Some topological reflections of the work of Michel André

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Haynes Miller

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1967: **Anno mirabilis for nonabelian derived functors**

$$F : \mathcal{B} \to \mathcal{A}$$

Three distinguishable approaches:

Jonathan Beck: Cotriple resolutions.

If the notion of projectives in $\mathcal{B}$ is given by a cotriple

$$T : \mathcal{B} \to \mathcal{B} \quad , \quad \epsilon : T \to I \qquad \delta : T \to T^2$$

Example: Commutative algebras, TB given by the symmetric algebra on the set underlying B.

(T, ϵ, δ) determines for each $B \in \mathcal{B}$ a simplicial object

$$T_\bullet B : \quad TB \Leftarrow T^2 B \Lleftarrow T^3 B \cdots$$

and an augmentation to B. This is to be thought of as a projective resolution. Define derived functors by

$$L_*^T F(B) = \pi_*(FT_\bullet B) = H_*(N(FT_\bullet B))$$

Michel André: Résolutions pas-à-pas.

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Daniel Quillen: Cofibrant replacements..

Quillen characterized what properties you expect of a projective resolution, and established an axiomatic system guaranteeing they exist and are unique up to homotopy: <u>Model categories</u>. A projective resolution is a "cofibrant approximation."

All three extend to defining derived functors of F applied to a <u>simplicial</u> object in $\mathcal{B}$.

Compare and contrast:

- **Beck's cotriple resolutions** are canonical.

- **André's résolutions pas-à-pas** are small.

- **Quillen's cofibrant replacements** are conceptual and flexible.

Abelianization: All three authors told us what the fundamental functor to derive is:

$$\mathsf{Ab} : \mathcal{B} \rightleftarrows \mathsf{Ab}\,\mathcal{B} : u$$

[See recent work of Martin Frankland for conditions guaranteeing that Ab exists and that $\mathsf{Ab}\,\mathcal{B}$ is an abelian category.]

Example: Given ring homomorphism $u : A \to C$, let $\mathcal{B} = \mathsf{Fac}(u)$, the category whose objects are factorizations

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr] & & C \\ & B \arrow[ur] & \end{tikzcd}$$

Then

$$\mathsf{Ab}\,\mathcal{B} = C\text{--mod} \quad , \quad \mathsf{Ab}\,B = \Omega_{B/A} \otimes_B C$$

Definitions: Given $B \in \mathcal{B}$ or $B \in s\mathcal{B}$, let $X \to B$ be a cofibrant replacement. The cotangent complex of B is

$$L_B = \mathsf{Ab}(X)$$

and the homology of B is

$$H_*(B) = \pi_*(L_B) = L_*\mathsf{Ab}(B)$$

Example: $\mathcal{B} = \mathsf{Fac}(A \to C)$: in André's notation,

$$H_*(A \to B \to C) = H_*(A, B, C)$$

Sub-example: $A = C$ and $u = 1$: $\mathcal{B}$ is the category of augmented A-algebras. Let $I = \ker(B \to A)$. Then

$$\mathsf{Ab}(B) = I/I^2 = QB$$

so

$$H_*(B) = L_*Q(B)$$

Topology. "Space" = "pointed simplicial set"

Goal: Compute $\pi_n(\text{map}_*(X,Y),*)$, knowing only the mod p cohomologies $H^*(X)$ and $H^*(Y)$.

$H^*(X)$ is almost the subject of commutative algebra. Two differences:

– $H^*(X)$ is a graded commutative $\mathbb{F}_p$-algebra

– $H^*(X)$ supports extra symmetries, natural endomorphisms generated by

$$\mathrm{P}^n : H^i \to H^{i+2(p-1)n} \quad , \quad p \neq 2$$

$$\beta : H^i \to H^{i+1}$$

or

$$\mathrm{Sq}^n : H^i \to H^{i+n} \quad , \quad p = 2$$

$$\mathrm{P}^0 = 1 \quad , \quad \mathrm{Sq}^0 = 1$$

These operations satisfy universal relations and generate the Steenrod algebra $\mathcal{A}$.

Their action on $H^*(X)$ satisfies added unstable conditions

$$\mathsf{P}^n x = 0 \quad \text{if} \quad n > |x|/2$$

$$\beta \mathsf{P}^n x = 0 \quad \text{if} \quad n \geq |x|/2$$

$$\mathsf{Sq}^n x = 0 \quad \text{if} \quad n > |x|$$

Write $\mathcal{U}$ for the category of $\mathcal{A}$ modules satisfying these conditions.

An <u>unstable $\mathcal{A}$-algebra</u> is a graded commutative algebra structure on an unstable $\mathcal{A}$ module such that

$$\mathsf{P}^n x = x^p \quad \text{if} \quad n = |x|/2$$

$$\mathsf{P}^n(xy) = \sum_{i+j=n} \mathsf{P}^i x \cdot P^j y$$

$$\beta(xy) = \beta x \cdot y \pm x \cdot \beta y$$

$$\mathsf{Sq}^n x = x^2 \quad \text{if} \quad n = |x|$$

$$\mathsf{Sq}^n(xy) = \sum_{i+j=n} \mathsf{Sq}^i x \cdot \mathsf{Sq}^j y$$

Write $\mathcal{K}$ for the category of augmented unstable $\mathcal{A}$-algebras.

There is an adjoint pair

$$G : \mathbb{F}_p\text{−mod} \rightleftarrows \mathcal{K} : u$$

We know this is the complete list of operations and relations, by virtue of the Serre-Cartan calculation of the cohomology of Eilenberg Mac Lane spaces:

$$H^n(X) = [X, K(\mathbb{F}_p, n)]$$

Write

$$K(V) = \prod_n K(V_n, n)\,,\ V \text{ a graded vector space}$$

Then (if V is of finite type)

$$H^*(K(V)) = G(V)$$

Goal: Compute $\pi_n(\mathsf{map}_*(X,Y),*)$, knowing only $H^*(X)$ and $H^*(Y)$.

$$\pi_n(\mathsf{map}_*(X,Y)) = [\Sigma^n X, Y]$$

$$\pi_0(\mathsf{map}_*(X,Y)) \to \mathsf{Map}_{\mathcal{K}}(H^*(Y), H^*(X))$$

$$\pi_n(\mathsf{map}_*(X,Y),*) \to \mathsf{Map}_{\mathcal{K}}(H^*(Y), H^*(\Sigma^n X))$$

If $Y = K(V)$, then (under finite type assumptions) these maps are isomorphisms.

The technology of Bousfield and Kan lets us "resolve" Y by $K(V)$'s, and we get the "Adams spectral sequence"

$$E_2^{s,n} = \mathrm{Ext}^s_{\mathcal{K}}(H^*(Y), H^*(\Sigma^n X))$$

$$\Longrightarrow \pi_{n-s}(\mathrm{map}_*(X,Y),*)$$

For $n > 0$, $H^*(\Sigma^n X)$ is an abelian object in $\mathcal{K}$, and the E_2-term is a form of "Quillen cohomology":

$$\mathrm{Ext}^s_{\mathcal{K}}(H^*(Y), H^*(\Sigma^n X))$$

$$= \pi^s(\mathrm{Map}_{\mathcal{K}}(P_\bullet, H^*(\Sigma^n X)))$$

where

$$H^*(Y) \leftarrow P_\bullet$$

is a cofibrant replacement in $s\mathcal{K}$.

This Ext looks hard to compute. But —

Products vanish in a suspension, so any map in $\mathcal{K}$ factors though the module of indecomposables: For $n > 0$,

$$\mathrm{Map}_{\mathcal{K}}(H^*(Y), H^*(\Sigma^n X)) =$$

$$\mathrm{Hom}_{\mathcal{V}}(QH^*(Y), \Sigma^n \overline{H}^*(X))$$

Here $\mathcal{V}$ is the abelian category of "strictly unstable" $\mathcal{A}$-modules, in which

$$\mathrm{P}^n x = 0 \quad \text{if} \quad n \geq |x|/2$$

$$\beta\mathrm{P}^n x = 0 \quad \text{if} \quad n \geq |x|/2$$

$$\mathrm{Sq}^n x = 0 \quad \text{if} \quad n \geq |x|$$

so that when $p = 2$,

$$M \in \mathcal{U} \Leftrightarrow \Sigma M \in \mathcal{V}$$

This is set up so that

$$Q : \mathcal{K} \to \mathcal{V}$$

This functor carries projectives to projectives, so we get a composite functor spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{V}}(L_*Q(H^*(Y)), \Sigma^n \overline{H}^*(X))$$
$$\Longrightarrow \mathrm{Ext}^{s+t,*}_{\mathcal{K}}(H^*(Y), H^*(\Sigma^n X))$$

This is characteristic of how André-Quillen homology enters in topology: You separate out the operations, and what is left is just (graded) commutative algebra. In characteristic zero there are no Steenrod operations and the link is tighter.

Case. If $H^*(Y)$ is polynomial, then

$$L_n Q(H^*(Y)) = 0 \quad \text{for} \quad n > 0$$

so the composite functor spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{V}}(L_t Q(H^*(Y)), \Sigma^n \overline{H}^*(X))$$

$$\Longrightarrow \mathrm{Ext}^{s+t,*}_{\mathcal{K}}(H^*(Y), H^*(\Sigma^n X))$$

collapses to

$$\mathrm{Ext}^{s,n}_{\mathcal{K}}(H^*(Y), H^*(\Sigma^n X))$$

$$= \mathrm{Ext}^s_{\mathcal{V}}(QH^*(Y), \Sigma^n \overline{H}^*(X))$$

A story. Northwestern, Spring, 1982. $p = 2$.

The category $\mathcal{U}$ of unstable $\mathcal{A}$ modules has injective objects. John Harper the elder and I were thinking about them.

Mark Mahowald observed that these were (the cohomology modules of the dual) Brown-Gitler spectra.

Now Gunnar Carlsson had just shown that $\overline{H}^*(\mathbb{R}P^\infty)$ splits off of a limit of these $\mathcal{A}$-modules. The result was

Theorem. $\overline{H}^*(\mathbb{R}P^\infty)$ is an injective in $\mathcal{U}$.

In his MIT notes "Geometric Topology, Localization, Periodicity, and Galois Symmetry," Dennis Sullivan had asked a question, of which a special case was the following:

For X a finite pointed complex, is

$$\text{map}_*(\mathbb{R}P^\infty, X) \simeq *$$

I realized that I could now prove this theorem; in fact

$$\text{map}_*(BG, X) \simeq *$$

for any finite group and any finite complex. I did not realize then how useful this theorem would be.

There were a few things to verify. The Adams spectral sequence technology and some tricks with the fundamental group showed that what I needed to show was that if $B \in \mathcal{K}$ is bounded above then for all $n \geq s \geq 0$

$$\mathrm{Ext}^s_{\mathcal{K}}(B, H^*(\Sigma^n \mathbb{R}P^\infty)) = 0$$

Injectivity of $H^*(\mathbb{R}P^\infty)$ showed that in the composite functor spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathcal{V}}(L_t Q(B), \Sigma^n \overline{H}^*(\mathbb{R}P^\infty))$$

$$\Longrightarrow \mathrm{Ext}^{s+t,*}_{\mathcal{K}}(B, H^*(\Sigma^n \mathbb{R}P^\infty))$$

the E_2 term would be zero <u>provided that</u> $L_t Q(B)$ is bounded for all t.

Under finite type hypotheses, this is a finiteness result that André had proved!

Actually, I needed something a bit stronger, and I used the homotopy theory of the category $s\mathcal{B}$ of simplicial commutative augmented k-algebras. For $B \in s\mathcal{B}$ is a Hurewicz map

$$\pi_*(B) \to H_*(B)$$

A bigraded vector space $V_{*,*}$ has an exponential bound c provided that $V_{s,n} = 0$ for all $n > cp^s$.

Theorem. Let $B_\bullet \in \mathcal{B}$. If $\pi_*(B_\bullet)$ is exponentially bounded then so is $H_*(B_\bullet)$.

In particular if B is a constant object which is zero in large degrees, then each of its André-Quillen homology groups is bounded.

The homotopy of B has a lot of structure, which is explicitly known when $k = \mathbb{F}_p$. It is a graded commutative algebra. Its ideal of elements of degree at least 2 has divided powers. In addition (Cartan, Bousfield, Dwyer) there are natural operations ($p = 2$)

$$\delta_n : \pi_i(B) \to \pi_{n+i}(B) \quad , \quad 2 \leq n \leq i$$

and

$$\delta_n x = \gamma_2 x \quad \text{if} \quad n = |x|$$

The Hurewicz map factors as

$$\begin{tikzcd} \pi_*(B) \arrow[r] \arrow[d] & H_*(B) \\ Q\pi_*(B) \arrow[r] & k \otimes_D Q\pi_*(B) \arrow[u, "h"'] \end{tikzcd}$$

and for B a free simplicial k-algebra, h is an isomorphism. Since you can resolve into frees, you get a spectral sequence

$$L_*(k \otimes_D Q)(\pi_*(B)) \Longrightarrow H_*(B)$$

Since Q carries frees to frees, we get a composite functor spectral sequence

$$\mathrm{Untor}_s^D(k, L_t Q(\pi_*(B))) \Longrightarrow L_*(k \otimes_D Q)(\pi_*(B))$$

and

$$L_* Q(\pi_*(B)) = H_*(\pi_*(B))$$

Applying this machinery to a constant algebra B will be useless. But homology commutes with suspension, so we can replace B by its suspension. In the category $s\mathcal{B}$, there is a cofiber sequence

$$B \to WB \to \overline{W}B$$

with WB contractible, so $\overline{W}B = \Sigma B$, and

$$\mathrm{Tor}_*^B(k,k) = \pi_*(\Sigma B)$$

This gives us nontrivial spectral sequences.

Much better, though: ΣB is a co-H-space, so its homotopy has a diagonal: it is a Hopf algebra. Hopf algebras are complete intersection algebras, and one of the things André had proven (by "very beautiful arguments"—Quillen) was that there are then only two nonzero homology groups. So the group

$$\mathrm{Untor}_*^D(k, H_*(\mathrm{Tor}_*^B(k,k)))$$

is not so hard to compute, and this leads to the boundedness result I needed.

The Hurewicz map for ΣB factors:

$$
\begin{tikzcd}
\mathsf{Tor}^B_*(k,k) \arrow[d] \arrow[dr] & \\
Q\mathsf{Tor}^B_*(k,k) \arrow[r] \arrow[d] & H_{*-1}(B) \\
Q\mathsf{Tor}^B_*(k,k)/PD \arrow[d] \arrow[ur] & \\
k \otimes_D \mathsf{Tor}^B_*(k,k) \arrow[uur, "\text{edge}"'] &
\end{tikzcd}
$$

André studied the map from $Q\mathrm{Tor}_*^B(k,k)/DP$. He gave an example showing that one of the δ_i operations was nontrivial on the PD-indecomposables. We can see that failure of injectivity can occur because of other operations or because of differentials in the spectral sequence.

Also, $L_s(k \otimes_D Q)(\mathrm{Tor}_*^B(k,k))$ for $s > 0$ holds potential classes in the cokernel of this map. There is a lot more to learn about this situation.