Notes on Clark Barwick's operator categories

Haynes Miller
August, 2008

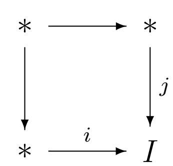
Definition. An *operator category* is an essentially small category Φ such that
(1) Hom sets are finite.
(2) There exists a terminal object, $*$.
(3) Fibers exist: for any map $J \to I$ and any $i : * \to I$, the pullback $J_i \to J$ exists.

The *set of points* of an object I in Φ is $|I| = \Phi(*, I)$. This is a functor from Φ to finite sets.

An *operator morphism* from Ψ to Φ is a functor $u : \Psi \to \Phi$ which preserves the terminal object and fibers, and is such that for every $I \in \Psi$ the natural map $|I| \to |u(I)|$ is surjective.

Lemma. If $u : \Psi \to \Phi$ is an operator morphism then $|I| \to |u(I)|$ is bijective for every $I \in \operatorname{ob} \Psi$.

Proof. Let $i, j : * \to I$ in Ψ and assume that $ui = uj : * = u(*) \to u(I)$. Let J be the fiber product of i and j. Since u preserves fibers, uJ is the fiber product of ui and uj. But these morphisms are equal, so $uJ = *$. Now $|J| \to |u(J)|$ is surjective, by assumption, so $|J| \neq \varnothing$. Let $k : * \to J$. Then we have a commutative diagram



and this implies that $i = j$.

Examples. The category `Fin` of finite sets is an operator category. The set of points functor for `Fin` is the identity functor. The category `Fin`$'$ of nonempty finite sets and surjections is also an operator category.

The category $*/$`Fin` of pointed finite sets is an operator category, but less interesting: for any I, there's only one map $* \to I$, and the fiber of $f : J \to I$ over it is the pre-image of $* \in I$ as a pointed set.

The category `Ord` of finite totally ordered sets is an operator category. The set of points is the underlying set. The category `Ord`$'$ of nonempty finite totally ordered sets and surjections is also an operator category.

For any operator category Φ, there is a natural operator map $\Phi \to \mathtt{Fin}$ given by the functor $|-|$. This morphism is initial in the category of morphisms from Φ to `Fin`: for any $u : \Phi \to \mathtt{Fin}$, the effect of u on $\Phi(*, -)$ induces a natural surjection

$$|I| = \Phi(*, I) \to \mathtt{Fin}(*, u(I)) = u(I)\,.$$

This is the only natural transformation from $|-|$ to u: the image of $(x : * \to I) \in |I|$ under a natural transformation is $u(x) : * \to u(I)$.

One might call a map an "interval inclusion" if it is isomorphic to the "inclusion" of a fiber. Since any map from a terminal object is a monomorphism, and any pullback of a monomorphism is a monomorphism, the word "inclusion" is justified: it is a monomorphism. Note that in an operator category pullbacks along inclusions of intervals exist: in

$$\begin{array}{ccccc} K_i & \xrightarrow{g|_{K_i}} & J_i & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle i} \\ K & \xrightarrow{g} & J & \xrightarrow{f} & I \end{array}$$

the map $g|_{K_i}$ exists uniquely since the right hand square is a pullback, and the left hand square is a pullback because the outer rectangle is a pullback.

Note that given $f : J \to I$,

$$|J| = \coprod_{i \in |I|} |J_i|$$

Definition. Given an operator category Φ and a symmetric monoidal category $(\mathbf{C}, \otimes, \mathbf{1})$, a Φ-*monoid* in $\mathbf{C}$ consists of an object $M \in \operatorname{ob} \mathbf{C}$ together with a morphism $\varphi_I : M^{\otimes |I|} \to M$ for every $I \in \operatorname{ob} \Phi$, such that $\varphi_* : M \to M$ is the identity map and for every $f : J \to I$ in Φ the diagram

$$\begin{array}{ccc} \bigotimes_{i \in |I|} M^{\otimes |J_i|} & \xrightarrow{=} & M^{\otimes |J|} \\ \downarrow{\scriptstyle \bigotimes_{i \in |I|} \varphi_{J_i}} & & \downarrow{\scriptstyle \varphi_J} \\ M^{\otimes |I|} & \xrightarrow{\varphi_I} & M \end{array}$$

commutes.

Perfection and the Leinster category

I think Barwick is interested in producing a version of the simplicial bar construction for such monoids, with the idea of getting at an A_∞ version. In order to construct the "Φ-simplicial category," he needs a little more structure on Φ.

Definition. Let Φ be an operator category. A *universal point* in Φ is a pair (T, o), where $o : * \to T$ is a point of T, such that given any $i : * \to I$, there is a unique map $\chi_i : I \to T$ (the "characteristic map") whose fiber over o is i.

For example `Fin` admits a universal point, namely $\{0, 1\}$, with $o(*) = 1$. Similarly, a universal point for `Ord` is given by $\{-, 0, +\}$, with $o(*) = 0$. But neither `Fin`′ nor `Ord`′ admit universal points.

The universal point is unique up to unique isomorphism. Suppose that $o : * \to T$ and $o : * \to U$ are both universal points. Then $o \in |T|$ determines a map $T \to U$ pulling o back to o, and $o \in |U|$ determines a map $U \to T$ pulling o back to o. The composite

$T \to T$ is the unique map pulling o back to itself—so is the identity on T. Similarly the composite $U \to U$ is the identity on U.

Definition. An operator category Φ is *perfect* if it admits a universal point and the functor $\mathrm{Fib} : \Phi/T \to \Phi$, assigning to $I \to T$ the fiber I_o, has a right adjoint.

Barwick writes the source of the right adjoint as $T(-)$. The structure map $T(*) \downarrow T$ is an isomorphism, since a map from $J \downarrow T$ to $T(*) \downarrow T$ is equivalent to giving a map $J_o \to *$, which carries no information, and this implies that the structure map of the object over T which we are mapping into is an isomorphism. Thus for any I, the unique map $p_I : I \to *$ induces

$$\begin{array}{ccc} T(I) & \xrightarrow{T(p_I)} & T(*) \\ & \searrow \quad \swarrow \cong & \\ & T & \end{array}$$

This shows that the canonical map $p_I : I \to *$ induces the structure map $T(I) \to T$. For this reason, only the source of the object in Φ/T needs a symbol.

It might be worthwhile writing out the adjunction in terms of the functor T. For any $f : J \to T(*)$ we have a factorization

$$\begin{array}{ccc} J & \xrightarrow{\alpha_f} & T(\mathrm{Fib} f) \\ & {\scriptstyle f}\searrow \quad \swarrow {\scriptstyle Tp_{\mathrm{Fib} f}} & \\ & T(*) & \end{array}$$

which is natural, and for any I we have a map $\mathrm{Fib}(Tp_I) \to I$, which is a natural isomorphism; and the composites

$$T(I) \xrightarrow{\alpha_{TI}} T(\mathrm{Fib}(Tp_I)) \xrightarrow{T\beta_I} T(I)$$

and

$$\mathrm{Fib} f \xrightarrow{\mathrm{Fib}\alpha_f} \mathrm{Fib}(Tp_{\mathrm{Fib} f}) \xrightarrow{\beta_{\mathrm{Fib} f}} \mathrm{Fib} f$$

are the identity maps. The left and right hand factors are thus inverse isomorphisms.

`Fin` is perfect. $T(I) = \{1\} \coprod I$ mapping to $T = \{0, 1\}$ by sending 1 to 1 and all of I to 0.

`Ord` is perfect. $T(I) = \{-\} \coprod I \coprod \{+\}$ mapping to $T = \{-, 0, +\}$ by sending $-$ to $-$, $+$ to $+$, and all of I to 0.

Lemma. The natural map $I \to \mathrm{Fib}(T(I) \downarrow T)$ is an isomorphism.

Proof. We must show that

$$\begin{array}{ccccc} I & \longrightarrow & \mathrm{Fib} T(I) & \longrightarrow & T(I) \\ \downarrow & & & & \downarrow \\ * & & \longrightarrow & & T \end{array}$$

is a pullback. A compatible pair of maps from J into the corners of the diagram is the same as a map in Φ/T from $J \xrightarrow{o} T$ to $T(I) \longrightarrow T$, which, by adjointness, is the same as a map $\mathrm{Fib}(J \xrightarrow{o} T) \to I$. Now the fiber of the composite $J \xrightarrow{o} T$ is just J, so we have verified the pullback property.

The functor $T : \Phi \to \Phi$ has a canonical triple structure. The unit in the triple is the composite in the top line of the diagram above. To construct the multiplication, note first that since T is a right adjoint, it carries the pullback diagram

$$\begin{array}{ccc} I & \longrightarrow & * \\ \downarrow{\scriptstyle \eta} & & \downarrow \\ T(I) & \longrightarrow & T(*) \end{array}$$

to a pullback, which we embed in the diagram of pullbacks

$$\begin{array}{ccccc} I & \longrightarrow & * & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle o} & & \\ T(I) & \longrightarrow & T(*) & & \downarrow{\scriptstyle o} \\ \downarrow & & \downarrow{\scriptstyle T(o)} & & \\ T^2(I) & \longrightarrow & T^2(*) & \longrightarrow & T(*) \end{array}$$

where the map $T^2(*) \to T(*)$ classifies the point $T(o) \circ o$ in $T^2(*)$. The multiplication $\mu_I : T^2(I) \to T(I)$ is the map over $T(*)$ which is adjoint to the identity map to I from the fiber of $T^2(I) \to T(*)$.

Definition. The *Leinster category* $\mathcal{L}(\Phi)$ of a perfect operator category Φ is the Kleisli category of the triple T.

Thus the objects of $\mathcal{L}(\Phi)$ are just the objects of Φ, while $\mathcal{L}(\Phi)(I, J) = \Phi(I, T(J))$. The identity map of I is $\eta_I : I \to T(I)$, and the composite of $f : I \to T(J)$ and $g : J \to T(K)$ is the composite

$$I \xrightarrow{f} T(J) \xrightarrow{T(g)} T^2(K) \xrightarrow{\mu_K} T(K)$$

The Leinster category will be the opposite of the Φ-analogue of the simplicial category.

For example, the Leinster category of `Fin` has finite sets as objects, and maps from I to J given by maps from I to $T(J)$, that is, J with a point added. Adjoining a basepoint to I and sending it to the basepoint of J shows that $\mathcal{L}(\texttt{Fin})$ is the category of finite pointed sets.

Similarly, $\mathcal{L}(\texttt{Ord})$ is the category of finite totally ordered sets with distinct maxima and minima which are preserved by morphisms. This is isomorphic to the opposite of the simplicial category, $\mathbf{\Delta}^{\mathrm{op}}$.

I think that if Φ is a perfect operator category, there is a functor

$$\mathrm{Mon}^{\Phi}(\mathbf{C}) \to \mathrm{Fun}(\mathcal{L}(\Phi), \mathbf{C})$$

whose essential image is described by a certain "Segal" condition.

Wreath product

An important construction is the "wreath product." I suppose this is a special case of a Grothendieck construction. Let Φ and Ψ be two operator categories. The wreath product $\Phi \wr \Psi$ has objects $(I : |J| \to \mathrm{ob}\,\Phi, J \in \mathrm{ob}\,\Psi)$. A morphism $(I', J') \to (I, J)$ consists of a morphism $g \in \Psi(J', J)$ and for each $j \in |J'|$ a morphism $f_j \in \Phi(I'(j), I(g \circ j))$.

A terminal object in $\Phi \wr \Psi$ is given by $(*, *)$, where the second $*$ denotes a terminal object of Ψ and the first $*$ denotes the function $|*| \to \mathrm{ob}\,\Phi$ sending $1 \in |*|$ to a terminal object in Φ. A point in (I, J) is thus a point j in J and a point i in $I(j)$;

$$|(I, J)| = \coprod_{j \in |J|} |I(j)|$$

Given $(f, g) : (I', J') \to (I, J)$, the fiber over a point $(i, j) \in |(I, J)|$ is given by $(j' \mapsto I'(j')_i, J'_j)$, where $I'(j')_i$ is the fiber of $g_{j'} : I'(j') \to I(j)$ over $i \in |I(j)|$, and $j' \in |J'_j|$, i.e. $j' : * \to J'$ such that $f \circ j' = j$.

The forgetful functor $\Phi \wr \Psi \to \Psi$ is an operator morphism: given (I, J) and a point $j \in |J|$, any choice of $i \in |I(j)|$ provides a point in (I, J) over j.

The functor $\Phi \to \Phi \wr \Psi$ sending I to $(I : * \to \mathrm{ob}\,\Phi, *)$ is an operator morphism.

If (T_Φ, o) and (T_Ψ, o) are universal points in Φ and Ψ, then we can contruct a universal point in the wreath product. It is given by (F, T_Ψ), where $F : |T_\Psi| \to \mathrm{ob}\,\Phi$ by sending o to T_Φ and the other points to o.

The category $\texttt{Ord}'_n$ defined inductively by $\texttt{Ord}'_1 = \texttt{Ord}'$, the category of nonempty finite ordered sets, and $\texttt{Ord}'_n = \texttt{Ord}'_{n-1} \wr \texttt{Ord}'$ is isomorphic to Batanin's category of "n-ordinals." He gives the following description of that category. An n-ordinal is a level-tree of uniform height n (so the levels are numbered 0 through n, level 0 is the root, and all branches grow up to level n) together with a total ordering of the leaves, with the property that if $a \le b \le c$ then b is at least as closely related to a and to c as a and c are to each other. This says that we have a planar tree with the leaves numbered consecutively. You could also describe this by giving a nested sequence of order-preserving relations, given by a composable sequence of order preserving surjections. Morphisms may not be order preserving (though Batanin defines a map to be order preserving if it is a morphism). To describe them, think of the object as a phylogenetic tree, expressing how closely related the various leaves are. Then a morphism never increases the distance between leaves, and if it reverses the order of a pair of leaves then it makes them more closely related.

When $n = 2$, the isomorphism assigns to a 2-ordinal T the object (I, J) of $\mathtt{Ord}' \wr \mathtt{Ord}'$ with J given by the set of nodes of level 1 (with its given order) and $I(a)$ the set of leaves above a (with the subset order). The inverse is given by sending (I, J) to $\coprod_{j \in J} I(j)$, with the lexicographic order, and tree structure putting J at level 1.

One can also say that an object of $\mathtt{Ord}'_2$ is a nonempty totally ordered set together with an order-preserving relation on it; and that a morphism $(J, \sim) \to (I, \sim)$ is a function $f : J \to I$ which respects the equivalence relations, is order preserving on each equivalence class, and such that if f reverses the order of j and j' then $f(j) \sim f(j')$.

If we add the empty set back in, we can identify the category $\mathtt{Ord}_2$. Its objects are the arrows in $\mathtt{Ord}$. A map from $\underline{p} \downarrow \underline{q}$ to $\underline{p'} \downarrow \underline{q'}$ is an order preserving map $\underline{q} \to \underline{q'}$ together with a map $\underline{p} \to \underline{p'}$ which is order preserving on fibers. These are trees which are not required to grow up to the top level.

Similarly, $\mathtt{Ord}_n$ is the category of $(n-1)$-chains in $\mathtt{Ord}$,

$$\underline{p_0} \xrightarrow{\alpha_1} \underline{p_1} \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} \underline{p_{n-1}}$$

and a morphism is a sequence of maps of underlying sets each of which is order preserving on the fibers of the α's.

Operads

Definition. Given an operator category Φ and a symmetric monoidal category $(\mathbf{C}, \otimes, \mathbf{1})$, a Φ-*operad* in $\mathbf{C}$ consists in
(1) An object $P(I) \in \mathbf{C}$ for every $I \in \Phi$,
(2) A map $\eta : \mathbf{1} \to P(*)$, and
(3) A map $\theta_f : P(I) \otimes P(f) \to P(J)$ for every morphism $f : J \to I$ in Φ,
where

$$P(f) = \bigotimes_{i \in |I|} P(J_i)\,,$$

such that for every $I, J \in \operatorname{ob} \Phi$ the diagrams

$$\begin{array}{ccc} \mathbf{1} \otimes P(I) & & \\ \downarrow{\scriptstyle \eta \otimes 1} & \searrow{\scriptstyle =} & \\ P(*) \otimes P(p_I) & \xrightarrow{\theta_{p_I}} & P(I) \end{array} \qquad \begin{array}{ccc} P(J) \otimes \mathbf{1} & & \\ \downarrow{\scriptstyle 1 \otimes \eta_J} & \searrow{\scriptstyle =} & \\ P(J) \otimes P(1_J) & \xrightarrow{\theta_{1_J}} & P(J) \end{array}$$

commutes, where $p_I : I \to *$ is the unique map to the terminal object (so that $P(I) = P(p_I)$) and

$$\eta_J : \mathbf{1} = \bigotimes_{j \in |J|} \mathbf{1} \xrightarrow{\otimes \eta} \bigotimes_{j \in |J|} P(*) = P(1_J)\,,$$

and for every $f : J \to I$ and $g : K \to J$ the diagram

$$\begin{array}{ccc}
P(I) \otimes P(f) \otimes P(g) & \xrightarrow{\theta_f \otimes 1} & P(J) \otimes P(g) \\
\downarrow{\scriptstyle 1\otimes\mu_{f,g}} & & \downarrow{\scriptstyle \theta_g} \\
P(I) \otimes P(fg) & \xrightarrow{\theta_{fg}} & P(K)
\end{array}$$

commutes, where $\mu_{f,g}$ is the composite

$$\left(\bigotimes_{i\in|I|} P(J_i)\right) \otimes \left(\bigotimes_{j\in|J|} P(K_j)\right) \cong \bigotimes_{i\in|I|} \left(P(J_i) \otimes \bigotimes_{j\in|J_i|} P(K_j)\right)$$

$$= \bigotimes_{i\in|I|} (P(J_i) \otimes P(f|_{K_i})) \xrightarrow{\otimes \theta_{f|_{K_i}}} \bigotimes_{i\in|I|} P(K_i)$$

A morphism of Φ operads, $\alpha : P \to P'$, is a collection of maps $\alpha_I : P(I) \to P'(I)$ for every $I \in \mathrm{ob}\,\Phi$ which commute with the structure maps.

Example. The one-morphism category is an operator category. A $*$-operad in $\mathbf{C}$ is a $\otimes$-monoid in $\mathbf{C}$.

Example. Let `Fin` be the operator category of finite sets. If $f : I \to I$ is a permutation then $P(f) \cong \bigotimes_{i\in I} P(*)$ is endowed with a canonical map from $\mathbf{1}$, and composition with the structure map $P(I) \otimes P(f) \to P(I)$ yields an automorphism of $P(I)$. Thus a `Fin` operad determines a symmetric sequence, and the rest of the `Fin` operad structure determines on P the structure of an operad in the traditional sense.

Example. Let `Ord` be the category of finite ordered sets. Now the intervals are precisely intervals in the usual sense. An `Ord` operad is a "non-symmetric" operad.

Sequences. A Φ operad doesn't determine a functor from Φ to $\mathbf{C}$, but it does determine a contravariant functor $\mathrm{Seq}\,\Phi \to \mathbf{C}$, where $\mathrm{Seq}\,\Phi$ is subcategory of Φ consisting of morphisms all of whose fibers are points. Pulling back along such a morphism $f : J \to I$ induces a map $|I| \to |J|$ which is inverse to $|f|$: so the functor of points takes quasi-isomorphisms to bijections. The subcategory $\mathrm{Seq}\,\Phi$ is the analogue of the category of sets and bijections in the traditional development of the theory of operads, so a functor from it to $\mathbf{C}$ is a "Φ sequence."

Intervals pull back (to intervals). Using the fact that fiber inclusions are monomorphisms, it is easy to show that if $\phi : J \to I$ is a quasi-isomorphism and $F \to I$ is a fiber inclusion, then the map $\phi^{-1}F \to F$ is again a quasi-isomorphism.

Let $\phi : J \to I$ be a quasi-isomorphism and P a Φ operad. Let $f : K \to J$. For any $i \in |I|$, let $j \in |J|$ be the unique point such that $\phi j = i$. Then there is a canonical isomorphism of fibers $K_j \to K_i$. Write $g = \phi f$. Combined with the functoriality $\phi^* :$

$P(I) \to P(J)$, we get a map along the top of the diagram

$$\begin{array}{ccc} P(I) \otimes \bigotimes_{i \in |I|} P(K_i) & \longrightarrow & P(J) \otimes \bigotimes_{j \in |J|} P(K_j) \\ & \searrow^{\theta_g} \quad {}^{\theta_f}\swarrow & \\ & P(J) & \end{array}$$

which commutes by the associativity diagram of the operad.

This suggests that we try to define a monoidal structure on Φ sequences by

$$(P \circ Q)(K) = \operatorname*{colim}_{\phi: K \to J} P(J) \otimes \bigotimes_{j \in |J|} Q(K_j)$$

where the colimit is taken over the category $K/\operatorname{Seq}\Phi$.

Note that if $\Phi = \mathtt{Fin}$, we get

$$(P \circ Q)(0) = \coprod_{n \geq 0} P(n) \otimes_{\Sigma_n} Q(0)^{\otimes n}$$

In a $\mathtt{Fin}$ operad, $P(0)$ is a P-algebra. If we assume that $Q(0) = o$, then the terms with ϕ not surjective don't contribute. If ϕ is surjective, then there are no automorphisms of it in $K/\operatorname{Seq}\Phi$, so

$$(P \circ Q)(K) = \coprod_{\sim} P(K/\sim) \otimes \bigotimes_{j \in K/\sim} Q(j)$$

where the coproduct runs over equivalence relations on K.

Pulling back operads. Let $u : \Psi \to \Phi$ be an operator morphism, and P a Φ-operad in $\mathbf{C}$. The pullback of P along u is the Ψ-operad u^*P in $\mathbf{C}$ with

$$(u^*P)(K) = P(u(K))$$

and structure maps given as follows. Let $f : J \to I$ in Ψ. For each $i \in |I|$, $(uf)^{-1}(ui) = u(f^{-1}(i))$ since u respects fibers, and u induces a bijection $|I| \to |u(I)|$ (by the Lemma). Thus the structure map

$$\theta_{uf} : P(uI) \otimes \bigotimes_{k \in |uI|} P((uf)^{-1}(k)) \to P(uJ)$$

precisely determines a map

$$\theta_f : (u^*P)(I) \otimes \bigotimes_{i \in |I|} (u^*P)(f^{-1}(i)) \to (u^*P)(J)$$

Also, $\eta : \mathbf{1} \to P(*)$ is the same as a map $\eta : \mathbf{1} \to (u^*P)(*)$, since $u* = *$. These define the structure of a Ψ-operad.

For example, suppose $u : \Psi \to \mathtt{Fin}$ is given by $uI = |I|$. Then a $\mathtt{Fin}$-operad P—that is, an operad in the usual sense—gives rise to a Ψ-operad u^*P given by $u^*P(I) = P(|I|)$, and structure map given by

$$P(|I|) \otimes \bigotimes_{i \in |I|} P(|f^{-1}(i)|) \to P(|J|),$$

using $|f^{-1}(i)| = |f|^{-1}(i)$.

Algebras over operads

Definition. Let P be a Φ-operad in $\mathbf{C}$. A *P-algebra* is an object A in $\mathbf{C}$ together with a map

$$\varphi_I : P(I) \otimes A^{\otimes |I|} \to A$$

for each $I \in \operatorname{ob} \Phi$ such that the diagram

$$\begin{array}{ccc} \mathbf{1} \otimes A & & \\ \downarrow{\scriptstyle \eta \otimes 1} & \searrow{\scriptstyle =} & \\ P(*) \otimes A & \xrightarrow{\varphi_*} & A \end{array}$$

commutes and for each $f : J \to I$ in Φ the diagram

$$\begin{array}{ccc} P(I) \otimes \bigotimes_{i \in |I|} P(J_i) \otimes A^{\otimes |J_i|} & \xrightarrow{1 \otimes \bigotimes \varphi_{J_i}} & P(I) \otimes A^{\otimes |I|} \\ \downarrow{\scriptstyle =} & & \\ \left(P(I) \otimes \bigotimes_{i \in |I|} P(J_i) \right) \otimes A^{\otimes |J|} & & \downarrow{\scriptstyle \varphi_I} \\ \downarrow{\scriptstyle \theta_f \otimes 1} & & \\ P(J) \otimes A^{\otimes |J|} & \xrightarrow{\varphi_J} & A \end{array}$$

commutes.

Definition. Let Φ be an operator category, $\mathbf{C}$ a closed symmetric monoidal category, and P a Φ operad in $\mathbf{C}$. The May-Thomason contruction provides us with a category enriched over $\mathbf{C}$, with the same object set as Φ. Its object of morphisms from J to I is

$$\bigotimes_{f : J \to I} P(J_i)$$

and the composition is given by the structure map for the operad. This is perhaps better viewed as a "Φ-graded" category.

Example. Suppose $\mathbf{C}$ contains a zero object $\varnothing$. Define a Φ operad Z in $\mathbf{C}$ by declaring that $Z(*) = \mathbf{1}$ and $Z(I) = \varnothing$ if $I \not\simeq *$. An algebra for this operad is precisely a Φ-sequence. This is because of the

Lemma. $f : J \to I$ is a quasi-isomorphism if and only if $J_i = *$ for every $i \in |I|$.

Multicategories and modules over them

Definition. Let Φ be an operator category and $\mathbf{C}$ a symmetric monoidal category. A Φ multicategory H enriched over $\mathbf{C}$ consists in the data of:
(0) A set S of "objects" of H;
(1) For each $I \in \operatorname{ob}\Phi$, object $x \in S$, and map $y : |I| \to S$, an object $H_I(y, x) \in \mathbf{C}$;
(2) For each $x \in S$, a map $\eta : \mathbf{1} \to H_*(x, x)$
(3) For each morphism $f : J \to I$ in Φ, $w \in S$, $x : |I| \to S$, and $y : |J| \to S$, a map

$$\theta_f : H_I(x, w) \otimes H_f(y, x) \to H_J(y, w)$$

where

$$H_f(y, x) = \bigotimes_{i \in |I|} H_{J_i}(y|_{|J_i|}, x_i)$$

To give the axioms, note that if $p_J : J \to *$ then $H_{p_J}(y, x) = H_J(y, x)$ and

$$H_{1_J}(y, x) = \bigotimes_{j \in |J|} H_*(y_j, x_j)$$

Define $\mu_{f,g} : H_f(y, x) \otimes H_g(z, y) \to H_{fg}(z, x)$ as the composite

$$\left(\bigotimes_{i\in|I|} H_{J_i}(y|_{|J_i|}, x_i)\right) \otimes \left(\bigotimes_{j\in|J|} H_{K_j}(z|_{|K_j|}, y_j)\right) \cong \bigotimes_{i\in|I|}\left(H_{J_i}(y|_{|J_i|}, x_i) \otimes \bigotimes_{j\in|J_i|} H_{K_j}(z|_{|K_j|}, y_j)\right)$$

$$= \bigotimes_{i\in|I|}\left(H_{J_i}(y|_{|J_i|}, x_i) \otimes H_{g_i}(z|_{|K_i|}, y|_{|J_i|})\right) \xrightarrow{\otimes\theta_{g_i}} \bigotimes_{i\in|I|} H_{K_i}(z|_{|K_i|}, x_i)$$

where g_i is the unique map making

$$\begin{array}{ccc} K_i & \xrightarrow{g_i} & J_i \\ \downarrow & & \downarrow \\ K & \xrightarrow{g} & J \end{array}$$

commutative. We require that

$$\begin{array}{ccc} \mathbf{1} \otimes H_K(z, y) & & \\ \downarrow{\scriptstyle \eta_J \otimes 1} & \searrow{\scriptstyle =} & \\ H_*(y, y) \otimes H_{p_K}(z, y) & \xrightarrow{\mu_{1,p}} & H_K(z, y) \end{array} \qquad \begin{array}{ccc} H_f(y, x) \otimes \mathbf{1} & & \\ \downarrow{\scriptstyle 1 \otimes \eta_J} & \searrow{\scriptstyle =} & \\ H_f(y, x) \otimes H_{1_J}(y, y) & \xrightarrow{\mu_{f,1}} & H_f(y, x) \end{array}$$

and

$$\begin{array}{ccc} H_I(x,w) \otimes H_f(y,x) \otimes H_g(z,y) & \xrightarrow{\theta_f \otimes 1} & H_J(y,w) \otimes H_g(z,y) \\ \Big\downarrow{\scriptstyle 1\otimes\mu_{f,g}} & & \Big\downarrow{\scriptstyle \theta_g} \\ H_I(x,w) \otimes H_{fg}(z,x) & \xrightarrow{\theta_{fg}} & H_K(z,w) \end{array}$$

commute.

For example, if $S = \{*\}$, then we have a Φ-operad in $\mathbf{C}$. If $\Phi = *$, then we have a category enriched over $\mathbf{C}$ with object set S.

Given any Φ multicategory in $\mathbf{C}$ with object set S, and any $x \in S$, we have the "endomorphism operad" End_x with

$$\mathrm{End}_x(I) = H_I(x_{|I|}, x)$$

where $x_{|I|}$ denotes the constant function from $|I|$ with value x.

Definition. Let H be a Φ multicategory enriched over $\mathbf{C}$. A *module* for H consists in
(1) a function $M : S \to \mathrm{ob}\,\mathbf{C}$, and
(2) for each $J \in \mathrm{ob}\,\Phi$ and each $y : |J| \to S$ and $x \in S$, a morphism

$$\varphi : H_J(y,x) \otimes \bigotimes_{j \in |J|} M(y_j) \to M(x)$$

such that the diagrams

$$\begin{array}{ccc} \mathbf{1} \otimes M(x) & & \\ \Big\downarrow{\scriptstyle \eta\otimes 1} & \searrow{\scriptstyle =} & \\ H_{1_x}(x,x) \otimes M(x) & \xrightarrow{\varphi} & M(x) \end{array}$$

and

$$
\begin{array}{ccc}
H_I(x,w) \otimes \bigotimes_{i\in|I|} \left(H_{J_i}(y|_{|J_i|}, x_i) \otimes \bigotimes_{j \in |J_i|} M(y_j) \right) & \xrightarrow{1\otimes\bigotimes\varphi} & H_I(x,w) \otimes \bigotimes_{i\in|I|} M(x_i) \\
\Big\downarrow{=} & & \\
H_I(x,w) \otimes \left(\bigotimes_{i\in|I|} H(y|_{|J_i|}, x_i) \right) \otimes \bigotimes_{j\in|J|} M(y_j) & & \Big\downarrow{\varphi} \\
\Big\downarrow{\theta\otimes 1} & & \\
H_J(y,w) \otimes \bigotimes_{j\in|J|} M(y_j) & \xrightarrow{\varphi} & M(w)
\end{array}
$$

commute.

For example, if $S = \{*\}$ this is the notion of an algebra over the operad.

Definition. Suppose that $\mathbf{C}$ has an initial object $\varnothing$ such that $\varnothing \otimes X \cong \varnothing$ for any X. Fix a universal point $o : * \to T$ in an operator category Φ. A Φ-chirality in $\mathbf{C}$ is a Φ multicategory H in $\mathbf{C}$ with object set $|T|$ such that:
(1) For any $I \in \operatorname{ob}\Phi$, $H_I(x, o) = \varnothing$ unless $x = |\chi_i| : |I| \to |T|$ for some $i \in |I|$.
(2) If $y \neq o$ in $|T|$ then for any $I \in \operatorname{ob}\Phi$, $H_I(x,y) = \varnothing$ unless $x : |I| \to |T|$ is the constant function with value y.

This is actually what Barwick calls a "pure chirality." In the general definition, the various different "generic" points of $|T|$ are allowed to interact with each other: (2) is replaced by the requirement that $o \notin \operatorname{im} x$. I hope that pure chiralities will suffice.

First let's study the objects $H_I(x,y)$.

By condition (2), when y is generic—i.e. $y \neq o$—the only nontrivial $H_I(x,y)$'s are the objects making up the endomorphism operad P_y of y,

$$P_y(I) = H_I(y_{|I|}, y)\,.$$

The other objects determined by a chirality are, for $J \in \operatorname{ob}\Phi$ and $j \in |J|$,

$$M(J, j) = H_J(|\chi_j|, o)\,.$$

All the other $H_J(x, o)$'s are trivial.

To analyze the endomorphism operad P_o we use:

Lemma. Let $J \in \operatorname{ob}\Phi$ and $j \in |J|$. The only element of $|J|$ mapped to o by χ_j is j itself.

Proof. Let $f : J \to I$, and suppose that f pulls $i : * \to I$ back to $j : * \to J$. Since $|-|$ preserves pullbacks, the inverse image of $i \in |I|$ under $|f|$ is $\{j\} \subseteq |J|$. Apply this with $J = T$.

Thus if χ_j is a constant map then $|J| = \{j\}$. Conditions (1) and (2) thus combine to show that the objects $H_I(o_{|I|}, o)$ are trivial unless $I = *$. The endomorphism operad of o thus reduces to the monoid

$$H_*(o,o) = P_o(*) = M(*, \mathrm{id}_*)$$

which we denote by A.

We analyze what data is given by the structure maps

$$\eta : \mathbf{1} \to H_*(x,x)$$

for $x \in |T|$ and

$$\theta_f : H_I(x,w) \otimes H_f(y,x) \to H_J(y,w)$$

for $f : J \to I$, $w \in |T|$, $x : |I| \to |T|$, and $y : |J| \to |T|$.

When $w = o$ and $I = *$, the map is trivial unless $x = o$ and the structure map gives us

$$H_*(o,o) \otimes H_J(\chi_j, o) \to H_J(\chi_j, o)$$

that is, an action by the monoid A on $M(J,j)$.

I'm finding it hard to interpret what the other structure morphisms give you. It seems to me that the following lemma should be useful.

Lemma. Let Φ be an operator category with a universal point $o : * \to T$, let $f : J \to I$ be a monomorphism (for example a fiber inclusion), and let $j \in |J|$. Then

$$\chi_{fj} \circ f = \chi_j : J \to T$$

Proof. Let $g : K \to J$ be such that $(\chi_{fj} f)g = op_K$. We want to show that $g = jp_K$. From the definition of χ_{fj}, $\chi_{fj}(fg) = op_K$ implies that $fg = (fj)p_K$, so what we want follows from associativity of composition and monomorphicity of f.

If I continue to take $w = o$, and expect to get something nontrivial in the source of θ_f, I had better take $x = |\chi_i|$ for some $i \in |I|$ and also $y = |\chi_j|$ for some $j \in |J|$, so the map is

$$\theta_f : M(I,i) \otimes H_f(|\chi_j|, |\chi_i|) \to M(J,j)$$

Now

$$H_f(|\chi_j|, |\chi_i|) = \bigotimes_{k \in |I|} H_{J_k}(|\chi_j||_{|J_k|}, \chi_i k)$$

We have to understand some things about the values of the characteristic functions of points. First, $\chi_i k = o$ if and only if $k = i$. So that tensor factor is a special case. The factor is nontrivial if and only if $|\chi_j||_{|J_i|} = |\chi_l|$ for some $l \in |J_i|$. If this occurs, then $\chi_j l = o$; but the only element carried to o by χ_j is j, so $l = j$. Conversely, by the lemma, if there is $l \in |J_i|$ such that $j = gl$ (where $g : J_i \to J$ is the interval inclusion) then $\chi_j \circ g = \chi_l$, and it follows that $|\chi_j||_{|J_i|} = |\chi_l|$.

The upshot is that the $k = i$ factor $H_{J_i}(|\chi_j||_{|J_i|}, o)$ is nontrivial only if $i = fj$. When this factor is trivial, it kills the entire object $H_f(|\chi_j|, |\chi_i|)$: so θ_f is interesting only when $i = fj$. In that case, the $k = i$ factor is $H_{J_{fj}}(|\chi_j|, o) = M(J_{fj}, j)$, so the structure morphism has the form

$$\theta_f : M(I, fj) \otimes M(J_{fj}, j) \otimes \bigotimes_k P_{\chi_{fj}k}(J_k) \to M(J, j)$$

as long as all points k of I other than fj have the property that $|\chi_j|$ is constant on J_k with value $\chi_{fj}k$. If this condition fails for any value of k, then the source is $\varnothing$ and the structure map has no content.

So the question is: Let $f : J \to I$, $k \in |I|$, and $j \in |J|$. Is $|\chi_j|$ constant on J_k with value $\chi_{fj}k$? This is true if f is a monomorphism. If $k = fj$ then $J_k = *$, mapping to J by j, so it is true then too.

But it certainly isn't true in general. Suppose $I = *$ for example. Then $J_k = J$, and χ_j is certainly not constant (as long as J has more than one point). In this case $fj = k$, so one definitely needs to assume that $fj \neq k$ for the hypothesis to stand a chance.

Clark mentions that there are (pathological) perfect operator categories for which T contains intervals which don't contain o but do contain more than one point. Perhaps χ_j is always interval-preserving; if so, then Clark's hypothesis would imply that χ_j is constant on intervals not containing j.

If $w \neq o$, then for $H_I(x, w)$ to be nontrivial we must have $x = w_{|I|}$, and for $H_J(y, w)$ to be nontrivial we must have $y = w_{|J|}$. So now we want to understand

$$H_f(w_{|J|}, w_{|I|}) = \bigotimes_{i \in |I|} H_{J_i}(w_{|J_i|}, w) = \bigotimes_{i \in |I|} P_w(J_i)$$

so the structure map is

$$P_w(I) \otimes \bigotimes_{i \in |I|} P_w(J_i) \to P_w(J)$$

which is just the operad structure on P_w.

Notes on points: Suppose $f : I \to *$ is a monomorphism. If $i \in |I|$, then the composite fi is the identity (since it's a self-map of the terminal object), and the composite if is too (since $f \circ if = fi \circ f = f = f \circ \mathrm{id}_I$, and f is a monomorphism). Thus the proper subobjects of $*$ have no points. A pointless object doesn't have to be initial, in interesting cases. For example, in $\mathtt{Ord}_2$ we have the object (J, I), where $I = \{1\}$ and $J(1) = \varnothing$.

In $\mathtt{Ord} \wr \mathtt{Ord}$, we claim that the universal point is the central point in the object with base $\underline{3}$ and fibers $\underline{1}, \underline{3}, \underline{1}$. But suppose I want to classify 1 in the object with base $\underline{2}$ and both fibers $\underline{1}$. It seems to me that there are three maps which pull o back to 1: send 1 to o, and send 2 to either of the other points in the middle fiber or to the unique point in the right fiber.

The fallacy is that when 2 gets sent to the wrong point, the pullback is not the terminal object $\underline{1} \downarrow \underline{1}$ but rather $\underline{1} \downarrow \underline{2}$.

Given a Φ-operad P there will generally be many chiralities with $P_x = P$ for all x, but there is a preferred one. It has

$$M(J, j) = P(J)$$

for all $J \in \operatorname{ob}\Phi$ and all $j \in |J|$. The structure map corresponding to $f : J \to I$, $j \in |J|$ is

$$\theta_f : P(I) \otimes \bigotimes_{k \in |I|} P(J_k) \to P(J)$$

as long as all $k \in |I|$ other than $k = fj$ have the property that $|\chi_j| = (\chi_{fj} k)_{|J_k|}$. This is a piece of the operad structure of P (and if the condition holds for all k, it is precisely the operad structure map).