ON THE ANTI-AUTOMORPHISM OF THE STEENROD ALGEBRA

M. G. Barratt and H. R. Miller

(Dedicated to J. Adem)

1. INTRODUCTION

Our purpose is to use the beautiful formulation of the Adem relations by S.R.Bullett and I.G.Macdonald ([3]) to establish some useful identities involving the canonical anti-automorphism χ of the mod.p Steenrod algebra, for all primes p. The first result is

Theorem 1. The Bootstrap Functions

$$Q(N,K,L) = \sum_j \binom{K-j}{L} P^j \chi P^{N-j}$$

are zero when $(p-1)N > pL - \alpha(L)$.

Here P^i means Sq^i when $p = 2$, and $\alpha(L)$ is the sum of the digits in the p-ary expression of $|L|$. The binomial coefficient $\binom{T}{S}$ is the coefficient of x^S in the formal power series for $(1+x)^T$, and is zero when S is negative. Thus Q(N,K,L) is zero when L is negative, and also, by the nature of χ, when $L = 0 < N$. The result is best possible.

2. PROOF OF THEOREM 1

Let $P(t)$ denote the formal power series $\sum_j P^j t^j$ so that

$$P(t).\chi P(t) = 1 = \chi P(t).P(t) \quad .$$

The Bullett-Macdonald expression of the Adem relations in [3] can be written

$$(2.1) \qquad \chi P(s(s-t)^{p-1}).P(t(s-t)^{p-1}) = P(t^p).\chi P(s^p) \quad ,$$

together with some analogous relations involving the Bochstein homomorphism, which will not be used here. On putting $s=1$ and $t = 1/(1+x)$, (2.1) becomes

$$(2.2) \qquad \chi P(x^{p-1}(1+x)^{1-p}).P(x^{p-1}(1+x)^{-p}) = P((1+x)^{-p}).\chi P(1) \quad .$$

If this is multiplied by $(1+x^p)^K$ and expanded in formal power series, equating the coefficients of various powers of x yields

Lemma (2.3) Let $Q^*(N,K,L) = \sum_j \binom{K-j}{L} \chi P^{N-j}.P^j$. Then

$$Q(N,K,L) = Q^*(N,\ N+p(K-N),\ N+p(L-N)\ ) \ ,$$

and, when $T \not\equiv 0 \mod. p$,

$$0 = Q^*(N,\ N+p(K-N),\ N+T\ ).$$

For $p = 2$ these are equivalent to the Adem relations.

Corollary (2.4) $Q(N,K,L) = 0$ when $(p-1)N > pL$ or $(p-1)N = pL > 0$.

(For $Q^*(N,K,L)$, like $Q(N,K,L)$, is zero when $L < 0$ or $L = 0 < N$.) Thus Theorem 1 is true when $\alpha(L) < 2$; the proof will be completed by induction on $\alpha(L)$, using the lemma

Lemma (2.5) $\binom{A+B}{A} Q(N,K,A+B) = \sum_T Q(T,K,A).Q(N-T,K-A-T,B).$

Assuming the truth of this, and of the theorem for $L = A$ and $L = B$, shows that each term in the summation on the right will be zero if

$$(p-1)N = (p-1)T + (p-1)(N-T) > (pA - \alpha(A)) + (pB - \alpha(B)).$$

When $\alpha(A) + \alpha(B) = \alpha(A+B)$ this is the desired inequality and the binomial coefficient on the left is prime to p (see, for example, Lemma 1 of [2]). Thus the theorem will be true for $L = A + B$, and so for all L. It remains to prove (2.5), which follows readily from the identity

$$\binom{A+B}{A}\binom{K-j}{A+B} = \binom{K-j}{A}\binom{K-j-A}{B}$$

and the trivial

Lemma (2.6) $\binom{K-j}{A} P^j = \sum_T Q(T,K,A).P^{j-T}$.

For the term on the left is repeated in $Q(j,K,A)$ on the right, and the remaining terms on the right group themselves into blocks

$$\binom{K-i}{A} P^i \left\{ \sum_T \chi P^{T-i}.P^{j-T} \right\} = 0 \qquad \text{for } i < j.$$

This completes the proof of the theorem, which now makes (2.6) more interesting, as the terms on the right with $(p-1)T > pA - \alpha(A)$ are zero, and can be omitted, whatever the size of j may be.

3. SOME APPLICATIONS OF THEOREM 1

The vanishing of $Q(N,L,L)$ implies

Theorem (3.1) When $(p-1)N > pL - \alpha(L)$,

$$-\chi P^N = (-1)^L \sum_{j>L} \binom{j-1}{L} P^j . \chi P^{N-j} .$$

Here the binomial identity $\binom{L-j}{L} = (-1)^L \binom{j-1}{L}$ has been used. For a given N , L can certainly be [N - N/p] ; larger values of L may be useful and available. For example, when p = 2, Q(24,12,12) has 5 terms while Q(24,13,13) is also zero and yields directly

$$\chi Sq^{24} = Sq^{14}.\chi Sq^{10} + Sq^{16}.\chi Sq^8$$

Other efficient choices of L come from the fact that when $\alpha(L)$ is nearly maximal the binomial coefficients in (3.1) are most often trivial mod. p. It is convenient to define

(3.2) $$p\{e\} = (p^{e+1} - 1)/(p-1) = p^e + \ldots + p + 1.$$

Theorem (3.3) If $L = p^e - 1 \geq A \geq 0$, and $N \geq p\{e\} - e - A$,

$$- P^A.\chi P^N = \sum_{t>0} P^{A+tp^e}.\chi P^{N-tp^e} .$$

This follows from Q(N,L+A,L) = 0, and implies the misleadingly succinct

(3.4) $$-\chi P^{p^{e+1}} = \sum_{t=1}^{p} P^{tp^e}.\chi P^{p^e(p-t)} .$$

Corollary (3.5) Let $N = p\{e\} - s$. For $0 \leq s \leq e$,

$$-\chi P^N = (-1)^{e+s} P^{p^e}.P^{p^{e-1}} \ldots P^{p^s}.\chi P^{p\{s-1\}-s}$$

and, for $s = e+1$,

$$-\chi P^N = P^{p^e}.\chi P^{p\{e-1\}-e-1} - Q$$

where $Q = Q(N,p^e-1,p^e-1) = P^{p^e-1}.P^{p^{e-1}-1} \ldots P^{p-1}$.

The first part of (3.5) comes by induction on e from (3.3). Alternatively, having proved the case $s = 0$ in this way, the others can be deduced by using L.Kristensen's Steenrod algebra derivation κ such that

$$\kappa P^i = P^{i-1}, \quad \kappa\chi P^i = -\chi P^{i-1}, \quad \kappa(uv) = (\kappa u)v+u\kappa v ,$$

and the Adem relations $P^{pn-1}.P^n = 0$. For the case $s = e+1$ the first part of the statement expresses the definition of $Q(N,p^e-1,p^e-1)$, and the second part can be proved by induction on e and the application of κ to the previous case $s = e$, or by using $Q(N,p^e-2,p^e-2) = 0$ in the manner described below. The cases when $p = 2$ were first proved by D.M.Davis in Theorem 2 of [2].

It is worth remarking that, for a given N in Theorem (3.3), e can always be chosen so that no more than $p + 1$ terms appear in the summation. In some cases a greater economy can be acheived by

Theorem (3.6) Let $d < e$ and $q \leq p-1$ (or $p-2$ if $d = 0$).
Let $L = p^e - qp^d - 1$, and $N > p\{e\} - q.p\{d\} - e$.
Then

$$-\chi P^N = (-1)^q \sum_{t>0} \sum_{r=0}^{q} \binom{p-1-r}{p-1-q} P^{tp^e-rp^d}.\chi P^{N-tp^e-rp^d}$$

This comes from $Q(N,L,L) = 0$ with a simplification of the binomial coefficients using the well-known relation, proved by expanding $(1+x)^{\Sigma p^i c_i} = \Pi(1+x^{p^i})^{c_i}$,

$$\binom{\Sigma\, p^i c_i}{\Sigma\, p^i a_i} = \prod \binom{c_i}{a_i} .$$

4. COMMUTATION RELATIONS

Putting $s=1$, $t=-v$ in the Bullett-Macdonald relation (2.1) yields

(4.1) $$\chi P((1+v)^{p-1}).P(-v(1+v)^{p-1}) = P(-v^p).\chi P(1) \quad ;$$

also, putting $s(s-t)^{p-1} = -v$ and $t(s-t)^{p-1} = 1$ gives

(4.2) $$\chi P(-v).P(1) = P((1+v)^{1-p}).\chi P(-v^p(1+v)^{1-p}) \; .$$

Expansion yields certain commutation formulae which can be written

(4.3) $$(-1)^A P^A.\chi P^{N-A} = \sum_{m \leq pA} \chi P^{N-m}.P^m \, (-1)^m \binom{N(p-1)}{pA - m} ,$$

(4.4) $$\chi P^{N-B}.P^B = \sum_m P^m.\chi P^{N-m} \binom{pm-1-B}{N(p-1)-1}$$

where the summation in (4.4) is for $pm \geq B + N(p-1)$.

(4.1) yields other identities from the coefficients of v^T with $T \not\equiv 0$ mod.p.

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