SHEAVES, GRADINGS, AND THE EXACT FUNCTOR THEOREM

HAYNES MILLER

1. Introduction

In [4] Pierre Conner and Edwin Floyd proved that the Todd orientation $MU \to K$ induces an isomorphism

$$K_* \otimes_{MU_*} MU_*(X) \to K_*(X).$$

K homology is thus algebraically determined from complex cobordism as an MU_* module. Their proof used the fact that $K_*(-)$ is already known to be a homology theory, but the theorem raised the question of what it was about K_* as an MU_* algebra which makes $X \mapsto K_* \otimes_{MU_*} MU_*(-)$ a homology theory. This question was answered by Peter Landweber [9], who gave conditions on an MU_* module R which are necessary and sufficient to make $M \mapsto R \otimes_{MU_*} M$ an exact functor on the category of MU_*MU comodules. This has the effect of creating new spectra from the multiplicative structure of the spectrum MU. Among the most important spectra produced in this way are the spectra representing Elliptic cohomology theories.

Landweber's proof is based on ingenious use of the theory of primary decomposition of modules. It is natural to hope for a proof using more standard localization methods, and incorporating Lazard's classification of formal groups over separably closed fields in a more direct way. In the spring of 1999, Mike Hopkins gave a course at MIT which contained such a proof of the most useful cases of Landweber's theorem, expressed in the language of stacks. The relies on ideas of Jack Morava (as does all work in this area) and of Neil Strickland.

Our purpose in this note is to free the proof of exotic language, But we precede this very direct proof with musings on several aspects of the theory of formal groups which are motivated by the stack perspective. It is common to employ a setting in which formal groups may be considered without specifying a parameter or coordinate. The resulting category is however equivalent to the category of formal group laws, so the gain is merely notational. The descent conditions associated to stacks leads to a true generalization, however, and we describe that here.

In formulations originating with Morava and Hopkins, one begins with an ungraded formal group law over a ring R and defines a graded formal group law over $R[u^{\pm 1}]$ by conjugating with the parameter u (which is to be regarded as a generator for the bundle of invariant differentials). Using objects which are given by a formal group law only locally in the flat topology allows us to extend these ungraded methods to handle spectra whose coefficient ring is not periodic.

We also describe the construction of the symmetric monoidal category of MU_*MU comodules in the ungraded setting in which we choose to work. This leads to the notion of a "spin formal group," which is a formal group together with a choice of square root of its line bundle of invariant differentials.

The last half of the paper describes the proof of a version of the exact functor theorem. The proof is based directly on the theorem of Lazard (as modified by Strickland) classifying formal groups in terms of their height.

It is a pleasure to thank the residents of the Homotopy Hut at the Institute for Advanced Study for their interest in this project.

2. Formal group laws and comodules

We will adopt, at the start, the most pedestrian attitude towards formal group laws. For more detail see [5], [1].

Definition 2.1. A *formal group law* over a ring R is a formal power series $F(x, y) \in R[[x, y]]$ such that

$$F(x, 0) = x\,, \qquad F(0, y) = y,$$

$$F(x, F(y, z)) = F(F(x, y), z)\,, \qquad F(x, y) = F(y, x)$$

A *homomorphism* $F \to G$ of formal groups over R is a formal power series $\theta(x) \in R[[x]]$ such that $\theta(0) = 0$ and

$$\theta(F(x, y)) = G(\theta(x), \theta(y))$$

An easy induction shows that there is automatically an "inverse," a power series $[-1](x)$ such that $F(x, [-1](x)) = x = F([-1](x), x)$. Formal groups laws over R form a pre-additive category, in which composition of morphisms is given by composition of formal power series, and the sum of $\alpha, \beta : F \to G$ is $G(\alpha(x), \beta(x))$. The set of endomorphisms of a formal group thus forms a ring, and the natural map $\mathbb{Z} \to \mathrm{End}_R(F)$ is written $n \mapsto [n]$.

Standard examples include the *additive* formal group law $G_a(x,y) = x+y$ and the *multiplicative* formal group law $G_m(x,y) = x+y-xy$.

Let $\mathcal{F}_R$ denote the category of formal group laws over R and their *isomorphisms*. These are the homomorphisms $\theta(x)$ for which $\theta'(0) \in R^\times$. Given a formal group law $F(x,y)$ and an invertible power series $\theta(x)$, the target of $\theta(x)$ as a homomorphism from F is uniquely determined; it is

$$^{\theta}F(x,y) = \theta(F(\theta^{-1}(x), \theta^{-1}(y))).$$

A ring homomorphism $f : R \to S$ determines a functor which I will also denote by f, $f : \mathcal{F}_R \to \mathcal{F}_S$, by applying f to all coefficients.

This groupoid-valued functor of rings is representable by a "Hopf algebroid" [8]. The functor of objects is represented by the "universal formal group law" $G_u(x,y)$ over the *Lazard ring* L. The universal morphism is represented by the power series

$$\theta_u(x) = \sum_{i=0}^{\infty} b_i x^{i+1}$$

over

$$W = L[b_0^{\pm 1}, b_1, b_2, \ldots].$$

In this expression, the L algebra structure is provided by the map classifying the source. The various structure maps defining the groupoid $\mathcal{F}_R$ are represented by ring homomorphisms

$$\begin{aligned} \eta_L : L \longrightarrow W \longleftarrow L : \eta_R\,, &\qquad \text{source, target} \\ \Delta : W \longrightarrow W \otimes_L W\,, &\qquad \text{composition} \\ \epsilon : W \longrightarrow L\,, \quad c : W \longrightarrow W\,, &\qquad \text{identity, inverse} \end{aligned}$$

Definition 2.2. An *even cobordism comodule* is a comodule for the Hopf algebroid (L, W).

This means we have an L module M together with an L module map $\psi : M \to W \otimes_L M$ which is unital and associative. Even cobordism comodules form an abelian category. Here are two examples:

Example 2.3. $L[2n]$ is the L module L with coaction determined by $\psi(1) = b_0^n \otimes 1$.

Example 2.4. W is a comodule with the coaction $\Delta : W \to W \otimes_L W$.

The various categories $\mathcal{F}_R$ come together to form the category $\mathcal{F}$ of formal group laws, in which an object is a pair F/R consisting of a ring R and a formal group law over it, and a morphism from F/R to G/S is a ring homomorphism $f : R \to S$ together with an isomorphism of formal group laws $\theta : fF \to G$ over S.

An even cobordism comodule M defines a functor on this category, which we will denote by $\widetilde{M}$. Its value on F/R is $R \otimes_L M$, and its value on the morphism $(f, \theta) : F/R \to G/S$ is given by the composite

$$(f, \theta)_* : R \otimes_L M \xrightarrow{1 \otimes \psi} R \otimes_L (W \otimes_L M) = (R \otimes_L W) \otimes_L M \xrightarrow{(f, \theta_*) \otimes 1} S \otimes_L M,$$

where (f, θ_*) is the ring homomorphism $R \otimes_L W \to S$ induced by $f : R \to S$ and the ring homomorphism $\theta_* : W \to S$ inducing θ.

Definition 2.5. A formal group law F/R is *Landweber exact* if $M \mapsto R \otimes_L M = \widetilde{M}(F/R)$ is an exact functor on the category of even cobordism comodules.

The following lemma will be important.

Lemma 2.6. (1) The map $\eta_R : L \to R \otimes_L W$ is a natural transformation from the constant functor on $\mathcal{F}$ with value L to the functor $\widetilde{W}$.
(2) The map $R \otimes_L W \to S \otimes_L W$ induced by $(f, \theta) : F/R \to G/S$ is flat if f is and is faithfully flat if f is.

Proof. The first statement comes from the fact that Δ is a map of right L modules. For the second, notice that $(f, \theta)_*$ can be obtained as the composite

$$R \otimes_L W \xrightarrow{f \otimes 1} S \otimes_L W \xrightarrow{(1, \theta)_*} S \otimes_L W$$

and that this is (faithfully) flat if f is, since W is faithfully flat (in fact free) as a left L module and $(1, \theta)_*$ is an isomorphism (with inverse $(1, \theta_*^{-1})$. ∎

The condition of Landweber exactness can be expressed in terms of a property of this natural transformation evaluated at the even cobordism comodule W:

Lemma 2.7 (G. Laures)**.** The formal group law F/R is Landweber exact if and only if the map $\eta_R : L \to R \otimes_L W$ is flat.

Proof. The maps ψ and ϵ render the functor $M \mapsto R \otimes_L M$ a retract of the functor $M \mapsto R \otimes_L (W \otimes_L M) = (M \otimes_L W) \otimes_L M$, which is exact when η_R is flat.

Conversely, let $A \to B$ be a monomorphism of L modules. The inversion isomorphism $c : W \to W$ swaps η_R and η_L. W is visibly flat over L under η_L, so it is also flat over L under η_R. Thus $W \otimes_L A \to W \otimes_L B$ is a monomorphism of even cobordism comodules. If F/R is Landweber exact, it follows that $R \otimes_L (W \otimes_L A) \to R \otimes_L (W \otimes_L B)$ is a monomorphism, whence $\eta_R : L \to R \otimes_L W$ is flat. ∎

3. Even cobordism sheaves

Definition 3.1. An *even cobordism sheaf* is a functor $\mathcal{M} : \mathcal{F} \to \mathbf{Ab}$ together with an R module structure on the abelian group $\mathcal{M}(F/R)$ for each $F/R \in \mathcal{F}$, with the property that the homomorphism $(f,\theta)_* : \mathcal{M}(F/R) \to \mathcal{M}(G/S)$ induced by $(f,\theta) : F/R \to G/S$ is R linear and induces an isomorphism

$$S \otimes_R \mathcal{M}(F/R) \xrightarrow{\cong} \mathcal{M}(G/S).$$

Even cobordism sheaves form a category in which morphisms are natural transformations which are R linear when evaluated at any object F/R.

We have constructed a functor from even cobordism comodules to even cobordism sheaves, sending M to $\widetilde{M}$.

Proposition 3.2. The functor $M \mapsto \widetilde{M}$ is an equivalence of categories.

Proof. If $\mathcal{M}$ is an even cobordism sheaf, let $M = \mathcal{M}(G_u/L)$. This is an L module, and we proceed to define on it the structure of an even cobordism comodule. There are canonical morphisms in $\mathcal{F}$,

$$(\eta_L, \theta_u) : G_u/L \longrightarrow \eta_R G_u/W \longleftarrow G_u/L : (\eta_R, 1),$$

inducing the top maps in the diagram

$$\begin{tikzcd}
\mathcal{M}(G_u/L) \arrow[r] \arrow[dr, "\psi"] & \mathcal{M}(\eta_R G_u/W) & \mathcal{M}(G_u/L) \arrow[l] \arrow[dl] \\
& W \otimes_L \mathcal{M}(G_u/L) \arrow[u, "\cong"] &
\end{tikzcd}$$

Here the right diagonal is the inclusion of $\mathcal{M}(G_u/L)$ into its extension to a W module, and the vertical map is the extension of $(\eta_R, 1)_*$ to a W module map. We receive the map ψ since the vertical map is an isomorphism. We refer to [6] for the check that this defines the structure of an even cobordism comodule on M, and that this defines a quasi-inverse to $M \mapsto \widetilde{M}$. ∎

Example 3.3. We denote by ω^n the even cobordism sheaf corresponding to the even cobordism comodule $L[2n]$.

4. Formal groups

To any evenly graded commutative and associative cohomology theory $E^*(-)$ we may associate a formal group law over the graded ring $E^*(\mathbb{C}P^\infty)$ by means of the map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifying the tensor product of line bundles, after a choice of "complex orientation" $x \in E^2(\mathbb{C}P^\infty)$. One can carry out this theory by various means without choosing such an orientation, but from a certain point of view these devices are redundant, since it is easy to see that the formal group law is *canonical*, in the following sense. There is a functor from the category whose objects are complex orientation of $E^*(-)$, with exactly one morphism between any two objects, to the category $\mathcal{F}_{E_*}$; that is, the formal group laws associated to the various complex orientations are *canonically and compatibly isomorphic*. This is as well defined as things get in mathematics.

On the other hand, there are genuine extensions of the notion of a formal group law, which are well understood in the context of algebraic geometry and which do in fact occur in topology. These objects, which we will call *formal groups*, admit an intrinsic description, and constitute the objects of a "stack," but we content ourselves here with a somewhat ad hoc definition of the objects.

To make this definition, we begin by defining the *descent Hopf algebroid* associated to any ring homomorphism $R \to S$ as $(S, S \otimes_R S)$. The structure maps are forced by $\eta_L s = s \otimes 1$ and $\eta_R s = 1 \otimes s$. The structure map for a comodule M over this Hopf algebroid is an R module map $\psi : M \to S \otimes_R M$ which is unital and associative in the sense that

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[dr, "1"'] & S \otimes_R M \arrow[d, "\varphi"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\psi"] & S \otimes_R M \arrow[d, "\delta\otimes 1"] \\ S \otimes_R M \arrow[r, "1\otimes\psi"] & S \otimes_R S \otimes_R M \end{tikzcd}$$

where φ is the S module structure and $\delta s = s \otimes 1$.

Define the *primitives* of the $(S, S \otimes_R S)$ comodule M to be

$$\mathrm{Prim}(M) = \{x \in M : \psi x = 1 \otimes x\}.$$

The following fact constitutes "faithfully flat descent for modules."

Lemma 4.1. The primitives give a functor from $(S, S \otimes_R S)$ comodules to R modules which is an exact equivalence if $R \to S$ is faithfully flat.

Proof. If $x \in M$ is primitive and $r \in R$ then rx is again primitive since $\psi(rx) = r\psi x = r(1 \otimes x) = 1 \otimes rx$ because the tensor product is over R. For a quasi-inverse, let N be an R module and define an $S \otimes_R S$ comodule structure on the S module $S \otimes_R M$ by $\psi(s \otimes x) = s \otimes 1 \otimes x$. We leave as an exercise the standard check (due to Grothendieck) that if $R \to S$ is faithfully flat then this is a quasi-inverse. ∎

Definition 4.2. A *formal group* over a ring R is a faithfully flat map $R \to S$ together with a map of Hopf algebroids $(L, W) \to (S, S \otimes_R S)$.

A formal group over R is thus a formal group law F over some faithfully flat extension S of R, together with an isomorphism of formal group laws $\theta : \eta_L F \to \eta_R F$ over $S \otimes_R S$ which satisfies appropriate cocycle conditions.

A formal group law over R determines a formal group over R, by taking $S = R$ and $\theta(x) = x$.

If $G = (S, F, \theta)$ is a formal group over R and $f : R \to T$ is a ring homomorphism, $T \to S \otimes_R T$ is faithfully flat and $fG = (S \otimes_R T, fF, (f \otimes f)\theta)$ is a formal group over T. This gives a functor f from formal groups over R to formal groups over T.

If $G = (S, F, \theta)$ is a formal group over R and M is an even cobordism comodule, then $S \otimes_L M$ is a comodule over $(S, S \otimes_R S)$ and we may define the R module

$$\widetilde{M}(S, F, \theta) = \mathrm{Prim}(S \otimes_L M).$$

We leave it as an exercise to check that this definition extends the definition on formal group laws, and continues to satisfy the sheaf condition

$$T \otimes_R \widetilde{M}(G/R) \xrightarrow{\cong} \widetilde{M}(fG/T).$$

We are concerned with the property of Landweber exactness. This condition extends verbatim to apply to a formal group. The sheaf condition shows that if $f : R \to T$ is faithfully flat, then G/R is Landweber exact if and only if fG/T is; this follows from the exactness clause in Lemma 4.1.

5. Line bundles and invertible modules

We recall another example in which the descent problem can be solved explicitly. The relevant Hopf algebroid is $(\mathbb{Z}, \mathbb{Z}[t^{\pm 1}])$ with $\Delta t = t \otimes t$.

Definition 5.1. A *line bundle* over a ring R is a faithfully flat extension $R \to S$ together with a map $(\mathbb{Z}, \mathbb{Z}[t^{\pm 1}]) \to (S, S \otimes_R S)$ of Hopf algebroids.

A line bundle is thus a faithfully flat extension S of R together with a unit (the image of t) $c = \sum c'_i \otimes c''_i \in (S \otimes_R S)^\times$ which satisfies the "cocycle conditions"

$$\sum c'_i c''_i = 1\,, \qquad \sum c'_i \otimes c'_k c''_i \otimes c''_k = \sum c'_j \otimes 1 \otimes c''_j .$$

Line bundles of the form (S, c) are said to *split* over S.

A line bundle (S, c) determines an $(S, S \otimes_R S)$ comodule structure on S, $\psi_c : S \to (S \otimes_R S) \otimes_S S \cong S \otimes_R S$, given by the left S module map sending 1 to c: $x \mapsto (x \otimes 1)c$. Write $S[c]$ for S with this comodule structure. Write $L(c)$ for the R module of primitives in this comodule,

$$L(c) = \mathrm{Prim}(S[c]) = \{s \in S : (s \otimes 1)c = 1 \otimes s\}.$$

This is an R submodule of S with the property that the inclusion extends to an isomorphism $S \otimes_R \mathrm{Prim}(S[1]) \xrightarrow{\cong} S$.

In order to recognize what this is, we recall that if $\mathfrak{p}$ is a prime ideal in A then we may localize R at $\mathfrak{p}$ by inverting elements not in $\mathfrak{p}$, to obtain a local ring $R_\mathfrak{p}$. Denote the residue field by $k(\mathfrak{p})$. We may localize M to obtain an $R_\mathfrak{p}$-module $M_\mathfrak{p}$, and hence a vector space $M_\mathfrak{p}/\mathfrak{p}$ over $k(\mathfrak{p})$. The *rank* of M at $\mathfrak{p}$ is the dimension of $M_\mathfrak{p}/\mathfrak{p}$ over $k(\mathfrak{p})$. The rank is a locally constant function from $\mathrm{Spec}R$ to cardinals.

A projective P of constant rank n is said to be *split* over the faithfully flat R algebra S if $S \otimes_R P$ is free of rank n over S.

Definition 5.2. An R module M is *invertible* if it is projective of constant rank 1.

We adopt this definition despite the fact that it is subject to the following criticism. It would be more natural to say that an R module P is "invertible" if the natural map $P \otimes_R \mathrm{Hom}_R(P, R) \to R$ is an isomorphism. This is the case if P is projective of rank 1, but for the converse one must apparently assume in advance that P is finitely generated ([2] II §5.4 Theorem 3, p. 114). In our context it is natural because it is local in the flat topology:

Theorem 5.3 ([2] II. §5 Ex. 8, pp. 147 f)**.** A module P over R is projective of constant rank n if and only if there is a faithfully flat map $R \to S$ such that $S \otimes_R P$ is free of rank n over S. Such a module is automatically finitely generated. ∎

We conclude that $L(c)$ is an invertible R module, and that for any invertible R module P there is a faithfully flat extension $R \to S$ and an R module inclusion $P \hookrightarrow S$ which extends to an isomorphism $S \otimes_R$

$P \xrightarrow{\cong} S$. Let $c = \sum c_i' \otimes c_i'' \mapsto 1$ under this isomorphism, and regard $c_i'' \in S$. Then $c \in (S \otimes_R S)^\times$ is a cocycle, and $P \cong L(c)$ as R modules.

It is easy to check that if b and c are two units in $S \otimes_S R$ satisfying the cocycle conditions, then bc and b^{-1} satisfy them as well, and

$$L(bc) = L(b) \otimes_R L(c)\,, \qquad L(1) = R.$$

$L(c)$ is free of rank one if and only if $c = a \otimes a^{-1}$ for some unit $a \in S^\times$.

6. Gradings and spin formal groups

Even cobordism comodules admit a natural grading, via the following observation.

Lemma 6.1. The R module underlying an $(R, R[t^{\pm 1}])$ comodule M splits naturally as

$$M = \bigoplus_{n \in \mathbb{Z}} M_n\,, \quad M_n = \{x \in M : \psi x = t^n \otimes x\}.$$

This assignment constitutes a natural equivalence from the category of $(R, R[t^{\pm 1}])$ comodules to the category of $\mathbb{Z}$ graded R-modules.

Proof. If M is a comodule, the equation

$$\psi x = \sum t^i \otimes \pi_i x$$

determines R linear operators π_i on M. Associativity forces these to be commuting idempotents, and unitality forces them to sum to the identity on M. We let $M_i = \pi_i M$. Conversely, a grading determines projection operators π_i, which give a coaction by the same formula. ∎

The category of comodules over any Hopf algebroid (A, V) admits a symmetric monoidal product defined by forming the tensor product as left A modules. In working with Hopf algebroids, the notation $- \otimes_A -$ invariably refers to a right A action on the left variable, so we write the tensor product of left A modules using the nonstandard notation $- \otimes^A -$. Then the coaction on $M \otimes^A N$ is given by

$$\psi(x \otimes y) = \sum x_i' y_j' \otimes x_i'' \otimes y_j''.$$

With $(A, V) = (\mathbb{Z}, \mathbb{Z}[t^{\pm 1}])$, this gives the usual graded tensor product, with the signless symmetric monoidal structure.

Define a Hopf algebroid map $(L, W) \to (\mathbb{Z}, \mathbb{Z}[t^{\pm 1}])$ by classifying the additive formal group law $G_a(x, y) = x + y$ over $\mathbb{Z}$ and the automorphism $t^2 x : G_a \to G_a$. Any even cobordism comodule M thereby admits a natural even grading as an abelian group, with

$$M_{2n} = \{x \in M : \psi x = b_0^n \otimes x + \cdots\}.$$

In particular, L itself, being a comodule, admits an even grading, and the L module structure on M respects that grading.

In the context of topology, L with its even grading is the complex bordism ring MU_*. Cobordism comodules are supposed to capture $MU_*(X)$, but *even* cobordism comodules fail to do so because $MU_*(X)$ may not be evenly graded. The easy out is to consider the even part and odd part as independent evenly cobordism comodules. This fails to account adequately for the symmetric monoidal structure, however. The remedy is to work over a larger Hopf algebroid. Let

$$W^s = L[e_0^{\pm 1}, b_1, b_2, \ldots],$$

and regard W as embedded in W^s by sending b_0 to e_0^2 and b_i to b_i for $i > 0$. Define a Hopf algebroid structure on (L, W^s) so that (L, W) is a sub Hopf algebroid and

$$\Delta e_0 = e_0 \otimes e_0.$$

We have thus adjoined a square root of b_0.

Definition 6.2. A *cobordism comodule* is a comodule over (L, W^s).

The Hopf algebroid map $(L, W^s) \to (\mathbb{Z}, \mathbb{Z}[t^{\pm 1}])$ classifying the additive formal group law and sending $e_0 \mapsto t$ and $b_i \mapsto 0$ for $i > 0$ endows cobordism comodules with full $\mathbb{Z}$ gradings, and the category of cobordism comodules is naturally equivalent to the category of (graded) (MU_*, MU_*MU) comodules.

To define the correct symmetric monoidal structure on the category of cobordism comodules, we may use the theory of cobraidings of Hopf algebras as described in [7], extended to the case of Hopf algebroids. Rather than elaborate this theory in generality here, we merely point out that there is unique left L bilinear pairing

$$\langle -, - \rangle : W^s \times W^s \longrightarrow L$$

such that

$$\langle xy, z\rangle = \sum \langle x, z_i'\rangle\langle y, z_i''\rangle\,, \qquad \langle x, yz\rangle = \sum \langle x_i', y\rangle\langle x_i'', z\rangle$$

and

$$\langle e_0, e_0\rangle = -1\,, \qquad \langle e_0, b_i\rangle = \langle b_i, b_j\rangle = 0.$$

The symmetry is then given on $M \otimes^L N$ by

$$c(x \otimes y) = \sum \langle x_i', y_j'\rangle y_j'' \otimes x_i''.$$

The Hopf algebroid (L, W^s) defines a stack just as the Hopf algebroid (L, W) defined the stack of formal groups.

Definition 6.3. A *spin formal group* over R is a faithfully flat map $R \to S$ together with a Hopf algebroid map $(L, W^s) \to (S, S \otimes_R S)$.

Thus a spin formal group over R consists in a formal group (S, F, θ) over R, together with a choice e of square root of $\theta'(0) \in (S \otimes_R S)^\times$.

The Hopf algebroid map $(\mathbb{Z}, \mathbb{Z}[t^{\pm 1}]) \to (L, W^s)$ sending t to e_0 associates to any spin formal group $G = (S, F, \theta, e)$ over R a line bundle over R. Write λ_G for the corresponding invertible module; thus $\lambda_G = L(e)$.

Only the square of this module, $\omega_G = \lambda_G^2$, is determined by the underlying formal group. A spin structure consists of a square root of the canonical line bundle ω_G attached to a formal group G. Of course, if G is a formal group *law*, then ω_G comes equipped with a trivialization and hence with a preferred square root.

A cobordism comodule M determines a *cobordism sheaf* $\widetilde{M}$, sending a spin formal group (S, F, θ, e) to the R module

$$\widetilde{M}(S, F, \theta, e) = \mathrm{Prim}(S \otimes_L M).$$

7. Laurent algebras

In the topological context, we have a *graded* formal group law in hand. We will now explain how such objects relate to formal groups as we have defined them. The graded rings involved will be of the following form.

Definition 7.1. A *Laurent algebra* over a ring R is an evenly graded R algebra E_* such that $E_0 = R$, E_2 is an invertible R module, and for every $m, n \in \mathbb{Z}$ the multiplication $E_{2m} \otimes_R E_{2n} \to E_{2(m+n)}$ is an isomorphism.

The invertible R module $\omega = E_2$ determines the Laurent algebra; E_{2n} is naturally isomorphic to ω^n for $n \in \mathbb{Z}$. We may write $R(\omega)$ for the Laurent algebra with ω in degree two.

Note that if $R \to S$ is a faithfully flat map spitting ω—so that there is an inclusion of R modules $\omega \hookrightarrow S$ which extends to an isomorphism $S \otimes_R \omega \xrightarrow{\cong} S$—then embedding ω^n into $S\langle u^n \rangle$ defines an injection of graded R algebras

$$R(\omega) \hookrightarrow S[u^{\pm 1}], \qquad |u| = 2.$$

The fact that this is an embedding implies that $R(\omega)$ is commutative. Since $S \otimes_R \omega \xrightarrow{\cong} S$, $S \otimes_R R(\omega) \xrightarrow{\cong} S[u^{\pm 1}]$, and from the fact that $R \to S$ if faithfully flat it follows that $R(\omega) \to S[u^{\pm 1}]$ is faithfully flat as well.

If $f : R \to S$ is a ring homomorphism and ω is an invertible R module, then $S \otimes_R \omega$ is an invertible S module which we will write $f\omega$. There are natural isomorphisms $(f\omega)^n \cong f(\omega^n)$, and f induces a graded R algebra map $R(\omega) \to S(f\omega)$ with the property that its S module extension $S \otimes_R R(\omega) \to S(f\omega)$ is an isomorphism.

The formalism of comodules over $(\mathbb{Z}, \mathbb{Z}[t^{\pm 1}])$ provides a systematic way to grade objects. Thus a graded ring is a comodule algebra over this Hopf algebra. It is evenly graded if in fact it is a comodule over the sub Hopf algebra $(\mathbb{Z}, \mathbb{Z}[t^{\pm 2}])$. A formal group law $F(x,y) = \sum a_{i,j} x^i y^j$ over the evenly graded ring E_* is graded if $a_{i,j} \in E_{2(i+j-1)}$.

A graded formal group law $F(x,y)$ over a Laurent algebra $R(\omega)$ determines a formal group over R in the following way. Let S split ω and let $\omega = L(c)$ be determined by the cocycle $c \in (S \otimes_R S)^\times$. Since $\omega^n \subseteq S$ for all n, we may regard the formal power series $F(x,y)$ as defined over S, and as such it defines a formal group law over S. The fact that $a_{i,j} \in \omega^{i+j-1}$ means that

$$(a_{i,j} \otimes 1) c^{i+j-1} = 1 \otimes a_{i,j},$$

and these together say that

$$\eta_L F(cx, cy) = c \eta_R F(x,y).$$

That is to say, (S, F, cx) is a formal group over R, with canonical invertible module given by ω. A different choice of cocycle c gives a canonically isomorphic formal group.

Lemma 7.2. The formal group (S, F, cx) is Landweber exact if and only if the formal group law $F/R(\omega)$ is Landweber exact.

Proof. F is Landweber exact over $R(\omega)$ if and only if it is Landweber exact over $S[u^{\pm 1}]$, since $R(\omega) \hookrightarrow S[u^{\pm 1}]$ is faithfully flat. Over $S[u^{\pm 1}]$, F is isomorphic to F^{cx}, which is the image under the faithfully flat map $R \to S$ of the formal group (S, F, cx), so they are Landweber exact together by Lemma 4.1. ∎

8. Scales

We will now set up some general ideas about sequences of ideals of the type occuring in the statement of Landweber's theorem.

Definition 8.1. An increasing sequence $0 = I_0 \subseteq I_1 \subseteq \cdots$ of ideals in a ring R is a *scale* if there exist elements $v_0, v_1, \ldots$ in R such that for each $n \geq 0$,

$$I_{n+1} = I_n + R v_n.$$

Such a sequence of elements is a *defining sequence* for the scale.

Here are two easy lemmas.

Lemma 8.2. Let $v_0, v_1, \dots$ define a scale $0 = I_0 \subseteq I_1 \subseteq \cdots$ in R. A sequence $w_0, w_1, \dots$ of elements in R defines the same scale if and only if for each n there is an element $u \in R$ which is a unit modulo I_n and is such that $w_n \equiv uv_n \mathrm{mod}\ I_n$. ∎

Lemma 8.3. Suppose two sequences $v_0, v_1, \dots$ and $w_0, w_1, \dots$ define the same scale in R, and let M be an R-module. Then $v_n|(M/I_nM)$ is monic for all n if and only if $w_n|(M/I_nM)$ is monic for all n. ∎

We will say that a scale acts regularly on M in this case. If $M = R$ we will call the scale regular. We will also say that the scale acts *finitely* on M if I_nM becomes constant for large n, and that the scale itself is finite if it acts finitely on R.

Given a scale with defining sequence $v_0, v_1, \dots$, we have a diagram of R-modules

$$\begin{array}{ccccc} M & \xrightarrow{v_0} & M & \longrightarrow & v_0^{-1}M \\ & & \downarrow & & \\ M/I_1 & \xrightarrow{v_1} & M/I_1 & \longrightarrow & v_1^{-1}M/I_1 \\ & & \downarrow & & \\ M/I_2 & \xrightarrow{v_2} & M/I_2 & \longrightarrow & v_2^{-1}M/I_2 \\ & & \downarrow & & \\ & & \vdots & & \end{array}$$

in which the vertical maps are the cokernels of the left horizontals, and the right horizontal maps are the localization homomorphisms. This diagram is independent, up to isomorphism, of the choice of defining sequence. If the scale acts regularly on M then the left horizontal maps are monic, so we have short exact sequences

$$0 \longrightarrow M/I_{n-1} \xrightarrow{v_{n-1}} M/I_{n-1} \longrightarrow M/I_n \longrightarrow 0.$$

If the scale is finite on M, the modules in the diagram are eventually zero.

Lemma 8.4. If the scale $I_0 \subseteq I_1 \subseteq \cdots \subseteq R$ is regular then, for each $n \geq 0$, R/I_n has flat dimension at most n over R.

Proof. We use upward induction on n, starting with the fact that $R/I_0 = R$ is flat over R. The long exact sequence associated to

$$0 \longrightarrow R/I_{n-1} \xrightarrow{v_{n-1}} R/I_{n-1} \longrightarrow R/I_n \longrightarrow 0$$

reads in part

$$\mathrm{Tor}^R_{n+1}(R/I_{n-1}, -) \longrightarrow \mathrm{Tor}^R_{n+1}(R/I_n, -) \longrightarrow \mathrm{Tor}^R_n(R/I_{n-1}, -).$$

The inductive assumption is that the left and right terms are zero so the middle term is zero as well, establishing the induction. ∎

We will actually be interested in the localizations of these quotients,

$$M_n = v_n^{-1}M/I_n.$$

This is naturally a module over the ring $R_n = v_n^{-1}R/I_n$, in a way independent of choice of defining sequence. Since localization is exact $\mathrm{Tor}_k^R(R_n, -) = v_n^{-1}\mathrm{Tor}_k^R(R/I_n, -)$, so the flat dimension of R_n is at most n:

Corollary 8.5. If the scale $I_0 \subseteq I_1 \subseteq \cdots \subseteq R$ is regular then R_n has flat dimension at most n over R. ∎

Lemma 8.6. Assume that the scale $I_0 \subseteq I_1 \subseteq \cdots \subseteq R$ acts regularly and finitely on M. If in addition M_n has flat dimension at most n over R for all $n \geq 0$, then M is flat over R.

Proof. We prove that $\mathrm{Tor}_n^R(M/I_{n-1}, -) = 0$ by downward induction on $n \geq 1$. The induction ends with $\mathrm{Tor}_1^R(M, -) = 0$, which is equivalent to the flatness of M. The assumption $M/I_n = 0$ for large n grounds the induction, so suppose $\mathrm{Tor}_{n+1}^R(M/I_n, -) = 0$. The short exact sequence

$$0 \longrightarrow M/I_{n-1} \xrightarrow{v_{n-1}} M/I_{n-1} \longrightarrow M/I_n \longrightarrow 0.$$

leads to a long exact sequence in Tor^R, which reads in part

$$\mathrm{Tor}_{n+1}^R(M/I_n, -) \longrightarrow \mathrm{Tor}_n^R(M/I_{n-1}, -) \xrightarrow{v_{n-1}} \mathrm{Tor}_n^R(M/I_{n-1}, -).$$

By the inductive assumption the first term is zero, so v_{n-1} induces a monomorphism on $\mathrm{Tor}_n^R(M/I_{n-1}, -)$. On the other hand, localization is exact, so the assumption that M_{n-1} has flat dimension at most n implies that

$$v_{n-1}^{-1}\mathrm{Tor}_n^R(M/I_{n-1}, -) = \mathrm{Tor}_n^R(v_{n-1}^{-1}M/I_{n-1}, -) = 0.$$

Putting these two facts together gives us the next step in the induction: $\mathrm{Tor}_n^R(M/I_{n-1}, -) = 0$. ∎

The action of R on M_n factors through an action of R_n, so there is a spectral sequence ([3], XVI §5 $(2)_2$)

$$\mathrm{Tor}_p^{R_n}(M_n, \mathrm{Tor}_q^R(R_n, -)) \Longrightarrow \mathrm{Tor}_{p+q}^R(M_n, -).$$

By Lemma 8.5, the spectral sequence vanishes above the $(n+1)$st row if the scale is regular, so if M_n is flat over R_n then it has flat dimension at most n over R and we can apply Lemma 8.6 to obtain

Proposition 8.7. If a regular scale $I_0 \subseteq I_1 \subseteq \cdots \subseteq R$ acts regularly and finitely on the R-module M, and M_n is flat as an R_n-module for each $n \geq 0$, then M is flat as an R-module. ∎

9. Statement of the exact functor theorem

Let F/R be a formal group law. For a prime number p and an integer $n \geq 0$, define

$$I_{p,n} = I_{p,n}(F/R) = (a_0, \dots, a_{p^{n-1}-1})$$

to be the ideal generated by the coefficients of the p-series

$$[p](x) = \sum_{i=1}^{\infty} a_{i-1} x^i$$

through the coefficient of $x^{p^{n-1}}$. Then $I_{p,0} = 0$, and, since $a_0 = p$, $I_{p,1} = (p)$. In fact,

Lemma 9.1. For any prime p, $I_{p,0} \subseteq I_{p,1} \subseteq \cdots \subseteq R$ is a scale.

This is standard, but we pause to give the proof anyway. It is based on the following observation. Let

$$v(x) = x^p$$

and if R is a ring of characteristic p write $\phi : R \to R$ for the Frobenius, $a \mapsto a^p$. Then $v : \phi F \to F$ is a homomorphism.

Lemma 9.2. Let R be a ring of characteristic p, let F and G be formal group laws over R, and let $\theta : F \to G$ be a homomorphism. If $\theta'(0) = 0$ then there is a unique power series $\theta_1(x)$ such that

$$\begin{tikzcd} \phi F \arrow[rr, "\theta_1"] & & G \\ & F \arrow[ul, "\theta"] \arrow[ur, "v"'] & \end{tikzcd}$$

commutes.

Proof. Differentiate $\theta(F(x,y)) = G(\theta(x), \theta(y))$ with respect to y and set $y = 0$:

$$\theta'(F(x,0))F_2(x,0) = G_2(\theta(x),0)\theta'(0).$$

$F(x,0) = x$, and $F_2(x,0)$ is a unit in the power series ring, and by assumption $\theta'(0) = 0$, so $\theta'(x) = 0$. This implies that $\theta(x) = \theta_1(x^p)$ for an obviously unique power series $\theta_1(x)$, which is the result. ∎

Proof of Lemma 9.1. $a_0 = p$ so $I_1 = pR$. Apply Lemma 9.2 to the endomorphism $[p](x)$ of F over R/I_1. We learn that the coefficients of $[p](x) \in R[[x]]$ up to the coefficient of x^p lie in pR. Continue this argument by induction. ∎

In fact, it is easy to check that $I_{p,0} \subseteq I_{p,1} \subseteq \cdots \subseteq R$ is a *sheaf of scales*:

Lemma 9.3. (1) If $(f,\theta) : F/R \to G/S$, then

$$S \otimes_R I_{p,n}(F/R) \xrightarrow{\cong} I_{p,n}(G/S).$$

(2) For any formal group F over the $\mathbb{Z}_{(p)}$ algebra R, there is a natural isomorphism $R_n \otimes_L W \xrightarrow{\cong} (R \otimes_L W)_n$ such that the diagram

$$\begin{tikzcd} L \arrow[r, "\eta_R"] \arrow[d] & R_n \otimes_L W \arrow[d, "\cong"] \\ L_n \arrow[r, "(\eta_R)_n"] & (R \otimes_L W)_n \end{tikzcd}$$

commutes. ∎

Here is the main theorem, first proven by a different method in [9].

Theorem 9.4 (P. S. Landweber)**.** Let G/R be a formal group such that for every prime p the scale $I_{p,0}(G/R) \subseteq I_{p,1}(G/R) \subseteq \cdots$ is finite and regular. Then G/R is Landweber exact.

Example 9.5. The multiplicative group $G_m/\mathbb{Z}$ satisfies these hypotheses and hence is Landweber exact. First, p is a non zero divisor in $\mathbb{Z}$. Next, $[p](t) = 1 - (1-t)^p$, so the coefficient of t^p in $[p](t)$ is 1 and $I_{p,2}(G_m/\mathbb{Z}) = \mathbb{Z}$.

Remark 9.6. In fact Landweber proves something slightly different. Recall that an L-module M is *coherent* if it is finitely generated and every finitely generated submodule is finitely presented. Landweber restricts himself to the category of comodules which are coherent as L-modules, and shows that the functor $M \mapsto M \otimes_L R$ is exact on this category if and only if the height scale in R is regular for every prime. A result from [10] shows that every comodule is a union of coherent subcomodules, and it follows that one may dispense with the coherence condition here. The theorem we prove is thus less general; for us, the height scales have to be finite at every prime.

On the other hand, suppose that for some prime p the scale $I_{p,\bullet}(F/R)$ is not finite. Over $R_{(p)}$, F is isomorphic to a p-typical formal group law, which is represented by a map $BP_* \to R_{(p)}$.

Conjecture 9.7. If $I_{p,\bullet}(F/R)$ is not finite then $BP_* \to R_{(p)}$ is flat.

If this is the case, then F/R is necessarily Landweber exact at p, and Landweber's full assertion follows. In any case, the height scales are finite in the most important applications.

10. Proof of the exact functor theorem

The proof of Theorem 9.4 will use the criterion of Lemma 2.7: we must show that

$$\eta_R : L \to R \otimes_L W$$

is flat. We can proceed one prime at a time, by virtue of the following lemma, in which $A_{(p)}$ denotes the localization of an abelian group A at the prime number p: $A_{(p)} = A \otimes \mathbb{Z}_{(p)}$.

Lemma 10.1. Let R be a ring. An R module M is flat over R if and only if $M_{(p)}$ is flat over $R_{(p)}$ for all prime numbers p.

Proof. This is based on the observation that for any abelian group A the map

$$s : A \to \prod_p A_{(p)} ,$$

whose factors are the localization maps at the rational prime numbers, is injective. To see this, let $a \in \ker s$. Let $n_0 \in \mathbb{Z}$ generate the annihilator ideal of a. For each prime p, a dies in $A \otimes \mathbb{Z}_{(p)}$, so there exists an integer n_p which is not divisible by p and for which $n_p a = 0$. If $p | n_0$, then $p | n_p$ since $n_0 | n_p$. Thus n_0 is not divisible by any primes, and the conclusion is that $n_0 = \pm 1$ and so $a = 0$.

Now let $i : A \to C$ be a monomorphism of R-modules. Then $i_{(p)} : A_{(p)} \to C_{(p)}$ is again a monomorphism, since localization is exact, and hence $M_{(p)} \otimes_{R_{(p)}} i_{(p)} = (M \otimes_R i)_{(p)}$ is a monomorphism by hypothesis. Thus the bottom row of the diagram

$$\begin{tikzcd} M \otimes_R A \arrow[r] \arrow[d] & M \otimes_R C \arrow[d] \\ \prod_p (M \otimes_R A)_{(p)} \arrow[r] & \prod_p (M \otimes_R C)_{(p)} \end{tikzcd}$$

is a monomorphism, since in the category of abelian groups a product of monomorphisms is a monomorphism. We just checked that the vertical maps are too, so the top horizontal map is monomorphic, and this shows that M is flat over R. ∎

To show that $\eta_R : L \to R \otimes_L W$ is flat it will thus suffice to show that $\eta_{R\,(p)}$ is flat for each prime number p. Since $(R \otimes_L W)_{(p)} = R_{(p)} \otimes_L W$, we can restrict out attention to formal group laws over $\mathbb{Z}_{(p)}$ algebras.

The ingredients necessary to apply Proposition 8.7, using the height scale at p, are contained in Proposition 10.2 below. To lighten notation, we will tacitly replace L by $L_{(p)}$. This leaves the meaning of L_n unchanged except when $n = 0$, but by L_0 we will intend $p^{-1}L_{(p)} = L \otimes \mathbb{Q}$ rather than $p^{-1}L$. We will suppress the prime p from notation for the height scale at p. What we need to know is:

Proposition 10.2. For any formal group law F over a $\mathbb{Z}_{(p)}$ algebra R, the map $(\eta_R)_n : L_n \to (R \otimes_L W)_n$ is flat for all $n \geq 0$.

The ring homomorphism $L \to L_n$ is universal among maps classifying formal groups over $\mathbb{Z}_{(p)}$ algebras which are of "strict height n."

Definition 10.3. A formal group F over the $\mathbb{Z}_{(p)}$ algebra R is of *strict height* n provided that its p series has the form

$$[p](x) = a_{p^n-1}x^{p^n} + \cdots, \qquad a_{p^n-1} \in R^{\times}.$$

Let $\mathcal{F}^n(S)$ denote the groupoid of formal group laws over S, a $\mathbb{Z}_{(p)}$ algebra, which are of strict height n at p. This condition is unchanged by conjugation. Consequently the groupoid $\mathcal{F}^n(S)$ is *split*; that is to say, it is isomorphic to the translation groupoid of the group $\Gamma(S)$ of formal power series $f(t) \in S[[t]]$ with $f(0) = 0$ and $f'(0) \in S^{\times}$, under composition, acting by conjugation on the set $\mathcal{F}^n_0(S)$ of objects. Such a groupoid valued functor is represented by a split Hopf algebroid. The map $L \to S$ representing a formal group law of strict height n at p factors through $L_n = v_n^{-1}L/I_n$, by definition, and since $L \to L_n$ is an epimorphism in the category of rings this factorization is unique. This shows that the functor $\mathcal{F}^n_0$ is represented by the ring L_n. Denote by W_n the ring representing the morphisms in the groupoid $\mathcal{F}^n$. Then

$$W_n = L_n \otimes B,$$

where $B = \mathbb{Z}[b_0^{\pm 1}, b_1, \ldots]$. As usual there are structure maps $\eta_R, \eta_L : L_n \to W_n$, representing source and target, and they are swapped by the conjugation $c : W_n \to W_n$, representing inverse. Since W_n is clearly flat—even free—over L_n using η_L to define the module structure, it follows that the ring homomorphism $\eta_R : L_n \to W_n$ is flat as well.

This relates to our earlier work, since

$$W_n \cong L_n \otimes_L W \cong (L_n \otimes_L W)_{(n)}.$$

Thus the universal formal group law of strict height n, over L_n, provides one example in which Proposition 10.2 holds. This single example in fact suffices, because of the following important fact (in which we continue to assume our rings are localized at a prime p).

Proposition 10.4 (M. Lazard, N. S. Strickland)**.** Let G/R and H/S be formal groups over $\mathbb{Z}_{(p)}$ algebras, both of strict height $n \geq 0$. Then there is a formal group K/T and morphisms from G/R to K/T and from H/S to K/T with the property that both the ring homomorphisms $R \to T$ and $S \to T$ are faithfully flat.

Sketch of proof. If $n = 0$, both R and S are $\mathbb{Q}$ algebras, and we may take $T = R \otimes_{\mathbb{Q}} S$, since any two formal groups over a $\mathbb{Q}$ algebra are isomorphic. Henceforth assume $n > 0$. Then both R and S are $\mathbb{F}_p$ algebras. They map to the common algebra $R \otimes_{\mathbb{F}_p} S$ by faithfully flat maps. The images of F and G may not yet be isomorphic there, but at least we get to assume that F and G are defined over a single ring, say R. Now the proof of Lazard's theorem, as found in [5], has two steps. First one shows that, up to isomorphism, over an $\mathbb{F}_p$ algebra any two formal groups both having $[p](t) = t^{p^n}$ are isomorphic. For the second step, one enlarges the ring of definition so as to make a given formal group of strict height n isomorphic to one with p series of this form. Lazard and Froehlich work over a field, and adjoin roots of irreducible polynomials. Following Strickland, one can instead simply adjoin a polynomial generator x and divide by the ideal generated by the appropriate polynomial (which is of the form $x^{p^n-1} - a$ for a unit a, or $x^{p^n} - x + a$). This is a free extension, so the direct limit of all of them is faithfully flat. If one performs this construction first for F and then for the G over the new ring, one achieves the result. ∎

Proof of Proposition 10.2. Let F be any formal group law of strict height n over a $\mathbb{Z}_{(p)}$ algebra R. Let $i : R \to T$ and $j : L_n \to T$ be faithfully flat maps as in Proposition 10.4, and $\theta : iF \to G_u$ the isomorphism. Using naturality of the map η_R, together with the fact that $L \to L_n$ is an epimorphism in the category of rings, we obtain a commutative diagram

$$\begin{tikzcd}
 & L_n \arrow[dl] \arrow[d] \arrow[dr] & \\
R \otimes_L W \arrow[r, "(i,1)_*"] & S \otimes_L W & L_n \otimes_L W \arrow[l, "(j,\theta)_*"']
\end{tikzcd}$$

The right diagonal is flat, as we have seen, and $(j, \theta)_*$ is flat by Lemma 2.6, so the vertical arrow is flat. The map $(i, 1)_*$ is faithfully flat by Lemma 2.6, and it follows that the left diagonal is flat. ∎

References

[1] J. F. Adams, *Stable Homotopy and Generalized Homology*, Univ. of Chicago Press, 1974.

[2] N. Bourbaki, *Elements of Mathematics: Commutative Algebra*, Chapters 1–7, Springer, 1985.
[3] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Univ. Press, 1956.
[4] P. E. Conner and E. E. Floyd, *The Relation of Cobordism to K-Theories*, Springer Lecture Notes in Math. 28, 1966.
[5] A. Fröhlich, *Formal Groups*, Springer Lecture Notes in Math. 74, 1968.
[6] M. Hovey, Morita theory for Hopf algebroids and presheaves of groupoids, Amer. J. Math 124 (2002) 1289–1318.
[7] C. Kassel, *Quantum Groups*, Springer, 1995.
[8] P. S. Landweber, BP_*BP and typical formal groups, Osaka J. Math. 12 (1975) 357–363.
[9] P. S. Landweber, Homological properties of comodules over MU_*MU and BP_*BP, Amer. J. Math. 98 (1976) 273–610.
[10] H. R. Miller and D. C. Ravenel, Morava stabilizer algebras and the localization of Novikov's E_2-term, Duke Math. J. 44 (1977) 433–447.
[11] N. P. Strickland, Formal schemes and formal groups, in Homotopy Invariant Algebraic Structures, Contemp. Math. 239 (1999) 263–352.

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02459
E-mail address: hrm@math.mit.edu