of stable maps forms an abelian group.

Spanier and Whitehead (around 1955) defined what is now called the *homotopy category of finite spectra*, $\mathcal{S}$. An object has the form $K[q]$ for $K \in \mathcal{F}$ and $q \in \mathbb{Z}$. Morphisms are defined by

$$[K[q], L[r]] = \lim_{n \to \infty} [\Sigma^{q+n} K, \Sigma^{r+n} L]$$

The direct system begins with n large enough so that both $q + n$ and $r + n$ are nonnegative.

If $K \in \mathcal{F}$, write K also for the object $K[0] \in \mathcal{S}$. We can now make sense of S^n for any $n \in \mathbb{Z}$: $S^n = S^0[n]$.

For $X, Y \in \mathcal{S}$, $[X, Y]$ is an abelian group. In fact $\mathcal{S}$ is an additive category, with

$$K[q] \oplus L[r] = (\Sigma^{n+q} K \vee \Sigma^{n+r} L)[-n]$$

for any n making both $n + q$ and $n + r$ nonnegative. Even better, it is *triangulated*. The distinguished triangles are those isomorphic to *cofiber sequences*. In $\mathcal{F}$ the cofiber of $f : K \to L$ is the pushout in

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d] & L \arrow[d] \\
CK \arrow[r] & L \cup_f CK
\end{tikzcd}$$

The cofiber maps to ΣK by collapsing K to a point. The shift functor in this triangulation is given by $\Sigma K[q] = K[q + 1]$. There is an isomorphism $(\Sigma K)[q] \xrightarrow{\cong} K[q + 1]$. The

The category $\mathcal{S}$ of "finite spectra" offers a useful "first approximation" to homotopy theory, in the words of Spanier and Whitehead. It embeds in a larger triangulated category of spectra as the compact objects.

Define the homotopy and homology of $K[q]$ by

$$\pi_r(K[q]) = [S^r, K[q]] = \lim_{n \to \infty} \pi_{r+n}(\Sigma^{q+n} K)$$

and

$$H_r(K[q]) = \overline{H}_{r-q}(K)$$

2. Constructions. Let p be a prime number and consider the homotopy class of maps $S^n \to S^n$ of degree n. These suspend to each other, and define a map $p : S^0 \to S^0$ in $\mathcal{S}$. It is *non-nilpotent*: no iterate is ever null-homotopic. This is seen by applying homology to compute the *degree*.

The first element of p-torsion in $\pi_*(S^0)$ is in degree $2p - 3$, and is written $\alpha_1 : S^{2p-3} \to S^0$. (When $p = 2$ this comes from the famous *Hopf map* $S^3 \to S^2$.)

The identity $p\alpha_1 = 0$ has a geometric implication. A map $S^n \to S^n$ of degree p can be used to attach an $(n + 1)$-cell to S^n: $S^n \cup_p e^{n+1}$. This construction is compatible with suspension, and defines a finite spectrum written S^0/p or $S^0 \cup_p e^1$.

Let $q = 2p - 2$ and consider

$$\begin{tikzcd}
S^q \cup_p e^{q+1} \arrow[r, dotted] \arrow[dr, dotted] & S^0 \cup_p e^1 \arrow[d] \\
S^q \arrow[u] \arrow[r, "\Sigma\alpha_1"] & S^1 \\
S^q \arrow[u, "p"] &
\end{tikzcd}$$

The diagonal arrow records the null-homotopy of $p\alpha_1 = \alpha_1 \circ p$. The top arrow exists by virtue of a harder computation, due to Frank Adams, which is valid only when $p > 2$.

We now have a *self-map* of S^0/p,

$$v_1 : \Sigma^q S^0/p = S^q/p \to S^0/p$$

Even though it has nonzero dimension, you should think of it as an analogue of $p : S^0 \to S^0$. In particular you can iterate it: neglecting to indicate suspensions of maps,

$$\begin{tikzcd}
S^{qt}/p \arrow[r, "v_1"] & S^{q(t-1)}/p \arrow[r, "v_1"] & \cdots \arrow[r, "v_1"] & S^0/p \arrow[d] \\
S^{qk} \arrow[u] \arrow[rrr, "\Sigma\alpha_t"] & & & S^1
\end{tikzcd}$$

The composite α_t defines a new and high dimensional element in the homotopy of the sphere spectrum. It is actually a well-known class, in the "image of the J-homomorphism."

Larry Smith showed that this may be repeated, as long as $p \geq 5$: let

$$S^0/(p, v_1) = S^0/p \cup_{v_1} C(S^q/p)$$

This has "cells" in dimensions $0, 1, 2p - 1$ and $2p$. There is a map

$$v_2 : S^{2(p^2-1)}/(p, v_1) \to S^0/(p, v_1)$$

which we may use to create more elements in high dimensional homotopy:

$$
\begin{tikzcd}
S^{2(p^2-1)t}/(p, v_1) \arrow[r, "v_2^k"] & S^0/(p, v_1) \arrow[d] \\
S^{2(p^2-1)t} \arrow[u] \arrow[r, "\beta_t"] & S^{2p}
\end{tikzcd}
$$

3. Detection. This is the constructive side of the story. It is not clear though that these elements are nonzero!—you need some analogue of the *degree*.

This is provided by a certain family of homology theories, known as *Morava K-theories*. These are functors on $\mathcal{S}$ that satisfy a Meyer-Vietoris property making them computable.

The zeroth member of this family is just $K(0)_*(X) = H_*(X; \mathbb{Q})$.

Thereafter there is a family for each prime p. $K(1)_*$ is very closely related to topological complex K-theory, but in general $K(n)_*$ is more novel. $K(2)_*$ is related to "elliptic cohomology."

The maps v_n are such that $K(n)_*$ is nonzero on the spectra and $K(n)_*(v_n)$ is an isomorphism. This shows that v_n is non-nilpotent.

It takes more work to show that α_t and β_t are nonzero, but they are.

4. The Periodicity Theorem. These are hard computations in the Adams spectral sequence, and they seem to represent very special situations. This turns out not to be the case, however! This is the content of work of Ethan Devinatz, Mike Hopkins, and Jeff Smith, from the 1980's.

If $X \in \mathcal{S}$ is such that $H_*(X; \mathbb{Q}) = 0$, then the homotopy type of X splits as a finite wedge of spectra each of which has homology which is p-torsion for a single prime p. Let $\mathcal{S}_p$ denote the category of p-torsion finite spectra. Let $\mathcal{S}_{p,n}$ be the category of p-torsion finite spectra for which $K(n-1)_*(X) = 0$. Its objects are said to be of *type n*.

Theorem. [Ravenel [4]] If $X \in \mathcal{S}$ then $K(n)_*(X) = 0 \Rightarrow K(n-1)_*(X) = 0$.

This implies that

$$\mathcal{S}_{p,1} \supseteq \mathcal{S}_{p,2} \supseteq \mathcal{S}_{p,3} \supseteq \cdots$$

It is far from clear that the strata are nonempty, and a construction is needed to show this.

Theorem. [Devinatz, Hopkins, Smith [1]] Let $X \in \mathcal{S}_{p,n}$. There exists a map $\phi : \Sigma^{2(p^n-1)p^k} X \to X$ such that $K(n)_*(\phi)$ is an isomorphism and $K(i)_*(\phi) = 0$ for $i > n$.

The map ϕ is a v_n*-self-map.* The maps v_1 and v_2 are examples.

Theorem. [Devinatz, Hopkins, Smith [1]] Let $X, Y \in \mathcal{S}_{p,n}$, and $f : X \to Y$. Let $\phi : \Sigma^{2(p^n-1)p^k} X \to X$ and $\theta : \Sigma^{2(p^n-1)p^l} Y \to Y$ be v_n-self-maps. There exists an integer m with $m \geq k$ and $m \geq l$ such that the diagram

$$\begin{tikzcd} \Sigma^{2(p^n-1)p^m} X \arrow[r, "f"] \arrow[d, "\phi^{p^{m-k}}"'] & \Sigma^{2(p^n-1)p^m} Y \arrow[d, "\theta^{p^{m-l}}"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes.

In particular, take f to the identity map on X: this says that any two v_n-self-maps have homotopic iterates. On the category $\mathcal{S}_{p,n}$, there is an "ideal operator" acting, well defined up to iterates. The labor involved in constructing the self-maps v_1, v_2, wasn't in vain; but some power of them exists automatically by this theorem.

5. Localization. This leads to the following challenge: *For a finite p-torsion spectrum X of type n, compute*

$$\phi^{-1}\pi_*(X) = \varinjlim \left(\pi_*(X) \xrightarrow{\phi} \pi_*(\Sigma^{-2(p^n-1)p^k} X) \xrightarrow{\phi} \cdots \right)$$

The only known examples occur when $n = 0$ or $n = 1$.

When $n = 0$, we are looking at a finite spectrum whose homology contains an infinite cyclic summand (and, say, no p'-torsion). A v_0 self-map is given by multiplication by p, and Serre proved that

$$p^{-1}\pi_*(X) = H_*(X; \mathbb{Z}[1/p])$$

When $n = 1$ we have the computation (for $p > 2$) [3]

$$v_1^{-1}\pi_*(S^0/p) = \mathbb{F}_p[v_1^{\pm 1}]\langle i, a\rangle$$

where

$$i : S^0 \longrightarrow S^0/p \quad \text{and} \quad a : S^{2p-3} \xrightarrow{\alpha_1} S^0 \xrightarrow{i} S_0/p$$

References

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[4] D. C. Ravenel, Localization with respect to certain periodic homology theories, Amer. J. Math. 106 (1984) 351–414.