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UNIVERSAL BERNOULLI NUMBERS AND THE S^1 - TRANSFER

Haynes Miller

Several authors ([4], [6], [7], [8], [12]) have considered a stable "transfer" map

$$t: \ \mathbb{C}P_0^\infty \wedge S^1 \to S^0,$$

and it is of interest to develop techniques by which to compute its effect in stable homotopy. In this note we begin an attack on this problem via the Novikov spectral sequence ([2], [13]) $E_r(-)$.

We shall study t by means of its unique factorization through the Moore spectrum for $\mathbb{Q}/\mathbb{Z}$:

$$\begin{array}{ccc} \mathbb{C}P_0^\infty \wedge S^2 & \xrightarrow{\ \bar{u}\ } & S\mathbb{Q}/\mathbb{Z} \\ & \searrow^{t \wedge S^1} & \downarrow \partial \\ & & S^1 \end{array}$$

The behavior of ∂ in the Novikov spectral sequence is quite well understood [13], and our principal result here, Cor. 3.10, describes $\bar{u}_*$ on the standard generators in $MU_*\mathbb{C}P_0^\infty$. The analogous result for $K_*\mathbb{C}P_0^\infty$ appears in [8], and I understand that Knapp now has a proof for the present case also.

In [16], D. M. Segal gave generators for $E_2^0(\mathbb{C}P_0^\infty)$. In Section 4 we describe these elements together with a simplified proof of their properties. Their images under $\bar{u}_*$ turn out to be "universal Bernoulli numbers," in the sense that they are to the formal group for MU, which is universal, as the usual Bernoulli numbers are to the multiplicative formal group. A construction of these classes, and a computation of their denominators, is carried out in Section 1.

As a corollary, we recover the fact, due to Becker and Schultz, that for $k > 0$ neither μ_{8k+1} nor the generator of the image of the J-homomorphism in dimension $8k - 1$ lies in the image of t. It is to be hoped that this work will lead to a complete computation of the image of t_* in $E_2^2(S^0)$, providing a context for the germinal result of K. Knapp [7].

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Section 1. BERNOULLI NUMBERS ATTACHED TO A FORMAL GROUP.

We assume the reader is familiar with the basic properties of formal groups as exposed for instance in [2] or [3]. As an illustration, for any element u in a ring A, one has a formal group

$$G_u(X,Y) = X + Y - uXY.$$

The additive formal group G_a is then the case $u = 0$, and the multiplicative formal group G_m is the case $u = 1$. If A is torsion-free, embed it in $A_{\mathbb{Q}} = A \otimes \mathbb{Q}$, let $\log_F\colon F \to G_a$ be the (unique) isomorphism of formal groups over $A_{\mathbb{Q}}$, and let $\exp_F(T)$ its inverse. For example, $\log_{G_m}(T) = -\ln(1 - T)$ and $\exp_{G_m}(T) = 1 - e^{-T}$.

DEFINITION 1.1. Let A be a torsion-free ring and F a formal group over A. The Bernoulli numbers associated to F are the coefficients $B_n(F) \in A_{\mathbb{Q}}$ in the powerseries expansion

$$\frac{T}{\exp_F(T)} = \sum_{n=0}^{\infty} \frac{B_n(F)}{n!} T^n.$$

The divided Bernoulli numbers are $B_n(F)/n$.

EXAMPLE 1.2. (a) $B_0(F) = 1$ for all F. (b) $B_n(G_a) = 0$ for all $n > 0$. (c) $B_n(G_m)$ is the usual Bernoulli number, occuring in

$$\frac{T}{1-e^{-T}} = \sum_{n=0}^{\infty} \frac{B_n(G_m)}{n!} T^n.$$

Recall [5] that the reduction of $B_n(G_m)/n$ in $\mathbb{Q}/\mathbb{Z}$ has order d_n, where

$$\begin{aligned} d_n &= \prod_{(p-1)\mid n} p^{\nu_p(n)+1} && \text{for even } n \\ &= 2 && \text{for } n = 1 \\ &= 1 && \text{for odd } n > 1. \end{aligned}$$

These denominators are universal:

THEOREM 1.3. If F is a formal group over a torsion-free ring A, then

$$d_n \frac{B_n(F)}{n} \in A.$$

We begin our proof of this theorem with a lemma.

LEMMA 1.4. If F and F' are formal groups isomorphic over a torsion-free ring A, then $B_n(F)/n \equiv B_n(F')/n \mod A$.

PROOF. Let $\varphi\colon F \to F'$ be the isomorphism, so that $\exp_{F'}(T) = \varphi(U)$ where $U = \exp_F(T)$. Then

$$\frac{T}{\exp_{F'}(T)} = \frac{T}{\varphi(U)} = \frac{T}{U}\,\frac{U}{\varphi(U)} = \frac{T}{\exp_F(T)} + T\sum_{n=1}^{\infty} a_n U^{n-1}$$

where $U/\varphi(U) = \sum_{n=0}^{\infty} a_n U^n$, $a_0 = 1$, $a_n \in A$. The lemma is equivalent to the assertion that the value at 0 of the $i^{\underline{th}}$ derivative of U^n lies in A for all $i \geq 0$ and $n \geq 1$. This is well-known for $U = \exp_F(T)$, and follows by Leibnitz' formula for larger n. □

A ring-homomorphism $f\colon A \to B$ carries a formal group F over A to a formal group f_*F over B. There is an initial object in the category of pairs (A,F), namely [2] the Lazard group G over the Lazard ring L. We now recall a well-known integrality statement for L. Let $A = \mathbb{Z}[b_1, b_2, \ldots]$ and let $\varphi(T) = \sum_{i=0}^{\infty} b_i T^{i+1}$ with $b_0 = 1$. Let ${}^{\varphi}G_m(X,Y) = \varphi(G_m(\varphi^{-1}(X), \varphi^{-1}(Y)))$; this is a formal group over A. Let $\eta\colon L \to A$ classify ${}^{\varphi}G_m$. Then we have:

THEOREM 1.5. (Stong-Hattori) In this situation, the diagram

$$\begin{array}{ccc} L & \xrightarrow{\ \eta\ } & A \\ \downarrow & & \downarrow \\ L_{\emptyset} & \xrightarrow{\ \eta_{\emptyset}\ } & A_{\emptyset} \end{array}$$

is a Cartesian square. In particular, L is torsion-free. □

For a proof, see [10]. Theorem 1.3 follows as a corollary. For $\varphi\colon G_m \to {}^{\varphi}G_m$ is an isomorphism, so by Lemma 1.4 $B_n(G_m)/n$ and $B_n({}^{\varphi}G_m)/n$ have the same denominators; but $B_n({}^{\varphi}G_m)/n = \eta_* B_n(G)/n$, so the theorem holds for G by the Stong-Hattori theorem. The general case follows immediately. □

REMARK 1.6. Consequently, if F is a formal group over a ring A, possibly <u>with</u> torsion, then "Bernoulli numerators" $N_n(F) \in A$ are defined, viz.,

$$N_n(F) = \eta_*\Big(d_n \frac{B_n(G)}{n}\Big)$$

where $\eta\colon L \to A$ classifies F.

Section 2. THE TRANSFER.

We give a brief description of the transfer construction, focussing on examples useful to us here. The reader is referred to [12, 14] for more exhaustive accounts of the transfer. We end with a proof (due to S. Mitchell [14]) of the fact, stated in [7], that $t\colon \mathbb{C}P_0^{\infty} \wedge S^1 \to S^0$ is the cofiber of a natural collapse map.

Let $\pi\colon E \to B$ be a smooth map, and let $\xi \downarrow E$ and $\zeta \downarrow B$ be vector bundles. A <u>relative framing</u> is a lift $j\colon E \to \mathbb{R}^k_B$ of π to an embedding (with normal bundle $\nu(j)$) into a trivial vector bundle over B together with a bundle-isomorphism

$$\phi\colon\ \xi \oplus \mathbb{R}^k_E \to \pi^*\zeta \oplus \nu(j).$$

Given this data, application of the Pontrjagin-Thom collapse gives a stable map

$$t\colon\ B^\zeta \to E^\xi \tag{2.1}$$

of Thom spaces, called the <u>transfer</u>. An obvious modification allows us to suppose that ζ and ξ are merely virtual bundles.

EXAMPLE 2.2. Let L_n denote the complex dual of the tautologous complex line-bundle over $\mathbb{C}P^n$. For $-\infty < q \le r < \infty$, define

$$\mathbb{C}P^r_q = (\mathbb{C}P^{r-q})^{qL_{r-q}}.$$

For $r \le s$, the bundle-map $L_{r-q} \to L_{s-q}$ induces a map

$$i\colon\ \mathbb{C}P^r_q \to \mathbb{C}P^s_q$$

of Thom spaces. For $p \le q$, recall that the inclusion $j\colon\ \mathbb{C}P^{r-q} \to \mathbb{C}P^{r-p}$ has normal bundle $(q-p)L_{r-q}$. Taking $\xi = qL_{r-q}$ and $\zeta = pL_{r-p}$, we obtain a transfer map

$$c\colon\ \mathbb{C}P^r_p \to \mathbb{C}P^r_q.$$

It is not hard to see that if p, q, r, and s are all nonnegative then under the usual homeomorphism

$$\mathbb{C}P^r_q \cong (\mathbb{C}P^r/\mathbb{C}P^{q-1})$$

these maps coincide with the natural inclusion and collapse maps. Also, their obvious compatibility allows us to include the possibility of $r = \infty$ or $s = \infty$.

Given $i + j = q \in \mathbb{Z}$ and $k \ge 0$, the bundle map $\overline{\Delta}\colon\ qL_k \to iL_k \times jL_k$ covering the diagonal of $\mathbb{C}P^k$ induces

$$\Delta\colon\ \mathbb{C}P^{q+k}_k \to \mathbb{C}P^{i+k}_i \wedge \mathbb{C}P^{j+k}_j \tag{2.3}$$

These maps are clearly associative, unitary, commutative, and behave well with respect to i and c.

EXAMPLE 2.4. Let $\pi\colon\ S^{2n+1} \to \mathbb{C}P^n$ be the usual projection map. The bundle $\tau(\pi)$ of tangents along the fiber is complementary to the normal bundle of π, and is trivialized by the infinitesimal generator of the S^1 - action. Thus we obtain a stable transfer map

$$\bar{t}: \mathbb{C}P_0^n \wedge S^1 \to S_+^{2n+1}.$$

The projection to S^{2n+1} has degree 1 and is of no further interest, by Hopf's theorem, so we consider the other factor, $\mathbb{C}P_0^n \wedge S^1 \to S^0$. These maps are compatible over n, and yield a stable map

$$t: \mathbb{C}P_0^\infty \wedge S^1 \to S^0.$$

This map was studied by J. C. Becker and R. E. Schultz, who proved:

THEOREM 2.5. [4] There is a stable map $j_{S^1}: U \to \mathbb{C}P_0^\infty \wedge S^1$ such that the composite $t \circ j_{S^1}$ is adjoint to the composite

$$j_{\mathbb{C}}: U \xrightarrow{J_{\mathbb{C}}} Q_1 S^0 \xrightarrow{*[1]} QS^0. \qquad \square$$

Recall also

PROPOSITION 2.6. [12] If $\lambda: \mathbb{C}P_0^\infty \wedge S^1 \to U$ carries (ℓ, z) to multiplication by z in the line ℓ, then the composite $j_{S^1} \circ \lambda$ is homotopic to the identity. $\square$

We end this section with:

LEMMA 2.7. [7] The diagram

$$\mathbb{C}P_{-1}^\infty \wedge S^1 \xrightarrow{c} \mathbb{C}P_0^\infty \wedge S^1 \xrightarrow{t} S^0$$

is a cofibration sequence.

PROOF. [14] First note that $\bar{t}c: \mathbb{C}P_{-1}^n \wedge S^1 \to S_+^{2n+1}$ is the transfer associated to the composite $S^{2n+1} \to \mathbb{C}P^n \to \mathbb{C}P^{n+1}$. Since this composite is null-homotopic, $\bar{t}c$ factors up to homotopy through the Thom space S^{2n+1} of a bundle over a point. Since the null-homotopies are compatible as n increases, the composite $\bar{t}c$ factors through the contractible spectrum S^∞, and hence is null-homotopic. Therefore tc is null-homotopic.

There results a map

$$\alpha: \mathbb{C}P_{-1}^\infty \wedge S^2 \to C(t)$$

to the mapping-cone of t. It is clearly a homology-isomorphism in positive dimensions. In Remark 3.5(c) we shall see that P^1 is nontrivial on $H^0(\mathbb{C}P_{-1}^\infty \wedge S^2)$ (where $P^1 = Sq^2$ if $p = 2$). According to Theorem 2.5, the J-homomorphism $j_{\mathbb{C}}: U \to S^0$ factors through t, so the element α_1 of Hopf invariant 1 is carried on $C(t)$, and it follows that P^1 is nontrivial on $H^0(C(t))$ as well. Therefore $H_0(\alpha)$ is also an isomorphism. The map α is thus a homotopy-equivalence, and the result follows. □

REMARK 2.8. One may use ideas of Löffler and Smith [11] in place of the work of Becker and Schultz to see that P^1 detects t.

Section 3. TRANSFER, THOM CLASS, AND COACTION.

In this section we show how to extends Adams' treatment [2] of the complex bordism of $\mathbb{C}P^\infty$ to the case of $\mathbb{C}P_n^\infty$ for $n \in \mathbb{Z}$. We shall adopt the convention that the E-homology of X is $X_*E = \pi_*(X \wedge E)$, the stable homotopy of $X \wedge E$. (The homology (and cohomology) of a space is thus always reduced.) If E is a ring-spectrum (always associative and commutative) for which E_*E is flat over $E_* = \pi_*E$, then we have a right coaction map

$$\psi\colon\ X_*E \to X_*E \otimes_{E_*} E_*E.$$

This convention seems natural from several points of view; in particular, it fits well with [13].

Recall that an orientation of a ring-spectrum E is an element $x = x_E \in E^2(\mathbb{C}P^\infty)$ restricting to the canonical generator of $E^2(S^2)$. If E is oriented by x, it follows that $E^*(\mathbb{C}P_0^\infty) = E^*[[x]]$; and if $\mu\colon\ \mathbb{C}P_0^\infty \wedge \mathbb{C}P_0^\infty \to \mathbb{C}P_0^\infty$ is the multiplication map, then $\mu^*x = F(x \otimes 1, 1 \otimes x)$ defines a formal group F over E^*. The homeomorphism $\mathbb{C}P^\infty \cong MU(1)$ provides MU with a canonical orientation x_{MU}, and (MU, x_{MU}) is universal in the obvious sense. By a famous result of Quillen, the natural map $L \to MU^*$ is an isomorphism; see [2].

Now consider $\mathbb{C}P_n^\infty$ for $n \in \mathbb{Z}$. The diagonal (2.3) $\Delta\colon\ \mathbb{C}P_n^\infty \to \mathbb{C}P_0^\infty \wedge \mathbb{C}P_n^\infty$ makes $E^*(\mathbb{C}P_n^\infty)$ into a module over $E^*(\mathbb{C}P_0^\infty)$, which is free on one generator u by the Thom isomorphism theorem. If $n \geq 0$, we take u such that $c^*u = x^n \in E^{2n}(\mathbb{C}P_0^\infty)$, using the notation of (2.2). If $n < 0$, we take u such that $x^{-n}u = c^*1 \in E^0(\mathbb{C}P_n^\infty)$. In either case it makes good sense for $i \geq n$ to write x^i for $x^{i-n}u$, and henceforth we do so.

Let $\beta_i = \beta_i^E \in \pi_{2i}(\mathbb{C}P_n^\infty \wedge E)$ be dual to x^i, and write

$$\hat\beta(T) = \hat\beta_n(T) = \sum_{i\geq n} \beta_i T^i.$$

Then the module structure dualizes to give

$$\Delta_*\hat\beta(T) = \beta(T) \otimes \hat\beta(T) \tag{3.1}$$

where $\beta(T) = \hat\beta_0(T)$.

Our approach to the coaction for $\mathbb{C}P^\infty$ differs from that of Adams. We may write

$$\psi\hat\beta(T) = \sum_{j\geq n} \beta_j \otimes f_j(T)$$

for suitable power series $f_j(T)$; these power series are independent of n, in view of the maps i and c. Now the Cartan formula of [1], p.71, applied to $\hat{\beta}(T)$, asserts that

$$\sum \beta_i \otimes \beta_j \otimes f_{i+j}(T) = \sum \beta_i \otimes \beta_j \otimes f_i(T) f_j(T).$$

Together with the obvious fact that $f_0(T) = 1$, this yields $f_j(T) = f_1(T)^j$. Write

$$f_1(T) = b(T) = \sum_{i \geq 0} b_i T^{i+1}.$$

The unital property of ψ shows that $b_0 = 1$.

We must next recall some generalities concerning the Kronecker pairing. For a ring-spectrum E and an E-module-spectrum F with structure-map $\varphi\colon F \wedge E \to F$, we have a pairing

$$\langle\ ,\ \rangle\colon \; F^p(X) \otimes X_q E \to F_{q-p}\colon$$

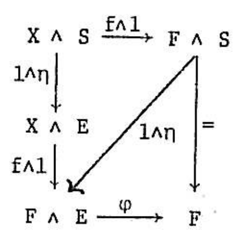
for $f\colon X \to \Sigma^p F$, $e\colon S^q \to X \wedge E$,

$$\langle f,e\rangle\colon \; S^q \xrightarrow{e} X \wedge E \xrightarrow{f\wedge 1} \Sigma^p F \wedge E \xrightarrow{\varphi} \Sigma^p F.$$

LEMMA 3.2. In addition to these notations, let $\mu\colon F_* \otimes_{E_*} E_*E \to F_*E$ be the evident multiplication. Assume that E_*E is flat over E_*. Then

$$f_* e = \mu\langle f, \psi e\rangle.$$

PROOF. Since



commutes, the top row of the following commutative diagram evaluates f_*.

$$\begin{array}{ccccccc}
X_*E & \xrightarrow{(1\wedge\eta)_*} & (X \wedge E)_*E & \xrightarrow{(f\wedge 1)_*} & (F \wedge E)_*E & \xrightarrow{\varphi_*} & F_*E \\
 & \searrow^{\psi} & \uparrow{\scriptstyle \mu\,\cong} & & \uparrow{\scriptstyle \mu\,\cong} & & \uparrow{\scriptstyle \mu\,\cong} \\
 & & X_*E \otimes_{E_*} E_*E & \xrightarrow{f_*\otimes 1} & F_*E \otimes_{E_*} E_*E & \xrightarrow{\varphi_*\otimes 1} & F_* \otimes_{E_*} E_*E
\end{array}$$

But the bottom row is $\langle f,-\rangle$. □

Now $b(T)$ is by definition $\langle x, \psi\beta(T)\rangle$; so we have proved:

PROPOSITION 3.3. In $\pi_*(\mathbb{C}P_n^\infty \wedge MU)$,

$$\psi\hat{\beta}(T) = \hat{\beta}(1 \otimes b(T)).$$

with $b(T) = x_*\beta(T)$. □

REMARK 3.4. This line of argument may be used to determine the form of the coaction in any space with polynomial integral cohomology. Consider $\mathbb{H}P^\infty$ for example. For the generator $y \in MU^4\mathbb{H}P_0^\infty$ we may take the second Connor-Floyd Chern class of the canonical bundle over $\mathbb{H}P^\infty$ thought of as a complex 2-plane bundle. Under the natural inclusion $i\colon \mathbb{C}P^\infty \to \mathbb{H}P^\infty$, this bundle pulls back to $L \oplus L^*$, so, with $[n] = [n]_G$ as in [15],

$$i^*y = cf_2(L \oplus L^*) = cf_1(L)cf_1(L^*) = x[-1](x).$$

Let γ_i be a dual to y^i, and form a power series

$$\gamma(T) = \sum_{i \geq 0} \gamma_i T^{2i}.$$

Since i_* is a coalgebra map, we find that

$$i_*\beta(T) = \gamma(c(T))$$

for some power series $c(T)$. We compute:

$$c(T) = \langle y, i_*\beta(T)\rangle = \langle i^*y, \beta(T)\rangle$$

$$= \langle x[-1](x), \beta(T)\rangle = T[-1](T).$$

For the coaction, we compute:

$$\psi\gamma(c(T)) = \psi i_*\beta(T) = i_*\psi\beta(T)$$

$$= i_*\beta(1 \otimes b(T)) = \gamma(1 \otimes c_L(b(T)))$$

where $c_L(T) = \eta_{L*}c(T)$. This implicitly determines the coaction in $\mathbb{H}P^\infty$; cf. [16].

REMARK 3.5. (a) Since we are using the <u>right</u> coaction, our elements $b_i \in BP_{2i}BP$ are <u>conjugate</u> to those of Adams [2 : I].

(b) Let $G_L = \eta_{L*}G$ and $G_R = \eta_{R*}G$; then by [2 : I (11.4)] the power series $b(T)$ is an isomorphism of formal groups from G_R to G_L; that is,

$$b(\exp_R(T)) = \exp_L(T) \tag{3.6}$$

where again $\exp_R(T) = \eta_{R*}\exp_G(T)$, etc. Also $\rho_*\colon MU_*MU \to H\mathbb{Q}_*MU$ carries

G_L to G_a, so

(3.7) $$\rho_* b(T) = \log_G(T).$$

(c) The same results hold in any oriented ring-spectrum E such that E_*E is flat over E_*. In particular we find in mod p homology that

$$\psi\hat{\beta}(T) = \hat{\beta}(1 \otimes \zeta(T)),$$

where $\zeta(T) = \sum \zeta_i T^{p^i}$ with $\zeta_i = \chi\xi_i$ for p odd and $\zeta_i = \chi\xi_i^2$ for $p = 2$.

Now let $t\colon \mathbb{C}P_0^\infty \wedge S^1 \to S^0$ be the transfer map constructed in Section 2. Since MU_* is evenly graded, $MU_*(t) = 0$; so the cofibration sequence

$$S^0 \xrightarrow{i} \mathbb{C}P_{-1}^\infty \wedge S^2 \xrightarrow{c} \mathbb{C}P_0^\infty \wedge S^2$$

induces a long exact sequence in $E_2(-)$, with boundary homomorphism

$$t_*\colon\ E_2^s(\mathbb{C}P_0^\infty \wedge S^2) \to E_2^{s+1}(S^0).$$

In [9] it is shown that if x is a permanent cycle in the first spectral sequence, then t_*x represents tx in the second.

Let $u\colon\ \mathbb{C}P_{-1}^\infty \wedge S^2 \to H\mathbb{Q} = S\mathbb{Q}$ represent the Thom class $x_{H\mathbb{Q}}^{-1}$. We then have a map of cofibration sequences:

(3.8) $$\begin{array}{ccccc} S^0 & \xrightarrow{i} & \mathbb{C}P_{-1} \wedge S^2 & \xrightarrow{c} & \mathbb{C}P_0 \wedge S^2 \\ \downarrow = & & \downarrow u & & \downarrow \bar{u} \\ S^0 & \longrightarrow & S\mathbb{Q} & \longrightarrow & S\mathbb{Q}/\mathbb{Z} \end{array}.$$

This yields a factorization

$$\begin{array}{ccc} E_2^s(\mathbb{C}P_0 \wedge S^2) & \xrightarrow{t_*} & E_2^{s+1}(S^0) \\ & \bar{u}_* \searrow \quad \nearrow \partial & \\ & E_2^s(S\mathbb{Q}/\mathbb{Z}) & \end{array}$$

of t_*, in which ∂ is the boundary - homomorphism induced by the bottom sequence in (3.8).

According to the program of [13], it is via the map ∂ that elements in $E_2^s(S^0)$ are best described; so it is very natural to compute $u_*\colon\ \pi_*(\mathbb{C}P_{-1}^\infty \wedge S^2 \wedge MU) \to \mathbb{Q} \otimes MU_*$.

Since u factors as

$$\mathbb{C}P_{-1}^\infty \wedge S^2 \xrightarrow{x^{-1}} MU \xrightarrow{\rho} H\mathbb{Q},$$

the first step is to compute

$$\begin{aligned} x_*^{-1}\hat{\beta}(T) &= \mu\langle x^{-1}, \psi\hat{\beta}(T)\rangle && \text{by (3.2)} \\ &= \mu\langle x^{-1}, \hat{\beta}(1 \otimes b(T))\rangle && \text{by (3.3)} \\ &= \mu(1 \otimes b(T)^{-1}) \\ &= b(T)^{-1}. \end{aligned}$$

Now using (3.7), we find:

THEOREM 3.9. $u_* T\hat{\beta}(T) = \dfrac{T}{\log_G(T)} \in \mathbb{Q} \otimes MU_*[[T]].$

Since $u_*\beta_{-1} = 1$, we have also:

COROLLARY 3.10. $\bar{u}_* T\beta(T) = \dfrac{T}{\log_G(T)} - 1 \in \mathbb{Q}/\mathbb{Z} \otimes MU_*[[T]].$ □

Section 4. THE IMAGE OF THE PRIMITIVES.

We begin by recalling the primitive generators in $\pi_*(\mathbb{C}P_0^\infty \wedge MU)$. Write $\exp(T)$ for $\exp_G(T)$.

PROPOSITION 4.1. (D. M. Segal [16]) If we define $p_n \in \pi_{2n}(\mathbb{C}P_0^\infty \wedge MU) \otimes \mathbb{Q}$ by means of the expansion

$$\beta(\exp(T)) = \sum_{n=0}^{\infty} \frac{p_n}{n!} T^n,$$

then p_n lies in $\pi_{2n}(\mathbb{C}P_0^\infty \wedge MU)$ and generates the subgroup of primitives.

PROOF. We first check that p_n is primitive, by means of the following calculation (due in different guise to Segal).

$$\begin{aligned} \psi\beta(\exp(T)) &= \beta(1 \otimes b(\exp_R(T))) && \text{by (3.3)} \\ &= \beta(1 \otimes \exp_L(T)) && \text{by (3.6)} \\ &= \beta(\exp(T)) \otimes 1. \end{aligned}$$

Next, note that $n!$ times the coefficient of T^n in $(\exp(T))^k$ is integral. It follows that p_n is integral. On the other hand, the coefficient $n!b_{n-1}$ of β_1 in p_n generates a summand in MU_*; this follows from the fact that its image in $\mathbb{Z}$ under the map classifying G_m is -1. Therefore p_n is a generator of $E_2^{0,2n}(\mathbb{C}P_0^\infty)$. But this group embeds into $E_2^{0,2n}(\mathbb{C}P_0^\infty \wedge S\mathbb{Q})$ since $\mathbb{C}P_0^\infty$ is torsion-free, and the latter group is just $\pi_{2n}(\mathbb{C}P_0^\infty \wedge S\mathbb{Q}) = \mathbb{Q}$ since $\mathbb{C}P_0^\infty \wedge S\mathbb{Q}$ is an MU-module-spectrum. □

REMARK 4.2. Since $\exp(x_{MU\mathbb{Q}}) \in MU\mathbb{Q}^2(\mathbb{C}P_0^\infty)$ reduces to $x_{H\mathbb{Q}} \in H\mathbb{Q}^2(\mathbb{C}P_0^\infty)$, we find that p_n reduces to $n!\beta_n^H \in \pi_{2n}(\mathbb{C}P_0^\infty \wedge H)$. In the H-structure of $\mathbb{C}P_0^\infty$, $n!\beta_n^H = (\beta_1^H)^n$; and it follows that

$$p_n = \beta_1^n.$$

That is,

$$\beta(\exp(T)) = e^{\beta_1 T}.$$

Therefore, incidentally, $\beta(\exp(S))\beta(\exp(T)) = \beta(\exp(S+T))$, and, replacing S and T by $\log(S)$ and $\log(T)$, we obtain the formula of Ravenel and Wilson [15]:

(4.3) $$\beta(S)\beta(T) = \beta(G(S,T)).$$

This line of argument may of course be reversed.

REMARK 4.4. Analogous primitive generators may be constructed for $\mathbb{H}P^\infty$. The power series $\frac{1}{2}\exp(T)\exp(-T)$ is even, so we may define, following [16], $q_n \in \pi_{4n}(\mathbb{H}P_0^\infty \wedge MU) \otimes \mathbb{Q}$ by

$$\frac{1}{2}\gamma_0 + \frac{1}{2}\gamma(\exp(T)\exp(-t)) = \sum_{n\geq 0} \frac{q_n}{(2n)!} T^{2n}.$$

An analogue of the above proof shows that q_n is integral and a primitive generator.

We now evaluate

$$\bar{u}_*: E_2^0(\mathbb{C}P_0^\infty \wedge S^2) \to E_2^2(S\mathbb{Q}/\mathbb{Z})$$

in terms of the Bernoulli numbers introduced in Section 1. Since MU_* is the Lazard ring, the universal Bernoulli number B_n lies in $MU_{2n} \otimes \mathbb{Q}$.

THEOREM 4.5.

$$\bar{u}_* p_n = -\frac{B_{n+1}}{n+1}.$$

PROOF. Replacing T by $\exp(T)$ in Corollary 3.10,

$$\bar{u}_*\beta(\exp(T)) = \frac{1}{T} - \frac{1}{\exp(T)}.$$

The result follows upon expanding both sides. □

This together with Theorem 1.3 implies that $\bar{u}_* p_n$ has order d_n. On the other hand, recall from [13] that (if $\|$ means divides exactly)

$$E_2^{0,0}(S\mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$$

$$\begin{aligned} E_2^{0,2(p-1)u}(S\mathbb{Q}/\mathbb{Z}) \otimes \mathbb{Z}_{(p)} &= \mathbb{Z}/p^{n+1} && \text{if } p^n \| u > 0,\ p > 2 \\ &= \mathbb{Z}/2 && \text{if } 2 \nmid u > 0,\ p = 2 \\ &= \mathbb{Z}/4 && \text{if } u = 2,\ p = 2 \end{aligned}$$

$$= \mathbb{Z}/2^{n+2} \quad \text{if} \quad 2^n \,\|\, u > 2,\ n > 0, \quad p = 2.$$

Furthermore, $\partial: E_2^0(S\mathbb{Q}/\mathbb{Z}) \to E_2^1(S^0)$ merely kills the $\mathbb{Q}/\mathbb{Z}$. Comparing these orders with the numbers d_n, we find

THEOREM 4.6. The image of $t_*: E_2^{0,2u}(\mathbb{C}P_0^\infty \wedge S^2) \to E_2^{1,2u}(S^0)$ is the subgroup of index 2 except when $u = 1$ or 2, when the map is surjective. □

From [13] we then easily recover the theorem of Becker and Schultz:

THEOREM 4.7. [4] For $k > 0$, neither μ_{8k+1} nor the generator j_{8k-1} of the image of the J-homomorphism in dimension $8k - 1$ lies in the image of $t: \pi_*(\mathbb{C}P_0^\infty \wedge S^1) \to \pi_*(S^0)$. □

REMARK 4.8. [4] Since $2j_{8k-1}$ is in the image of the usual complex J-homomorphism, it does lie in $\mathrm{Im}(t)$ by the result (Theorem 2.5 above) of Becker and Schultz.

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Added in proof: Many of these results occur explicitly in K. Knapp's Bonn Habilitationsschrift, "Some applications of K-theory to framed bordism: e-invariant and transfer," Bonner Mathematishe Schriften, Heft 118, 1979. For instance, Lemma 2.7 occurs there as Theorem 2.9, and Corollary 3.10 occurs there as (5.21).

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON 98195