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THE SEGAL CONJECTURE FOR ELEMENTARY ABELIAN p-GROUPS

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§1. INTRODUCTION

CARLSSON's proof of the Segal conjecture [2, 3] depends on an input from calculation; the object of this paper is to provide the input needed.

More precisely, we originally confirmed by calculation that a non-equivariant form of Segal's conjecture, describing the cohomotopy of the classifying space BG, is true when G is an elementary abelian p-group. Our approach was to calculate the cohomotopy groups $\pi^r(BG)$ by an Adams spectral sequence, and so most of the work lay in computing the requisite Ext groups over the Steenrod algebra A.

Carlsson [2, 3] invented an inductive argument, which proves the Segal conjecture in general, provided one can assume as input that an equivariant form of the Segal conjecture is true when G is an elementary abelian p-group. This he deduced from our non-equivariant result, by quoting work of Lewis, May and McClure [6].

Carlsson [2, 3] also observed that while his inductive argument by itself does not suffice to prove the case of an elementary abelian p-group, it does enable one to reduce the input from calculation. Instead of calculating the cohomotopy group $\pi^r(BG)$, it is sufficient to calculate the relevant group in Carlsson's "fundamental exact sequence", and to prove that the boundary map in this exact sequence is an isomorphism. For this we refer the reader to May and Priddy [10].

With this reduction, there is still work to be done in calculating an Ext group, but the work is less. Once this Ext group is calculated, there are two ways to calculate the relevant group in Carlsson's exact sequence. (i) One may follow Carlsson and reduce problems of equivariant homotopy theory to problems in non-equivariant homotopy theory; one then resolves the latter by using the classical Adams spectral sequence. (ii) Alternatively, one may set up an equivariant analogue of the Adams spectral sequence, capable of answering at least some of the problems of equivariant homotopy theory; one then uses an equivariant spectral sequence directly to calculate the relevant group in Carlsson's "fundamental exact sequence". The Ext group needed is the same either way.

In this paper we will not supply details for either of the arguments (i), (ii) above. We prefer not to write out (i) because it involves no essential novelty or serious difficulty, and because (ii) may well be preferable in the long run. We prefer not to write out (ii) because this method needs time to mature. We will therefore take as our object the calculation of the relevant Ext group, and we regard this as the theorem whose proof we have a duty to publish.

We also have results about other Ext groups which arise in studying the Segal conjecture, including those needed for our original calculation of $\pi^r(BG)$, but for these a statement and sketch proof will suffice.

The minimum which will serve Carlsson's purpose is provided by parts (a), (b) of the following result. We will explain it after stating it—but we assume that p is a fixed prime and V is an elementary abelian p-group of rank n.

THEOREM 1.1. (a) *The quotient map*

$$H^*(V)_{loc} \to F_p \otimes_A H^*(V)_{loc}$$

is a Tor-equivalence.

(b) $F_p \otimes_A H^*(V)_{loc}$ *is zero except in degree* $-n$, *where its rank is* $p^{n(n-1)/2}$.

(c) *The representation of* $Aut(V) = GL(V) = GL(n, F_p)$ *afforded by* $F_p \otimes_A H^*(V)_{loc}$ *is the Steinberg representation* [16].

(d) *A base for* $F_p \otimes_A H^*(V)_{loc}$ *is provided by the elements*

$$g(e_1 x_1^{-1} e_2 x_2^{-1} \dots e_n x_n^{-1}), \quad g \in Syl(V).$$

Here we write $H^*(G)$ for $H^*(BG; F_p)$. We use the letters U, V, $W \dots$ for elementary abelian p-groups because they often have to be regarded as vector spaces over F_p. We define $H^*(V)_{loc}$ by localizing $H^*(V)$ so as to invert $\beta h \in H^2(V)$ for every non-zero $h \in H^1(V)$. The ring $H^*(V)_{loc}$ is an algebra over the mod p Steenrod algebra A. If M is an A-module, $F_p \otimes_A M$ is regarded as a quotient A-module on which A acts trivially. We say that a map $\theta : L \to M$ of A-modules is a "Tor-equivalence" if the induced map

$$\theta_* : \mathrm{Tor}^A_{**}(F_p, L) \to \mathrm{Tor}^A_{**}(F_p, M)$$

is iso. The point of this emerges from the following result.

PROPOSITION 1.2. *If* $\theta : L \to M$ *is a Tor-equivalence, then the induced map*

$$\theta_* : Tor^A_{**}(K, L) \to Tor^A_{**}(K, M)$$

is iso for any (right) A-module K which is bounded above; the induced map

$$\theta^* : Ext_A^{**}(L, N) \leftarrow Ext_A^{**}(M, N)$$

is iso for any (left) A-module N which is bounded below and finite-dimensional over F_p *in each degree.*

The hypotheses of boundedness are essential. The proof will be omitted on the grounds that it is sufficiently obvious.

In (1.1) (d), the elements $e_1, e_2, \dots, e_n$ are a base chosen in $H^1(V)$, which may be identified with V^*, the dual of V. We then set $x_r = \beta e_r$, so that $x_1, x_2, \dots x_n$ are a corresponding base in $\beta V^* \subset H^2(V)$. Thus, whether $p > 2$ or $p = 2$, $H^*(V)$ contains a symmetric or polynomial algebra $S[\beta V^*]$ on generators $\{x_r\}$. Syl(V) is the subgroup of $GL(V)$ consisting of upper unitriangular matrices (with respect to these bases); it is a Sylow subgroup of $GL(V)$. We keep all this notation as standard, except that in the case $n = 1$, $V = Z_p$ we write e, x for the generators e_1, x_1.

The case $n = 1, p = 2$ of (1.1) is due to [8], while the case $n = 1, p > 2$ is due to [4]. Thus (1..1)) generalizes results previously known to be relevant to the Segal conjecture.

Our proof of Theorem 1.1 is based upon the "Singer construction" [14, 15, 7]. For the moment we need only explain three points about this. First, the Singer construction gives a functor $T(M)$ from A-modules to A-modules, which comes provided with a natural transformation $\varepsilon : T(M) \to M$. Secondly, the Singer construction allows one to reduce the calculation of Ext groups for a larger module, namely $T(M)$, to the calculation of Ext for a smaller module, namely M.

THEOREM 1.3. *The map* $\varepsilon : T(M) \to M$ *of Singer's construction is a Tor-equivalence.*

This reduction theorem was originally found by the second and third authors independently.

Thirdly, there is a relation between $H^*(V)_{loc}$ and the iterated Singer construction

$$T^n(F_p) = T(T(\dots T(F_p) \dots)).$$

THEOREM 1.4. *There is an isomorphism of A-algebras*

$$T^n(F_p) \cong H^*(V)^{Syl(V)}_{loc}.$$

Here M^G means the subobject of elements in M fixed under G, as usual. The localization required may be done by inverting

$$\Pi\beta h \mid h \in V^*, h \neq 0;$$

this element is fixed under $GL(V)$; it makes no difference whether we localize before or after passing to a subalgebra of fixed elements.

The case $p = 2$ of (1.4) is due to Singer [15], while the case $p > 2$ is modelled on a result of Li and Singer [7]. More precisely, Li and Singer prove the corresponding result for the subalgebra of invariants $H^*(V)_{\mathrm{loc}}^{\mathrm{Bor}(V)}$, where $\mathrm{Bor}(V)$ is the Borel subgroup of upper-triangular matrices in $GL(V)$. At this point we should explain that for $p > 2$ our version of the "Singer construction" is not quite the same as that of Li and Singer [7]. Theorem 1.3 is true for both versions; but for the purposes of our proof, a reduction theorem like (1.3) grows more useful as $T(M)$ grows larger. Our version of $T(M)$ is (roughly speaking) (p-1) times as large as that of Li and Singer [7], and our subalgebra of invariants is (roughly speaking) $(p\text{-}1)^n$ times as large as theirs; this allows us to get closer to $H^*(V)_{\mathrm{loc}}$. (See §2).

Priddy and Wilkerson [13] have shown how the deduction of (1.1) (a), (b) from (1.3) and (1.4) may be illuminated by their observation that $H^*(V)_{\mathrm{loc}}$ is projective as a module over $F_p[GL(V)]$. However, we will indicate our original argument, which is elementary.

If we localize $H^*(V)$ less than in (1.1) then it becomes harder to prove homological results about it, but we can still do so. Let S be a subset of $\beta V^* \subset H^2(V)$. We form $H^*(V)_S$ by localizing $H^*(V)$ so as to invert all the non-zero elements of S. The ring $H^*(V)_S$ is an algebra over A. We assume $S > \{0\}$ and suppose given a non-zero element $x = x_1 \in S$.

THEOREM 1.5. *The map*

$$H^*(V)_S \xrightarrow{\{res_W\}} \bigoplus_W H^*(W)_{S \cap \beta W^*}$$

is a Tor-equivalence.

Here W runs over certain quotients of V, so that W^* runs over certain subspaces of V^*. More precisely, βW^* runs over complements in βV^* for the subspace $\langle x \rangle$ generated by x; that is, we require $\beta V^* = \langle x \rangle \oplus \beta W^*$. There are p^{n-1} choices for W. The A-maps

$$H^*(V)_S \xrightarrow{\mathrm{res}_W} H^*(W)_{S \cap \beta W^*}$$

will be explained in §8; they raise degree by 1.

(1.5) enables one to reduce the calculation of Ext groups for any localized algebra $H^*(V)_S$ to the unlocalized case. In fact, if on the right we have an algebra $H^*(W)_{S \cap \beta W^*}$ with $S \cap \beta W^*$ non-zero, then we may choose a non-zero element $x_2 \in S \cap \beta W^*$ and apply the theorem again, and so on by induction.

As our work was originally conceived, we needed to compute Ext groups for the unlocalized case (at least in terms of more familiar Ext groups). The best version of the result is conceptual, and we will give this version in §9; but in this introduction we avoid explaining it, stating instead a form which is more explicit. We assume that U and V are elementary abelian p-groups and that M is an A-module, bounded below and finite-dimensional over F_p in each degree.

THEOREM 1.6. *Then the map*

$$\bigoplus_X Ext_A^{s-s(X), t-s(X)}(H^*(W(X)), M) \xrightarrow{\omega} Ext_A^{s,t}(H^*(V), M \otimes H^*(U))$$

is iso.

Here we explain that in §9 we shall associate to U and V a finite set of indices X. We shall also associate to each index X an integer $s(X)$ and an elementary abelian p-group $W(X)$. Finally we shall introduce the map ω.

For $s = 0$, $t = 0$, $M = F_p$ we can do without the indexing apparatus: our result reduces to the statement that the obvious map

$$F_p[\mathrm{Hom}_{F_p}(U, V)] \to \mathrm{Hom}^0_A(H^*(V), H^*(U))$$

is an isomorphism. However, for $s > 0$ we need more indices than are provided by the F_p-maps from U to V.

The level of generality in (1.6) is such as to compute the E_2-term of the Adams spectral sequence

$$\mathrm{Ext}^{**}_A(H^*(\mathbf{B}V), H^*(\mathbf{T}) \otimes H^*(\mathbf{B}U) \Rightarrow [\mathbf{T} \wedge \mathbf{B}U, \mathbf{B}V]_*$$

for any suitable choice of the test-object $\mathbf{T}$. (The bold-face letters stand for spectra, and $\mathbf{B}G = \Sigma^\infty(BG \sqcup P)$.)

We can justify this level of generality by considering the proof of (1.6). This proof flows by a simple and inevitable induction over the rank of U; we sketch the step from "rank 1" to "rank 2". Obviously, if you can compute $\mathrm{Ext}^{**}_A(H^*(V), M \otimes H^*(Z_p))$ for general M, then you can substitute $M = L \otimes H^*(Z_p)$; since $H^*(Z_p) \otimes H^*(Z_p) = H^*(Z_p \times Z_p)$, you can compute Ext^{**}_A $(H^*(V), L \otimes H^*(Z_p \times Z_p))$ in terms of groups

$$\mathrm{Ext}^{**}_A(H^*(W(X)), L \otimes H^*(Z_p)),$$

which you can compute by the same token. Notice that if you want to compute the cohomotopy groups $\pi^*(BU)$, so that you want the final result only for $V = 0$ and $M = F_p$, you still need the inductive hypothesis in the generality given.

The body of this paper is arranged as follows. The proofs of (1.3), (1.4) and (1.1) (a), (b), (d) will be given in §2, §3 and §4 respectively. These proofs involve a certain amount of forward reference. In particular, we proceed by stating and using any fact about Singer's functor T which we know to be true; in §5 we sketch an approach to T which allows one to prove all these facts. Similarly, in §4 we use a proof by induction, which involves algebras of invariants $H^*(V)^G_{\mathrm{loc}}$ for various subgroups $G \subset \mathrm{Syl}(V)$. In §6 we explain the subgroups G concerned, and obtain information about these algebras of invariants $H^*(V)^G_{\mathrm{loc}}$. As a corollary, we justify an explicit description of the algebra of invariants $H^*(V)^{\mathrm{Syl}(V)}_{\mathrm{loc}}$, which is stated at (3.3) and used in §3.

§7 deals with the Steinberg representation and proves (1.1)(c). §8 proves (1.5).

The final sections, §9–§12, are devoted to sketching the proof of (1.6). We wish to draw the reader's attention to the categorical considerations involved in giving a conceptual statement of (1.6), and in particular to a construct we call the "Burnside category" $\mathscr{A}$; we hope it may be of wider use. We therefore urge the reader to study §9.

We are grateful to W. M. Singer for keeping us informed of his work, and similarly to G. Carlsson and to Priddy and Wilkerson. We are grateful to the Sloan Foundation, to the University of Aarhus, and to the University of Chicago for enabling us to meet in spite of our usual geographical separation. Finally, we thank the editors and referees of *Topology* (ably seconded by J. P. May) for lifting from our consciences the duty of publishing our proofs in such complete detail as may be more appropriate to private archives.

§2. PROOF OF (1.3)

To prove (1.3), we shall need some facts about the Singer construction.

Additively, the Singer construction $T(M)$ is isomorphic to the tensor product $L \otimes M$ of M with a fixed object L. (However, the A-module structure on $T(M)$ is not given by the usual "diagonal" formula.) Just as one assigns to each cohomology theory K^* the "coefficient groups" $K^*(P)$, so to each functor T from A-modules to A-modules one assigns the "coefficient module" $T(F_p)$. In our case $T(F_p)$ is $L \otimes F_p$, that is, L; thus L becomes an A-module, and plays the role of a "coefficient module" for the Singer construction. It is usual to write T for this coefficient module, and to write $T(M) = T \otimes M$. In fact T is an A-algebra;

and the obvious action of T on $T(M) = T \otimes M$ is an A-action, that is, it satisfies the Cartan formula.

We can now explain the relationship between our version of the Singer construction (say $T(M)$) and the version of Li and Singer (say $T'(M)$). In our version the coefficient algebra is $T = H^*(Z_p)_{\mathrm{loc}}$, the case $n = 1$ of the algebra considered in (1.1). In the version of Li and Singer the coefficient algebra is $T' = H^*(\Sigma_p)_{\mathrm{loc}}$. Here Σ_p is the symmetric group on p letters; the inclusion $Z_p \to \Sigma_p$ induces an inclusion

$$H^*(\Sigma_p) \to H^*(Z_p).$$

We localize $H^*(\Sigma_p)$ by inverting x^{p-1}. If the version $T'(M)$ of the Singer construction is taken as known, one can define the version $T(M)$ by

$$T(M) = T \otimes_{T'} T'(M).$$

We shall need some facts about $T'(M)$. The instance $T'(A)$ must be a bimodule over A (for any $b \in A$, the map $a \mapsto ab$: $A \to A$ is a map of left A-modules, and $T'(-)$ is a functor). For homological purposes, the most convenient way to construct $T'(M)$ is to give an explicit description of the bimodule $T'(A)$, and then set

$$T'(M) = T'(A) \otimes_A M.$$

We shall give a description of this sort.

As in [8] we use the dual A_* of the mod p Steenrod algebra A [11]. This dual has exterior generators $\tau_0, \tau_1, \ldots$ and polynomial generators $\xi_1, \xi_2, \ldots$. (We omit the modifications necessary in the case $p = 2$, which are standard.) We have to use the usual finite subalgebras of the Steenrod algebra. We write $A_*(n)$ for the quotient $A_*/I(n)$, where the ideal $I(n)$ is generated by the τ_r with $r > n$ and the $\xi_r^{p^s}$ with $r + s \geq n + 1$. The quotient $A_*(n)$ is dual to a sub-Hopf-algebra $A(n)$ of A. The subalgebra $A(-1)$ is F_p; the subalgebra $A(0)$ is the exterior algebra generated by β.

We also introduce a localized quotient

$$B_*(n) = (A_*/J(n))\,[\xi_1^{-1}] \quad (n \geq 0)$$

where the ideal $J(n)$ is generated by the τ_r with $r > n$ and the $\xi_r^{p^s}$ with $r \geq 2, r + s \geq n + 1$. The object $A_*/J(n)$ is a left comodule over $A_*(n)$ and a right comodule over $A_*(n-1)$. Multiplication by $\xi_1^{p^n}$ preserves both comodule structures. Since $B_*(n)$ may be regarded as the direct limit of $A_*/J(n)$ under multiplication by $\xi_1^{p^n}$, it becomes a left comodule over $A_*(n)$ and a right comodule over $A_*(n-1)$. It is also an algebra, and is finite-dimensional over F_p in each degree.

We define $B(n)$ to be the dual of $B_*(n)$. This object is a bimodule; it is a left module over $A(n)$ and a right module over $A(n-1)$.

For example, $B_*(0)$ has a base consisting of the elements ξ_1^k and $\tau_0 \xi_1^k$ for $k \in Z$. We take the dual base in $B(0)$ and call its elements P^k and βP^k for $k \in Z$.

Since we have canonical maps $A_* \to B_*(n+1) \to B_*(n)$, wwe have canonical maps $B(n) \to B(n+1) \to A$ preserving all the relevant structure. The element written P^k in $B(0)$ maps to P^k in A if $k \geq 0$, to 0 if $k < 0$; similarly for βP^k.

LEMMA 2.1. (i) *$B(n)$ is free as a left module over $A(n)$; the elements P^k with $k \equiv 0 \bmod p^n$ may be taken as a base; the left-primitive submodule of $B_*(n)$ is $F_p[\xi_1^{p^n}, \xi_1^{-p^n}]$.*

(ii) *$B(n)$ is free as a right module over $A(n-1)$; the elements $P^k, \beta P^k$ with $k \in Z$ may be taken as a base. Equivalently, the map $B(0) \otimes A(n-1) \to B(n)$ is iso.*

We defer the proof in order to complete our description of $T'(M)$.

If M is an $A(n-1)$-module, we may now construct

$$B(n) \otimes_{A(n-1)} M;$$

this is an $A(n)$-module. If M is an $A(n)$-module, then the canonical map

$$B(n) \otimes_{A(n-1)} M \to B(n+1) \otimes_{A(n)} M$$

is iso, since both groups are isomorphic to $B(0) \otimes M$ by (2.1) (ii). Thus the construction is essentially independent of n. If M is an A-module, we may now construct the attained limit

$$\underset{n}{\varinjlim} \; (B(n) \otimes_{A(n-1)} M),$$

and this is an A-module.

Unfortunately this is not yet exactly what we want for $T'(M)$. The relevant isomorphism between $B(n) \otimes_{A(n-1)} F_p$ and $H_*(\Sigma_p)_{\text{loc}}$ is of degree -1, while for any purpose which involves products we want the isomorphism between T' and $H^*(\Sigma_p)_{\text{loc}}$ to have degree 0. While we defer any more detail to §5, we indicate that it is better to define a preliminary version $T''(M)$ of the Singer construction to be $B(n) \otimes_{A(n-1)} M$ or $\underset{n}{\varinjlim}\, B(n) \otimes_{A(n-1)} M$ according to the case, and to define $T'(M)$ as isomorphic to $T''(M)$ under an isomorphism which changes degrees by 1.

If M is an A-module, the map

$$B(n) \otimes_{A(n-1)} M \to A \otimes_{A(n-1)} M \to M$$

passes to the limit, and gives a map of A-modules

$$\varepsilon'' \colon T''(M) = \underset{n}{\varinjlim} \; B(n) \otimes_{A(n-1)} M \to M.$$

Replacing $T''(M)$ by the isomorphic A-module $T'(M)$, we get

$$\varepsilon' \colon T'(M) \to M$$

which is now an A-map of degree $+1$. To construct

$$\varepsilon \colon \; T(M) = T \otimes_{T'} T'(M) \to M,$$

one first projects $T = H^*(Z_p)_{\text{loc}}$ onto the direct summand $T' = H^*(\Sigma_p)_{\text{loc}}$ and then applies ε'.

This completes all we need explain about the Singer construction in order to prove (1.3).

Proof of (2.1). (i) It is clear that as a left comodule over $A_*(n)$, $B_*(n)$ is a direct sum of copies of $A_*(n)$ shifted by multiplication with the powers $\xi_1^{rp^n}$, $r \in Z$.

(ii) It is easy to show that in A, the elements P^k, βP^k with k sufficiently large (say $k \geq k_0$) are linearly independent under right multiplication by $A(n-1)$. Using the canonical map $B(n) \to A$, we see that the same result holds also in $B(n)$.

We can deduce that all the elements P^k, βP^k in $B(n)$ are linearly independent under right multiplication by $A(n-1)$. In fact, multiplication by $\xi_1^{-rp^n}$ gives a linear map $B_*(n) \to B_*(n)$ which is a map of bicomodules. Its dual is a linear map $B(n) \to B(n)$ which is a map of bimodules. Suppose we had any linear relation over $A(n-1)$ between the elements P^k, βP^k in $B(n)$; by applying this map for a suitable value of r, we could shift the relation up until it involved only elements P^k, βP^k with $k \geq k_0$.

Thus we see that the map

$$B(0) \otimes A(n-1) \to B(n)$$

is mono. On the other hand, the objects $B(0) \otimes A(n-1)$ and $B(n)$ have the same (finite) dimension over F_p in each degree; so the map is iso. This proves (2.1).

LEMMA 2.2. (i) *If M is A-free then $T(M)$ is A-flat.*

(ii) *If M is A-free then the map*

$$F_p \otimes_A T(M) \xrightarrow{1 \otimes \varepsilon} F_p \otimes_A M$$

is iso.

Proof. (i) If M is A-free then it is $A(n-1)$-free. If M is free over $A(n-1)$ then $B(n)\otimes_{A(n-1)}M$ is a direct sum of copies of $B(n)$, so it is free over $A(n)$ by (2.1) (i). This shows that $T'(M)$ is free over $A(n)$. Over $A(n), T(M) = T\otimes_{T'}T'(M)$ is a direct sum of $(p-1)$ copies of $T'(M)$, because multiplication by x^{p^n} gives a shift map commuting with $A(n)$. Therefore $T(M)$ is free over $A(n)$; this holds for all n. So

$$\mathrm{Tor}^{A}_{s,t}(K, T(M)) = \varinjlim_{n} \mathrm{Tor}^{A(n)}_{s,t}(K, T(M))$$
$$= 0 \text{ for } s > 0.$$

Thus $T(M)$ is A-flat.

(ii) It is sufficient to prove the special case $M = A$, for the general case follows by passing to direct sums. By (2.1) (i), $B(n)\otimes_{A(n-1)}A(n-1)$ is $A(n)$-free on generators P^k, $k \equiv 0 \bmod p^n$. Thus $F_p\otimes_{A(n)}T''(A(n-1))$ is F_p-free on generators P^k, $k \equiv 0 \bmod p^n$. Passing to the limit over n, we see that $F_p\otimes_A T''(A)$ is F_p-free on one generator P^0. Thus the map

$$F_p\otimes_A T''(A) \xrightarrow{1\otimes\varepsilon''} F_p\otimes_A A = F_p$$

is iso. Therefore the corresponding result holds for T'. For T, we can use x^{p^n} as a shift map, as above; we see that $F_p\otimes_{A(n)}T(A(n-1))$ is F_p-free on generators in degrees congruent to -1 mod $2p^n$. Passing to the limit over n, we see that $F_p\otimes_A T(A)$ is zero except in degree -1. This proves (2.2).

The deduction of (1.3) from (2.2) may be omitted as routine.

§3. PROOF OF (1.4)

Let us take $V = Z_p \times W$, so that

$$H^*(V) \cong H^*(Z_p)\otimes H^*(W).$$

THEOREM 3.1. *There is an isomorphism of A-algebras*

$$H^*(V)^{Syl(V)}_{loc} \cong T(H^*(W)^{Syl(W)}_{loc}).$$

Here Syl(V) and Syl(W) are groups of upper unitriangular matrices with respect to bases chosen so that e_1 is a base in Z_p^*, $e_2, e_3, \ldots, e_n$ are a base in W^*, and $x_r = \beta e_r$ as in §1.

If (3.1) is granted, (1.4) will follow immediately by induction over the rank n of V.

In order to construct the isomorphism in (3.1), we need to know more about the Singer construction. It comes provided with a structure map

$$T(M) = T\otimes M \xrightarrow{f} T\hat{\otimes} M.$$

Here $T\hat{\otimes}M$ is a completed tensor product; we get it by completing $T\otimes M$ with respect to a topology in which a typical neighbourhood of zero is

$$\left(\sum_{r\le -N} T^r\right)\otimes M.$$

A typical element of $T\hat{\otimes}M$ is a "downward-going formal Laurent series"

$$\sum_{r\le R} x^r\otimes m'_r + \sum_{r\le R} ex^r\otimes m''_r$$

where e, x are the generators in $T = H^*(Z_p)_{loc}$. To make A act on $T\hat{\otimes}M$, we take the usual (diagonal) action on $T\otimes M = H^*(Z_p)_{loc}\otimes M$ and pass to the completion.

The map f is an A-map and a map of T-modules; it is always mono. If M is an A-algebra, then the obvious product on $T\otimes M$ makes $T(M)$ and $T\hat{\otimes}M$ into algebras, and f becomes a map of algebras.

We apply this with $M = H^*(W)_{\mathrm{loc}}$. Since $V = Z_p \times W$ we have an embedding

$$H^*(V) \subset T \otimes H^*(W)_{\mathrm{loc}} \subset T \hat{\otimes} H^*(W)_{\mathrm{loc}}.$$

This embedding extends to the localization $H^*(V)_{\mathrm{loc}}$. (For any element c of degree 2 in $H^*(W)_{\mathrm{loc}}$ the element $x+c$ is invertible in $T \hat{\otimes} M^*(W)_{\mathrm{loc}}$, with inverse

$$x^{-1} - x^{-2}c + x^{-3}c^2 - \ldots .)$$

A more precise version of (3.1) is now as follows.

Theorem 3.2. *The image of $H^*(V)_{loc}^{Syl(V)}$ under the embedding*

$$H^*(V)_{loc} \to T \hat{\otimes} H^*(W)_{loc}$$

is the same as the image of $T(H^*(W)_{loc}^{Syl(W)})$ under the embedding f.

To prove (3.2), we need to know the algebras of invariants and to calculate f. We refer to §6 for the following.

Proposition 3.3. *$H^*(V)_{loc}^{Syl(V)}$ is a free module on the 2^n generators*

$$f_1^{i_1} f_2^{i_2} \ldots f_n^{i_n} \quad (\text{where each } i_r \text{ is } 0 \text{ or } 1)$$

over the algebra of finite Laurent series

$$F_p[y_1, y_1^{-1}] \otimes F_p[y_2, y_2^{-1}] \otimes \ldots \otimes F_p[y_n, y_n^{-1}].$$

Here the generators f_r, y_r are defined as follows.

$$f_r = \begin{vmatrix} x_1^{p^{r-2}} & \cdots & x_r^{p^{r-2}} \\ \vdots & & \\ x_1^p & \cdots & x_r^p \\ x_1 & \cdots & x_r \\ e_1 & \cdots & e_r \end{vmatrix}$$

$$y_r = \begin{vmatrix} x_1^{p^{r-1}} & \cdots & x_r^{p^{r-1}} \\ \vdots & & \\ x_1^{p^2} & \cdots & x_r^{p^2} \\ x_1^p & \cdots & x_r^p \\ x_1 & \cdots & x_r \end{vmatrix}$$

These elements and their constructions go back to Mui [12]. The elements f_r and y_r are easily seen to be invariant under Syl(V). The determinant y_r is a product of factors which are non-zero elements of βV^*; thus y_r is invertible in $H^*(V)_{\mathrm{loc}}$.

We write g_r, z_r for the generators in $H^*(W)^{\mathrm{Syl}(W)}$ constructed in the same way as the generators f_r, y_r in $H^*(V)^{\mathrm{Syl}(V)}$.

Proposition 3.4. *The map*

$$T \otimes H^*(W)_{loc} \xrightarrow{f} T \hat{\otimes} H^*(W)_{loc}$$

has

$$f(x_1 \otimes 1) = y_1, \quad f(e_1 \otimes 1) = f_1$$

and

$$f(x_1^{p^r} \otimes z_r) = y_{r+1}, \quad f(x_1^{p^{r-1}} \otimes g_r) = f_{r+1} \quad \text{for } r \geq 1.$$

If (3.3) and (3.4) are granted, then (3.2) follows at once.

In order to calculate the map f, we need to know the Steenrod operations on the generators g_r, z_r for $H^*(W)^{\mathrm{Syl}(W)}_{\mathrm{loc}}$.

LEMMA 3.5. (i) *We have $P^k z_r = 0$ unless $k = (p^r - p^j)/(p-1)$ for some j such that $0 \leq j \leq r$. In this case*

$$P^k z_r = \begin{vmatrix} x_2^{p^r} & \cdots & x_{r+1}^{p^r} \\ \vdots & & \\ x_2^{p^{j+1}} & \cdots & x_{r+1}^{p^{j+1}} \\ x_2^{p^{j-1}} & \cdots & x_{r+1}^{p^{j-1}} \\ \vdots & & \\ x_2 & \cdots & x_{r+1} \end{vmatrix}.$$

(ii) *We have $\beta P^k z_r = 0$.*

(iii) *We have $P^k g_r = 0$ unless $k = (p^{r-1} - p^j)/(p-1)$ for some j such that $0 \leq j \leq r-1$. In this case*

$$P^k g_r = \begin{vmatrix} x_2^{p^{r-1}} & \cdots & x_{r+1}^{p^{r-1}} \\ \vdots & & \\ x_2^{p^{j+1}} & \cdots & x_{r+1}^{p^{j+1}} \\ x_2^{p^{j-1}} & \cdots & x_{r+1}^{p^{j-1}} \\ \vdots & & \\ x_2 & \cdots & x_{r+1} \\ e_2 & \cdots & e_{r+1} \end{vmatrix}.$$

(iv) *We have $\beta P^k g_r = 0$ unless $k = (p^{r-1} - 1)/(p-1)$. In this case*

$$\beta P^k g_r = z_r.$$

We calculate (i) by applying the total Steenrod power $p = \sum_{k=0}^{\infty} P^k$ to the determinant for z_r and evaluating the resulting determinant; similarly for (iii). Parts (ii) and (iv) follow.

Proof of (3.4). We need to known that f is a map of T-modules and satisfies the following explicit formula.

$$f(1 \otimes m) = \sum_{k \geq 0} (-1)^k x^{-k(p-1)} \otimes P^k m + \sum_{k \geq 0} (-1)^{k+1} e x^{-k(p-1)-1} \otimes \beta P^k m. \quad (3.6)$$

Using the Steenrod operations given by (3.5), we calculate as follows.

$$f(x_1^{p^r} \otimes z_r) = \sum_{0 \leq j \leq r} (-1)^{r-j} x_1^{p^j} \begin{vmatrix} x_2^{p^r} & \cdots & x_{r+1}^{p^r} \\ \vdots & & \\ x_2^{p^{j+1}} & \cdots & x_{r+1}^{p^{j+1}} \\ x_2^{p^{j-1}} & \cdots & x_{r+1}^{p^{j-1}} \\ \vdots & & \\ x_2 & \cdots & x_{r+1} \end{vmatrix}$$

$$= y_{r+1}.$$

Similarly for $f(x_1^{p^{r-1}} \otimes g_r)$.

This proves (3.4), which completes the proof of (3.1) and (1.4), modulo the facts used.

§4. PROOF OF (1.1)(a), (b), (d)

As this proof is by induction, we must formulate the inductive hypothesis. For suitable subgroups G normal in $\mathrm{Syl}(V)$, we prove the following.

THEOREM 4.1. (a) *The quotient map*

$$H^*(V)^G_{loc} \xrightarrow{q} F_p \otimes_A H^*(V)^G_{loc}$$

is a Tor-equivalence.

(b) $F_p \otimes_A H^*(V)^G_{loc}$ *is zero except in degree* $-n$.

(c) *In degree* $-n$ *it is of rank* $|Syl(V):G|$.

(d) *More precisely, a base for* $F_p \otimes_A H^*(V)^G_{loc}$ *is provided by the sums*

$$\sum_{\gamma \in \Gamma} \gamma(e_1 x_1^{-1} e_2 x_2^{-2} \ldots e_n x_n^{-1})$$

where Γ *runs over the cosets of* G *in* $Syl(V)$.

The special case $G = 1$ of (4.1) proves (1.1) (a), (b), (d). For more detail on the other subgroups G considered, see §6. In part (d), the notation is as in (1.1). It is clear that the sum

$$\sum_{\gamma \in \Gamma} \gamma(e_1 x_1^{-1} e_2 x_2^{-1} \ldots e_n x_n^{-1})$$

is invariant under G.

The first step in proving (4.1) is to prove the special case $G = \mathrm{Syl}(V)$. In this case we have

$$H^*(V)^{\mathrm{Syl}(V)}_{\mathrm{loc}} \cong T^n(F_p)$$

by (1.4). By (1.3) we have n Tor-equivalences

$$T^n F_p \xrightarrow{\varepsilon} T^{n-1} F_p \longrightarrow \cdots \longrightarrow TF_p \xrightarrow{\varepsilon} F_p$$

each of degree $+1$. Thus we have a Tor-equivalence (of degree n)

$$H^*(V)^{\mathrm{Syl}(V)}_{\mathrm{loc}} \xrightarrow{\phi} F_p.$$

It is now easy to deduce (4.1) (a), (b), (c) for $G = \mathrm{Syl}(V)$ by commuting q with ϕ.

We now proceed by downwards induction over G. For the subgroups G with which we work (see §6), the inductive step presents itself as follows. We have a subgroup F normal in G, with quotient $G/F \cong Z_p$ generated by g. We suppose as our inductive hypothesis that (4.1) (a), (b) are true for G, and we wish to deduce them for F. We have a filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_p = H^*(V)^F_{\mathrm{loc}}$$

of $H^*(V)^F_{\mathrm{loc}}$ by A-submodules M_j, in which each subquotient M_j/M_{j-1} is isomorphic to $H^*(V)^G_{\mathrm{loc}}$. The definition of the filtration is

$$M_j = \mathrm{Ker}\,(g-1)^j : H^*(V)^F_{\mathrm{loc}} \longrightarrow H^*(V)^F_{\mathrm{loc}}$$

and the isomorphism $M_j/M_{j-1} \to M_1/M_0 = H^*(V)^G_{\mathrm{loc}}$ is given by $(g-1)^{j-1}$.

We suppose, as the hypothesis of a subsidiary induction over j, that the quotient map

$$M_j \xrightarrow{q} F_p \otimes_A M_j$$

is a Tor-equivalence and that $F_p \otimes_A M_j$ is zero except in degree $-n$. Consider the following diagram.

$$\begin{tikzcd} & 0 \arrow[r] & M_j \arrow[r] \arrow[d, "q_j"] & M_{j+1} \arrow[r] \arrow[d, "q_{j+1}"] & H^*(V)^G_{\mathrm{loc}} \arrow[r] \arrow[d, "q"] & 0 \\ \mathrm{Tor}^A_{1,*}(F_p, H^*(V)^G_{\mathrm{loc}}) \arrow[rr] & & F_p \otimes_A M_j \arrow[r] & F_p \otimes_A M_{j+1} \arrow[r] & F_p \otimes_A H^*(V)^G_{\mathrm{loc}} \arrow[r] & 0 \end{tikzcd}$$

By hypothesis, $F_p \otimes_A M_j$ is zero except in degree $-n$. By the main inductive hypothesis,

$$\begin{aligned} \mathrm{Tor}^A_{1,-n}(F_p, H^*(V)^G_{\mathrm{loc}}) &\cong \bigoplus \mathrm{Torr}^A_{1,0}(F_p, F_p) \\ &= 0. \end{aligned}$$

So the lower sequence is short exact. We see that $F_p \otimes_A M_{j+1}$ is zero except in degree $-n$. Now we use the Five Lemma; by our hypotheses, q_j and q are Tor-equivalences, and therefore q_{j+1} is a Tor-equivalence. This completes the subsidiary induction, which runs up to $j = p$ and proves (4.1) (a), (b) for F.

It is easy to carry (4.1) (c) through this induction, and it is not hard to carry through (4.1) (d) provided we have the necessary starting-point, as follows.

LEMMA 4.2. *The sum*

$$\sum_{\gamma \in Syl(V)} \gamma(e_1 x_1^{-1} e_2 x_2^{-1} \ldots e_n x_n^{-1})$$

gives a non-zero element of $F_p \otimes_A H^*(V)_{\mathrm{loc}}$.

The proof of (4.2) is best approached by further remarks about the Singer construction. The map ε used to state and prove (1.3) can be factored through the map f used to prove (1.4), to give the following diagram.

$$\begin{tikzcd} T(M) \arrow[r, "f"] \arrow[dr, "\varepsilon"'] & T \hat{\otimes} M \arrow[d, "\mathrm{res}"] \\ & M \end{tikzcd}$$

More precisely, the map "res" is defined by

$$\mathrm{res}\left(\sum_{r \le R} x^r \otimes m'_r + \sum_{r \le R} e x^r \otimes m''_r \right) = m''_{-1}.$$

It is reasonable to think of this map as a "residue", since it takes the coefficient of the term of degree -1 in a Laurent series. The map res is an A-map of degree $+1$.

We can restrict the map res to parts of $T \hat{\otimes} M$ constructed by localization. These remarks, taken with §3, suggest the following. Take $M = H^*(W)_{\mathrm{loc}}$, as in §3. Restrict the map

$$T \hat{\otimes} H^*(W)_{\mathrm{loc}} \xrightarrow{\mathrm{res}} H^*(W)_{\mathrm{loc}}$$

to the subalgebra $H^*(V)_{\mathrm{loc}}$, embedded in $T \hat{\otimes} H^*(W)_{\mathrm{loc}}$ as in §3. We get an A-map

$$H^*(V)_{\mathrm{loc}} \xrightarrow{\mathrm{res}} H^*(W)_{\mathrm{loc}}.$$

LEMMA 4.3. *The map res carries the sum*

$$\sum_{\gamma \in Syl(V)} \gamma(e_1 x_1^{-1} e_2 x_2^{-1} \ldots e_n x_n^{-1})$$

in $H^*(V)_{loc}$ *to the sum*

$$\sum_{\delta \in Syl(W)} \delta(e_2 x_2^{-1} \ldots e_n x_n^{-1})$$

in $H^*(W)_{loc}$.

(4.3) follows from simple properties of the residue, and (4.2) follows easily from (4.3) by induction over the rank n of V.

The same ideas allow an alternative proof of (1.1) (d) (or (4.1) (d)). For this we use $p^{n(n-1)/2}$ different iterated residues to show that the elements

$$\gamma(e_1 x_1^{-1} e_2 x_2^{-1} \ldots e_n x_n^{-1}) \quad (\gamma \in \mathrm{Syl}(V))$$

give linearly independent elements of $F_p \otimes_A H^*(V)_{loc}$ to the full number allowed by (1.1) (b).

§5. THE SINGER CONSTRUCTION

To a topologist, the way to understand Singer's functor T is that it computes the limiting cohomology of a certain construction on spectra. For this we refer the reader to [1].

To a conceptual algebraist, the way to understand Singer's functor T is via the derived functors of the functor when takes any A-module and assigns to it the quotient where the "unstable" axiom is satisfied. For this we refer the reader to [5].

Although we realize the value and interest of these viewpoints, we neglect them for brevity. In this paper we need to treat Singer's functor T as a matter of computational algebra.

Indeed, our approach in §2, §3, §4 has been to prove the theorems at issue, stating as we go any necessary facts about T which we know to be true. However, the approach in §2 already provides a self-contained account of $T'(M)$ so far as its A-module structure goes; it is natural to ask if it can be elaborated into a self-contained account of $T(M)$ which proves all the results we have used (most notably (3.6)). The answer is that such an approach is possible; indeed, we have carried it out in detail; but in this section we will merely sketch the ideas needed.

The approach in §2 is sufficiently detailed up to the point where we define a preliminary version $T''(M)$ of the Singer construction to be $B(n) \otimes_{A(n-1)} M$ or $\varinjlim_n B(n) \otimes_{A(n-1)} M$, according as M is given as a module over $A(n-1)$ or over A. This gives $T''(M)$ as a module over $A(n)$ or A according to the case. It also gives the map $\varepsilon'' : T''(M) \to M$. However, we need to see a map f'' which we can later process to give the structure map f used in §3. For this purpose we first introduce a diagonal map

$$T''(M \otimes N) \xrightarrow{\Psi} T''(M) \hat{\otimes} T''(N)$$

by dualizing the product map in $B_*(n)$. We then form the composite

$$\begin{array}{rl} T''(M) = T''(F_p \otimes M) \xrightarrow{\Psi} & T''(F_p) \hat{\otimes} T''(M) \\ & \Big\downarrow {\scriptstyle 1 \hat{\otimes} \varepsilon''} \\ & T''(F_p) \otimes M \end{array}$$

and take it for our map

$$T''(M) \xrightarrow{f''} T''(F_p) \hat{\otimes} M.$$

The map f'' has good properties; one can give explicit formulae for it, and show that it is mono.

Next we need to see a ring of coefficients acting on $T''(M)$. We will introduce such a ring, and later reconcile it with the account in §2.

Let $T'(n)$ be the right-primitive subobject of $B_*(n)$. The maps

$$\cdots \to B_*(n+1) \to B_*(n) \to \cdots \to B_*(0)$$

induce

$$\cdots \to T'(n+1) \to T'(n) \to \cdots \to T'(0)$$

and all these maps are iso, since each $T'(n)$ is dual to a quotient $B(n) \otimes_{A(n-1)} F_p$ with base P^k, βP^k for $k \in Z$. Let us write T' for the (attained) limit $\varprojlim_n T'(n)$; this is an algebra. It contains an algebra $F_p[\xi,\xi^{-1}]$ of finite Laurent series on one generator ξ, which maps to ξ_1 in $B_*(0)$ and (for example) to $\xi_1 - \xi_1^{-p}\xi_2$ in B_* (2). As a module over $F_p[\xi,\xi^{-1}]$, T' is free on two generators $1, \tau$ which map to $1, \tau_0$ in B_* (0). If we use cohomological degrees, we must give ξ degree $-2(p-1)$ and τ degree -1.

Multiplication by $t' \in T'$ gives a linear map $B_*(n) \to B_*(n)$ which is a map of right comodules. Its dual is a map of right modules

$$B(n) \xrightarrow{t'} B(n).$$

This defines

$$B(n) \otimes_{A(n-1)} M \xrightarrow{t'} B(n) \otimes_{A(n-1)} M$$

and passes to the limit to give

$$T''(M) \xrightarrow{t'} T''(M).$$

In this way T' comes to act on $T''(M)$.

We now need an explicit isomorphism between $T''(M)$ and $T' \otimes M$. With $M = F_p$, for example, $T''(F_p)$ is a free T'-module on one generator β, but it is not a free T'-module on the generator 1 (at least if $p > 2$). For this or other reasons, we define an isomorphism of T'-modules

$$T' \otimes M \xrightarrow{\theta} T''(M)$$

by

$$\theta(t' \otimes m) = (-1)^{\deg t'} t'(\beta \otimes m).$$

We now define $T'(M) = T' \otimes M$ and give it an action of $A(n)$ or A, as the case may be, by using θ to pull back the action of $A(n)$ or A on $T''(M)$. Of course, since θ is of degree 1, this introduces the usual signs.

By rewriting the source and target of f'', we obtain a map

$$T'(M) = T' \otimes M \xrightarrow{f'} T' \hat{\otimes} M.$$

This map has good properties, and it is still mono. Similarly we obtain

$$T'(M) \xrightarrow{\varepsilon'} M.$$

Finally, we need to identify T' with $H^*(\Sigma_p)_{loc}$.

LEMMA 5.1. *There is an isomorphism of algebras*

$$T' \xrightarrow{\phi} H^*(\Sigma_p)_{loc}$$

which is also an isomorphism of A-modules (provided that the A-action on T' is that which it gets as $T'(F_p)$). Explicitly,

$$\phi(\xi) = -x^{-(p-1)}, \quad \phi(\tau) = ex^{-1}.$$

The idea is as follows. We can define an $A(n)$-map of degree -1 by

$$B(n) \to A \xrightarrow{\gamma} H^*(\Sigma_p)_{\mathrm{loc}}$$

where γ is defined by

$$\gamma(a) = (-1)^{1+\deg(a)} a(ex^{-1}).$$

We use the unstable axiom to show that in sufficiently high degrees, this map factors through $B(n) \otimes_{A(n-1)} F_p$. Thus in sufficiently high degrees we get a composite

$$T'(F_p) \xrightarrow{\theta} T''(F_p) \to H^*(\Sigma_p)_{\mathrm{loc}}$$

which is an $A(n)$-map and coincides with the map ϕ of algebras given by the formulae in the enunciation. Now we use periodicity to show that the map ϕ is an A-map in all degrees.

Lemma 5.1 provides an A-map

$$H^*(\Sigma_p)_{\mathrm{loc}} \xrightarrow{\phi^{-1}} T'(F_p) \xrightarrow{\varepsilon'} F_p;$$

this shows that the map "res" of §4, §8 is an A-map (if the reader does not already have his preferred proof).

We mention briefly two more ideas. First, since the map

$$T'(M) = T' \otimes M \xrightarrow{f'} T' \hat{\otimes} M$$

is mono, suitable properties (such as the Cartan formula) can be checked after applying f'. Secondly, suppose we start with a structure map, such as f'' or ε'', which is given conceptually and by transparent explicit formulae. If we replace source and target using explicitly-given isomorphisms, we shall still have good explicit formulae; but if we iterate the process, the formulae may not stay so transparent. This leads to results such as (3.6).

This completes our sketch of a self-contained approach to the Singer construction.

§6. ALGEBRAS OF INVARIANTS

In this section our first object is to indicate the subgroups G which can be used in the argument of §4; we also indicate results about their algebras of invariants, including (3.3).

Our subgroups G can be considered as matrix groups, defined by restricting the matrix A to agree with the identity below a certain stepwise boundary line. More precisely, with notation as in (1.1), let U_r be the subspace of $U = \beta V^*$ generated by $x_1, x_2, \ldots, x_r$. Let q: $\{1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, n-1\}$ be a function which has $q(r) \le r-1$ and is non-decreasing, so that $r \le s$ implies $q(r) \le q(s)$. Let $G \subset \mathrm{Syl}(V)$ be the subgroup of matrices A which induce the identity map of $U_r/U_{q(r)}$ for each r; equivalently,

$$a_{ij} = \delta_{ij} \quad \text{for} \quad i > q(j).$$

Next we explain the lemma which we use to prove that pairs $F \subset G$ of such subgroups have the property needed in §4. We suppose given a group Z_p (such as G/F) with generator g, acting on a polynomial algebra $R[x]$ of characteristic p so that g fixes R and $g(x) = x + c$ where c is some invertible constant in R. We define

$$M_j = \mathrm{Ker}((g-1)^j : R[x] \to R[x])$$

as in §3.

LEMMA 6.1. *Then the map*

$$(g-1)^{j-1} : M_j/M_{j-1} \to M_1/M_0 = R[x]^{Z_p}$$

is iso for $1 \leq j \leq p$. *Moreover,* $R[x]^{Z_p}$ *is a polynomial algebra* $R[y]$, *where*

$$y = \prod_{0 \leq i < p} (g^i x) = x^p - c^{p-1} x.$$

The proof is elementary.

We must now explain why the action of $G/F = Z_p$ on the subalgebra $H^*(V)^F_{\mathrm{loc}}$ is such that we can apply this lemma.

First, since we can in fact apply (6.1), we can determine the subalgebras $H^*(V)^F_{\mathrm{loc}}$ by induction upwards over F, starting from $F = 1$ and computing $H^*(V)^G_{\mathrm{loc}}$ as $H^*(V)^F_{\mathrm{loc}})^{G/F}$. (There are enough subgroups F of the sort we consider to reach any one by an induction, either downwards from $\mathrm{Syl}(V)$ as in §4, or upwards from 1 as here.) To give the answer, we construct generators in $S[\beta V^*] \subset H^*(V)$ as follows. For each $r \in \{1, 2, \ldots, n\}$, we choose an F-orbit C_r in U_r which is not in U_{r-1}. Let $\pi(C_r)$ be the product of the elements in this orbit.

PROPOSITION 6.2. (a) *For each such choice, the algebra of invariant elements* $S[\beta V^*]^F_{\mathrm{loc}}$ *is*

$$F_p[\pi(C_1), \pi(C_2), \ldots, \pi(C_n)]_{\mathrm{loc}}.$$

(b) $H^*(V)^F_{\mathrm{loc}}$ *is a free module over* $S[\beta V^*]^F_{\mathrm{loc}}$ *on the* 2^n *generators*

$$f_1^{i_1} f_2^{i_2} \ldots f_n^{i_n}$$

described in §3.

Secondly, we have good control over the pair $F \subset G$. If $F \subset G$ are subgroups such as we consider with $G/F \cong Z_p$, then F differs from G only by the imposition of one extra condition $a_{ij} = 0$ for some pair (i, j) with $i < j$. A generator g for $G/F \cong Z_p$ is given by the elementary matrix which agrees with the identity matrix except for $a_{ij} = 1$. From this we see that G fixes all but one of the generators $\pi(C_r)$ in (6.2)(a). We can thus take

$$R = F_p[\pi(C_1), \ldots, \pi(C_{j-1})]_{\mathrm{loc}}[\pi(C_{j+1}), \ldots, \pi(C_n)].$$

Moreover, we can take $x = \pi(C_j)$, because g moves this generator in the required way. (This point does take some elementary algebra.)

If we want information about algebras of invariants, there is never any trouble in passing from less localized objects, such as $R[x]$, to information about more localized objects, such as $S[\beta V^*]^F_{\mathrm{loc}}$. We can thus prove (6.2) (a) by induction upwards over F.

As for (6.2) (b), there is never any trouble in throwing in the 2^n passive generators.

Both the last paragraphs apply also to proving $(g-1)^j$ iso, as is asserted for $R[x]$ by (6.1) and needed for $H^*(V)^F_{\mathrm{loc}}$ in §4.

Finally, the case $F = \mathrm{Syl}(V)$ of (6.2) can be rewritten to give (3.3).

§7. THE STEINBERG REPRESENTATION

(1.1) (c) states that $F_p \otimes_A H^*(V)_{\mathrm{loc}}$ affords the (mod p) Steinberg representation of $GL(V)$. We will sketch a proof of this by conceptual algebra, avoiding any explicit formula for a Steinberg idempotent. We subdivide the proof into two parts, (7.1) and (7.2) below, by introducing an alternative construction of the Steinberg module M. From our definition of M we prove the following.

PROPOSITION 7.1. *There is a canonical map from* $F_p \otimes_Z M$ *to* $H^*(V)_{\mathrm{loc}}$ *such that the composite*

$$F_p \otimes_Z M \to H^*(V)_{\mathrm{loc}} \xrightarrow{q} F_p \otimes_A H^*(V)_{\mathrm{loc}}$$

is iso.

If we assume the result of Priddy and Wilkerson [13] that $H^*(V)_{\mathrm{loc}}$ is projective over $F_p[GL(V)]$, then the splitting in (7.1) shows that $F_p \otimes_A H^*(V)_{\mathrm{loc}}$ is projective over

$F_p[GL(V)]$. By (1.1) (d), $F_p \otimes_A H^*(V)_{\mathrm{loc}}$ restricts to the regular representation of $\mathrm{Syl}(V)$ (over F_p). These two points show that $F_p \otimes_A H^*(V)_{\mathrm{loc}}$ satisfies one characterization of the mod p Steinberg representation; but we do not need to argue in this way.

PROPOSITION 7.2. *M is canonically isomorphic to $\tilde{H}_{n-2}(TB)$, the homology of the Tits building.*

We present the Z-module $M = M(V^*)$ by generators and relations, as follows. We take one generator $m(x_1, x_2, \ldots, x_n)$ for each base $(x_1, x_2, \ldots, x_n)$ of V^*. We prescribe the following relations.

(i) m is antisymmetric in its arguments, that is,

$$m(x_{\rho 1}, x_{\rho 2}, \ldots, x_{\rho n}) = \varepsilon(\rho) m(x_1, x_2, \ldots, x_n)$$

for each permutation ρ.

(ii) If λ is a non-zero scalar then

$$m(\lambda x_1, x_2, \ldots, x_n) = m(x_1, x_2, \ldots, x_n).$$

(iii) Suppose that V^* comes as the direct sum $V^* = X^* \oplus Y^*$ of a subspace X^* of dimension 2 and a subspace Y^* of dimension $n-2$. Suppose that any two of x_1, x_2, x_3 form a base for X^*, while $y_3, y_4, \ldots, y_n$ form a base for Y^*. Then

$$\begin{aligned} &m(x_1, x_2, y_3, y_4, \ldots, y_n) \\ &+ m(x_2, x_3, y_3, y_4, \ldots, y_n) \\ &+ m(x_3, x_1, y_3, y_4, \ldots, y_n) = 0. \end{aligned}$$

It is clear how $GL(V)$ acts on M.

We give the map

$$F_p \otimes_Z M \to H^*(V)_{\mathrm{loc}}$$

of (7.1) by giving it on the generators. For present purposes the vector-spaces V^* and βV^* can be identified under β; let $e_1, e_2, \ldots, e_n$ and $x_1, x_2, \ldots, x_n$ be corresponding bases in them. Then we send the generator

$$m(x_1, x_2, \ldots, x_n)$$

to

$$e_1 x_1^{-1} e_2 x_2^{-1} \ldots e_n x_n^{-1} \in H^*(V)_{\mathrm{loc}}.$$

We leave to the reader the exercise of checking that this map preserves the relations; the fact that it does so explains the construction of M.

We can analyse the structure of M. Let $(x_1, x_2, \ldots, x_n)$ be one base for V^*, and let g run over the corresponding group of upper uni-triangular matrices $\mathrm{Syl}(V)$.

PROPOSITION 7.3. *Then the generators $m(gx_1, gx_2, \ldots gx_n)$ form a Z-base for M.*

If this is granted, (7.1) follows: the map in (7.1) takes the base of $F_p \otimes_Z M$ given by (7.3), and sends it to the base for $F_p \otimes_A H^*(V)_{\mathrm{loc}}$ given by (1.1) (d).

We sketch the proof that the generators in (7.3) span M. This is done by induction over n, using the following lemma. Let W^* be a subspace of dimension $(n-1)$ in V^*.

LEMMA 7.4. *M is spanned by generators $m(y_1, y_2, \ldots y_n)$ in which all but one of the y_r lie in W^*.*

This is proved from the given relations by induction over the number of y_r which do not lie in W^*.

We sketch the proof that the generators in (7.3) are linearly independent over Z. This is done by setting up suitable homomorphisms

$$\theta : M = M(V^*) \to Z.$$

For each base $x_1, x_2, \dots, x_n$ in V^* and each maximal flag F in V^* there is at most one permutation ρ such that $x_{\rho 1}, x_{\rho 2}, \dots, x_{\rho n}$ is a base adapted to the flag F. We let θ_F carry $m(x_1, x_2, \dots, x_n)$ to $\varepsilon(\rho) = \pm 1$ if there is such a permutation ρ, to 0 otherwise.

We move on to (7.2). We recall that the Tits building TB (for V^*) is a certain finite simplicial complex. It has a vertex for each subspace of V^* other than the trivial subspaces 0 and V^*. Its top-dimensional simplexes are in $(1-1)$ correspondence with the maximal flags F in V^*. So by using as components the maps θ_F just introduced, we obtain a map

$$M = M(V^*) \xrightarrow{\{\theta_F\}} \bigoplus_F Z = C_{n-2}(TB);$$

from the proof of (7.3) we know that this map is mono. Since the subgroup of boundaries is zero, the reduced homology group $\tilde{H}_{n-2}(TB)$ is the subgroup of cycles $\tilde{Z}_{n-2}(TB)$. One checks directly that $\{\theta_F\}$ maps M into $\tilde{Z}_{n-2}(TB)$. It remains to show that $\{\theta_F\}$ maps M onto $\tilde{Z}_{n-2}(TB)$; the proof is combinatorial, and we omit it for brevity.

§8. DELOCALIZATION

In this section we sketch the proof of (1.5). The theme of our argument is that we take information about objects which are more localised, and deduce information about object which are less localized.

We first explain the map in (1.5). In §4 we said that a direct-sum splitting $V \cong Z_p \times W$ leads to an A-map

$$H^*(V)_{\mathrm{loc}} \xrightarrow{\mathrm{res}} H^*(W)_{\mathrm{loc}}.$$

We now write it res_w to indicate its dependence on W. If we localize less, this map carries $H^*(V)_S$ into $H^*(W)_{S \cap \beta W^*}$. We simplify the notation by dropping the symbol β, identifying the subspace $\beta V^* \subset H^2(V)$ with V^*, as in §7.

LEMMA 8.1. (1.5) *is true for* $S = V^*$.

The proof is based on the following diagram.

$$\begin{array}{ccc}
H^*(V)_{V^*} & \xrightarrow{\{\mathrm{res}_W\}} & \bigoplus_W H^*(W)_{W^*} \\
\Big\downarrow q & & \Big\downarrow \{q_W\} \\
F_p \otimes_A H^*(V)_{V^*} & \xrightarrow{\{1 \otimes \mathrm{res}_W\}} & \bigoplus_W F_p \otimes_A H^*(W)_{W^*}
\end{array}$$

The two vertical arrows are Tor-equivalences by (1.1) (a). We see that the lower horizontal arrow is iso by calculating its effect on the base given by (1.1) (d).

We prove (1.5) by induction over n. For $n = 1$ there is only one way to localize, and the result is true by (8.1). We therefore assume the result is true in dimension $(n-1)$. We now proceed by downwards induction over S. Lemma 8.1 shows that the result is true for $S = V^*$; for the induction step, we must assume that T contains just one more line than S, say $T = S \cup \langle y \rangle$, $y \notin S$, and assume that the result holds for T. We now have the following commutative diagram, in which ρ_W is defined by passing to the quotient from res_W.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^*(V)_S & \longrightarrow & H^*(V)_T & \longrightarrow & \dfrac{H^*(V)_T}{H^*(V)_S} & \longrightarrow & 0 \\
 & & \Big\downarrow \{\mathrm{res}_W\} & & \Big\downarrow \{\mathrm{res}_W\} & & \Big\downarrow \{\rho_W\} & & \\
0 & \longrightarrow & \bigoplus_W H^*(W)_{S \cap W^*} & \longrightarrow & \bigoplus_W H^*(W)_{T \cap W^*} & \longrightarrow & \bigoplus_W \dfrac{H^*(W)_{T \cap W^*}}{H^*(W)_{S \cap W^*}} & \longrightarrow & 0
\end{array}$$

Here the middle verticle arrow is a Tor-equivalence by the inductive hypothesis; we can prove that the left-hand vertical arrow is a Tor-equivalence by the Five Lemma, provided we prove the following.

LEMMA 8.2. *The map*

$$\frac{H^*(V)_T}{H^*(V)_S} \xrightarrow{\{\rho_W\}} \bigoplus_W \frac{H^*(W)_{T\cap W^*}}{H^*(W)_{S\cap W^*}}$$

is a Tor-equivalence.

Here we can restrict W^* to run over the p^{n-2} complements for $\langle x \rangle$ which contain $\langle y \rangle$, because $H^*(W)_{T\cap W^*}/H^*(W)_{S\cap W^*}$ is zero for the other choices of W^*.

The proof of (8.2) depends on a lemma.

LEMMA 8.3. *In* (8.2), *the truth or falsity of the conclusion depends only on the image of S in the quotient space $V^*/\langle y \rangle$.*

We sketch the proof of (8.3). It is sufficient to study the effect of replacing S by S', where $S \subset S'$ and both have the same image in $V^*/\langle y \rangle$. Of course we take $T' = S' \cup \langle y \rangle$. We now check that the map

$$\frac{H^*(V)_T}{H^*(V)_S} \longrightarrow \frac{H^*(V)_{T'}}{H^*(V)_{S'}}$$

is iso. The reason is that S' already acts invertibly on $H^*(V)_T/H^*(V)_S$. In fact, the series

$$s^{-1} - \lambda y s^{-2} + \lambda^2 y^2 s^3 - \ldots$$

provides an inverse for any element $s + \lambda y$ in S'; on any particular element of $H^*(V)_T/H^*(V)_S$ this series converges after a finite number of terms, because for any $z \in H^*(V)_T$ there is a power y^m of y such that $y^m z \in H^*(V)_S$. The same considerations show that the map

$$\frac{H^*(W)_{T\cap W^*}}{H^*(W)_{S\cap W^*}} \longrightarrow \frac{H^*(W)_{T'\cap W^*}}{H^*(W)_{S'\cap W^*}}$$

is iso. Therefore $\{\rho_W\}$ is a Tor-equivalence for S if and only if it is so for S'.

We sketch the proof of (8.2). Here we use (8.3) to clean up the position of S. Choose a complement $\bar{V}^*$ for $\langle y \rangle$ in V^* such that $\langle x \rangle \subset \bar{V}^*$. Then $\bar{V}^*$ provides one representative for each coset in $V^*/\langle y \rangle$, and so (8.7) allows us to suppose that $S \subset \bar{V}^*$.

The p^{n-2} complements W^* for $\langle x \rangle$ in V^* which contain $\langle y \rangle$ are now in $(1-1)$ correspondence with the p^{n-2} complements $\bar{W}^*$ for $\langle x \rangle$ in $\bar{V}^*$. By using our special choice of S, and properties of the residue, the map ρ_W can be thrown by isomorphisms of the source and target onto

$$H^*(\bar{V})_S \otimes \frac{H^*(Z_p)_{\mathrm{loc}}}{H^*(Z_p)} \xrightarrow{\mathrm{res}_{\bar{W}}\otimes 1} H^*(\bar{W})_{S\cap \bar{W}^*} \otimes \frac{H^*(Z_p)_{\mathrm{loc}}}{H^*(Z_p)}.$$

(Here $\langle y \rangle = (Z_p)^*$.) Since $\bar{V}$ is of rank $(n-1)$, the hypothesis of our main induction over n says that the map

$$H^*(\bar{V})_S \xrightarrow{\{\mathrm{res}_{\bar{W}}\}} \bigoplus_{\bar{W}} H^*(\bar{W})_{S\cap \bar{W}^*}$$

is a Tor-equivalence. An easy lemma, comparable with (1.2), says that Tor-equivalences remain Tor-equivalences if you tensor with an A-module bounded above, such as $H^*(Z_p)_{\mathrm{loc}}/H^*(Z_p)$. So the map

$$H^*(V)_S \otimes \frac{H^*(Z_p)_{\mathrm{loc}}}{H^*(Z_p)} \xrightarrow{\{\mathrm{res}_{\bar{W}}\otimes 1\}} \bigoplus_{\bar{W}} H^*(\bar{W})_{S\cap \bar{W}^*} \otimes \frac{H^*(Z_p)_{\mathrm{loc}}}{H^*(Z_p)}$$

is a Tor-equivalence. This proves (8.2).

§9. FUNCTION-OBJECTS

In this section we will explain the correct, conceptual version of (1.6) promised in §1.

We recall that the level of generality in (1.6) is such that the algebra corresponds to the topological problem of computing $[\mathbf{T} \wedge \mathbf{B}U, \mathbf{B}V]$, where the bold-face letters mean spectra. Here the group $[\mathbf{T} \wedge \mathbf{B}U, \mathbf{B}V]$ is a representable functor of the test-object $\mathbf{T}$; the representing object is the function-spectrum $\mathbf{F}$ "of maps from $\mathbf{B}U$ to $\mathbf{B}V$", and information about the functor is equivalent to information about $\mathbf{F}$. We need to formulate the algebraic analogue of a function-spectrum.

For this we need to work in a category $\mathscr{C}$ which is preadditive and monoidal [9], and we explain our first example.

In our "Ext category" $\mathscr{E}$ the objects are A-modules $L, M, N, \ldots$ which are bounded below and finite-dimensional over F_p in each degree. The hom-set $E(L, M)$ from L to M in $\mathscr{E}$ is the bigraded Ext group $\mathrm{Ext}_A^{**}(M, L)$. Thus $\mathscr{E}$ is the opposite of the usual Ext category; this makes some formulae look better; in particular, cohomology is a covariant functor with values in $\mathscr{E}$. Composition in $\mathscr{E}$ is the usual Yoneda product. The monoidal operation on objects is the usual tensor product $L \otimes M$; on morphisms it is the usual tensor product in Ext.

We will explain the notion of a "function-object" in such a category $\mathscr{C}$. Let L and M be given objects in $\mathscr{C}$; we plan to consider "functions from L to M". Suppose given further a finite number of objects W_i in $\mathscr{C}$ and morphisms

$$W_i \otimes L \xrightarrow{w_i} M.$$

If the hom-sets of $\mathscr{C}$ are bigraded then the morphisms w_i may be of any bidegrees (s_i, t_i). For each "test object" T in $\mathscr{C}$ we get a map

$$C^{s-s_i, t-t_i}(T, W_i) \xrightarrow{\omega_i} C^{s,t}(T \otimes L, M)$$

which carries $T \xrightarrow{f} W_i$ to

$$T \otimes L \xrightarrow{f \otimes 1} W_i \otimes L \xrightarrow{w_i} M.$$

With these maps as components we get a map

$$\bigoplus_i C^{s-s_i, t-t_i}(T, W_i) \xrightarrow{\omega} C^{s,t}(T \otimes L, M).$$

If this map ω is an isomorphism for all objects T in $\mathscr{C}$, we will say that the data $\{W_i, w_i\}$ are a "function-object" from L to M. In this case the data $\{W_i, w_i\}$ allow us to express the group $C(T \otimes L, M)$ in terms of representable functors of T.

Of course, if there were in $\mathscr{C}$ a categorical product of the objects W_i suitably regraded, then this object (with a suitable map) would be a function-object in the usual sense; but we do not assume that any such object exists in $\mathscr{C}$.

The content of (1.6) is that certain data constitute a function-object from $H^*(U)$ to $H^*(V)$ in $\mathscr{E}$, and thereby allow us to compute $\mathrm{Ext}_A^{**}(H^*(V), -\otimes H^*(U))$.

The task of saying what data constitute this function-object is usually called "book-keeping". The art of book-keeping is to establish a correspondence between entries in a ledger, where the information is easy to find, and certain aspects of the real world, where things may be harder. The analogue of the real world, for us, is the category $\mathscr{E}$ where we keep our unknown Ext groups. The analogue of the ledger is a category $\mathscr{A}^{gr}$ where things are easy. The objects of $\mathscr{A}^{gr}$ are the elementary abelian p-groups $U, V, W, \ldots$; the monoidal operation on objects is the Cartesian product $U \times V$; we explain the morphisms of $\mathscr{A}^{gr}$ later. As for the correspondence between the ledger and the world, its analogue is a certain functor β from $\mathscr{A}^{gr}$ to $\mathscr{E}$. On objects the functor β is given by $\beta(V) = H^*(V)$; since $H^*(U \times V) = H^*(U) \otimes H^*(V)$, β preserves the monoidal operation on objects; we explain the effect of β on morphisms later. We now explain how β allows us to transfer constructs from $\mathscr{A}^{gr}$ to $\mathscr{E}$.

Suppose given a suitable functor β from one preadditive monoidal category to another. Suppose given a function-object $\{W_i, w_i\}$ from U to V in the source category. We will say that β "preserves this function-object" if $\{\beta W_i, \beta w_i\}$ is a function-object from βU to βV in the target category. That is, in our applications, the appropriate induced map

$$\bigoplus_i \operatorname{Ext}_A^{s-s_i,\, t-t_i}(H^*(W_i), M)) \xrightarrow{\omega} \operatorname{Ext}_A^{s,t}(H^*(V), M \otimes H^*(U))$$

is to be iso for every A-module M which is bounded below and finite-dimensional in each degree.

We will sketch the proof of the following results.

PROPOSITION 9.1. *For each U and V there is a function-object $\{W(X), w(X)\}$ from U to V in $\mathscr{A}^{gr}$.*

THEOREM 9.2. *The functor β: $\mathscr{A}^{gr} \to \mathscr{E}$ of §10 preserves all function-objects.*

When we explain the function-object $\{W(X), w(X)\}$ in (9.1) and the functor β in (9.2), that will complete our explanation of (1.6).

We owe the reader details about $\mathscr{A}^{gr}$, and first we must explain a category $\mathscr{A}$ from which we construct $\mathscr{A}^{gr}$ by passing to an associated graded category.

If we consider the topological problem of computing $[\mathbf{T} \wedge \mathbf{B}G_1, \mathbf{B}G_2]$, it is natural to begin with the special case $\mathbf{T} = \mathbf{S}^0$ and study $[\mathbf{B}G_1, \mathbf{B}G_2]$. The most reasonable approach is to follow the ideas which Segal proposed for the special case $G_2 = 1$. The first step should be to define an algebraic construct $A(G_1, G_2)$ and a homomorphism

$$A(G_1, G_2) \xrightarrow{\alpha} [\mathbf{B}G_1, \mathbf{B}G_2].$$

Here the construct $A(G_1, G_2)$ should play the same role that the usual Burnside ring does in the special case $G_2 = 1$; it should be the closest approximation to $[\mathbf{B}G_1, \mathbf{B}G_2]$ that can be constructed by algebraic means (without using analytic methods such as completion).

In the special case $G_1 = G_2$, our construct $A(G, G)$ has already appeared in the work of C. M. Witten [17], for the same reason and purpose.

In general, these groups $A\ (G_1, G_2)$ should become the hom-sets of a category, under a product corresponding to the composition of maps of spectra. This category should be monoidal, with the monodial operation corresponding to the smash product in the category of spectra.

We therefore set up the "Burnside category" $\mathscr{A}$ as follows.

The objects of the Burnside category $\mathscr{A}$ will be the finite groups $G, H, \ldots$. We wish to describe the hom-set of morphisms from G to H in $\mathscr{A}$. We consider finite sets X which come provided with an action of G on the left of X and an action of H on the right of X, so that these two actions commute and the action of H on the right of X is free. Such sets X we call "(G, H)-sets". We take the (G, H)-sets and classify them into isomorphism classes. The operation of disjoint union passes to isomorphism classes, and turns the set of isomorphism classes into a commutative monoid. This monoid is a free cummutative monoid; we obtain a base by considering the isomorphism classes of (G, H)-sets X which are irreducible under disjoint union. (It is equivalent to say that the action of G on X/H is transitive.) We define $A(G, H)$ to be the Grothendieck group or universal group associated to this monoid. This is a free abelian group; we obtain a base by considering the same irreducibles as before.

For example, if $H = 1$, then a $(G, 1)$-set is essentially just a G-set, and so $A(G, 1)$ reduces to the usual group $A(G)$.

We define the set of morphisms in $\mathscr{A}$ from G to H to be $A(G, H)$. We have to define the composition product

$$A(G, H) \otimes A(H, K) \to A(G, K)$$

(where the notation reveals that for this purpose we shall compose morphisms from left to right). Let X be a (G, H)-set and Y an (H, K)-set; then $X \times_H Y$ is a (G, K)-set. This operation passes to isomorphism classes and is biadditive with respect to disjoint union; so it defines a product as stated. This product is associative and has units; $1_G \in A(G, G)$ is the class of G, considered as a (G, G)-set with the obvious left and right actions. This makes $\mathscr{A}$ into a category.

We now make $\mathscr{A}$ into a monoidal category. The product on objects in the Cartesian product $G \times H$ of groups. (In the ordinary category of groups and homomorphisms this is a categorical product; it is no longer a categorical product in $\mathscr{A}$.) The product on morphisms is defined as follows. Let X_1 be a (G_1, H_1)-set and let X_2 be a (G_2, H_2)-set; then $X_1 \times X_2$ is a $(G_1 \times G_2, H_1 \times H_2)$-set. This consutruction passes to isomorphism classes and is biadditive with respect to disjoint unions; so it defines a product

$$A(G_1, H_1) \otimes A(G_2, H_2) \to A(G_1 \times G_2, H_1 \times H_2)$$

as required.

We omit discussion of the good formal properties of the constructs just introduced.

For guidance it is useful to note that one can define a functor α from $\mathscr{A}$ to the category of spectra, that is, a set of homomorphisms

$$A(G_1, G_2) \xrightarrow{\alpha} [\mathbf{B}G_1, \mathbf{B}G_2]$$

which preserve the structure. This is done using transfer. It is not needed for the algebraic purposes of the present paper, and so we omit it.

We also define particular morphisms in $\mathscr{A}$. For each homomorphism $\theta: G \to H$ we introduce an element $\theta_* \in A(G, H)$; this is the class of H, with G acting on its left via θ and H acting on its right. For each monomorphism $\phi: H \to G$ we introduce an element $\phi^* \in A(G, H)$; this is the class of G, with G acting on its left and H acting on its right via ϕ. This action of H is free because we assume that ϕ is mono.

We can now give more motivation for the category $\mathscr{A}$. A functor T defined on $\mathscr{A}$ provides a functor defined on the usual category of finite groups: on objects G we take $T(G)$ and on homomorphisms $\theta: G \to H$ we take $T(\theta_*)$. But beyond this we get homomorphisms $T(\phi^*)$, which correspond to the possibility of "induction". (For example, the "homology of groups" is such a functor T, essentially because it factors as a composite of two functors: the functor α from $\mathscr{A}$ to spectra, and the homology-functor from spectra to graded groups.) If T is a functor defined on $\mathscr{A}$, then the homomorphisms $T(\theta_*)$ and $T(\phi^*)$ satisfy all the usual axioms for "induction" and "restriction", including the double coset formula. However, we do not have to state these axioms explicitly; they are implicit in the structure of the category $\mathscr{A}$. We regard the category $\mathscr{A}$ as the place where one can do "universal" calculations with induction and restriction subject to the usual axioms.

We will now move towards our associated graded category $\mathscr{A}^{gr}$. First we take the full subcategory of the Burnside category in which the objects are elementary abelian p-groups. Next we shall define a filtration on its hom-sets $A(U, V)$.

For guidance it is useful to note that the algebraic filtration of a morphism $f \in A(U, V)$ is in fact the Adams filtration of the resulting map of spectra

$$\mathbf{B}U \xrightarrow{\alpha f} \mathbf{B}V.$$

Indeed, for the purposes which we originally had in mind, it was important to know that our algebra had the correct relation to the topological world; if one uses geometrical means to set up a comparison map between two spectral sequences, then it is important to know that the geometrically-induced map of E_2-terms agrees with the map of Ext_A proved to be iso in the algebraic work. For the purposes of the present paper we need not worry.

If X is an irreducible (U, V)-set, we define $s(X)$ by

$$p^{s(X)} = |X/V|.$$

Clearly this depends only on the isomorphism class of X.

We define the filtration subgroup

$$F_s A(U, V) \subset A(U, V)$$

to be the subgroup generated by the elements $p^{\lambda}[X]$, where X runs over the irreducible (U, V)-sets and λ, X satisfy

$$\lambda + s(X) \geq s.$$

LEMMA 9.3. *Composition and cross product preserve this filtration. More precisely, if X is an irreducible (U, V)-set and Y is an irreducible (V, W)-set then $[X][Y] = p^{\lambda}[Z]$ where Z is an irreducible (U, W)-set with*

$$\lambda + s(Z) = s(X) + s(Y);$$

similarly for the cross product, with $\lambda = 0$.

We omit the proof.

We can now define the associated graded category $\mathscr{A}^{gr}$. The objects of $\mathscr{A}^{gr}$ are to be the elementary abelian p-groups $U, V, W, \ldots$. The hom-set $\mathscr{A}^{gr}(U, V)$ from U to V is to be a graded vector-space over F_p, whose sth component is

$$F_s A(U, V)/F_{s+1} A(U, V).$$

Lemma 9.3 shows that composition and cross product pass to the quotient and give operations on $\mathscr{A}^{gr}$.

Finally, we return to (9.1).

Let U, V be any two objects of $\mathscr{A}^{gr}$, that is, any two elemetary abelian p-groups. Let X run over a set of representatives for the isomorphism classes of irreducible (U, V)-sets. For each X, let $W(X)$ be the automorphism group of X; of course, we mean "automorphisms of X" to preserve the left action of $G = U$ and the right action of $H = V$. We can consider X as a $(W(X) \times U, V)$-set; let

$$w(X) \in \mathscr{A}^{gr}(W(X) \times U, V)$$

be the class of X. Then the data $\{W(X), w(X)\}$ constitute a function-object from U to V in $\mathscr{A}^{gr}$. The proof is an essentially straightforward exericse about sets with groups acting on them, and for brevity we omit it.

§10. THE FUNCTOR β

In this section we will describe the functor β promised in §9.

We suppose given an element

$$E \in \mathrm{Ext}_A^{1,1}(H^*(1), H^*(Z_p))$$

with the following properties.

(10.1) If $\theta: Z_p \to Z_p$ is an automorphism, then

$$\theta_* E = E.$$

(This reveals that we have reverted to the usual order of composition.)

(10.2) Let $\theta: Z_p \times Z_p \to Z_p \times Z_p$ be the homomorphism

$$\theta(x, y) = (x + \lambda y, y)$$

for some fixed $\lambda \in F_p$. Then

$$\theta_*(1 \times E) = (1 \times E).$$

PROPOSITION 10.3. *For each such element E there is a unique functor β: $\mathscr{A}^{gr} \to \mathscr{E}$ with the following properties.*

(a) *β is given on objects by $\beta(V) = H^*(V)$.*

(b) *β is additive and preserves the monoidal structure.*

(c) *For each morphism $\theta: U \to V$ we have*

$$\beta(\theta_*) = H^*(\theta): H^*(V) \to H^*(U).$$

(d) *For the injection* $i: 1 \to Z_p$ *we have*

$$\beta(i^*) = E \in Ext_A^{1,1}(H^*(1), H^*(Z_p)).$$

The idea is that with a suitable choice of E, we can secure the following: if $f \in A(U, V)$, then

$$\beta f \in \mathrm{Ext}_A^{**}(H^*(V), H^*(U))$$

gives the position of

$$\alpha f: \mathbf{B}U \to \mathbf{B}V$$

in the Adams spectral sequence for computing $[\mathbf{B}U, \mathbf{B}V]$. More formally, if f is of filtration s in $A(U, V)$, then

$$\beta f \in \mathrm{Ext}_A^{s,s}(H^*(V), H^*(U))$$

is a permanent cycle in the Adams spectral sequence

$$\mathrm{Ext}_A^{**}(H^*(V), H^*(U)) \Rightarrow [\mathbf{B}U, \mathbf{B}V],$$

and αf, βf have the same image in E_∞.

Unfortunately, in the absence of special information about our Adams spectral sequences, we cannot claim that a statement of this form defines βf uniquely except for $s = 0, 1$; *a priori*, differentials might cause some permanent cycles in E_2 to map to zero in E_∞. Moreover, one can hardly expect to obtain special information about our Adams spectral sequences until we have calculated their E_2 terms, and this would risk a circular argument.

However, we have a purely algebraic proof of (10.3), and its idea is as follows.

Given the objects of a preadditive monoidal category, one can present its morphisms by generators and relations, just as one presents a group by generators and relations. (For a group one builds up words from generators by multiplication, but here one builds up words from generators by the operations allowed in a preadditive monoidal category.) The category $\mathscr{A}^{gr}$ can be presented by generators and relations. A suitable presentation has as its generators all morphisms θ_*, and a single morphism i^* corresponding to the map $i: 1 \to Z_p$. (Each θ_* is of filtration 0, while i^* is of filtration 1.) As relations we have various formal relations which involve the generators θ_* but not i^*, and two special relations: if $\theta: Z_p \to Z_p$ is an automorphism (as in (10.1)) then $\theta_* i^* = i^*$, and if $\theta: Z_p \times Z_p \to Z_p \times Z_p$ is as in (10.2), then $\theta_*(1 \times i^*) = (1 \times i^*)$. Now, (10.3) (a)–(d) specify β on generators and ensure that β preserves the relations, so there is a unique β as asserted.

To carry out this proof would involve two things: to make precise the notion of a "presentation", and to show that $\mathscr{A}^{gr}$ can be presented by the presentation in question. In fact one need not do the first; the mathematics involved in the second can be rewritten as a proof that the requirements lead to a unique choice of β, with all the required properties, on successively larger parts of $\mathscr{A}^{gr}$. The work is somewhat long; it requires linear algebra and no new ideas, and we omit it.

We explain the choice of E we propose. In fact there is only one choice up to a scalar factor; and if we replace E by λE, we multiply β by λ^s in degree s, so as long as we don't take $\lambda = 0$ we don't alter the truth of (9.2) or (1.6). Evidently we should take E to be the position in the appropriate Adams spectral sequence of the map $\alpha(i^*): \mathbf{B}Z_p \to \mathbf{B}1$. This choice agrees (up to sign) with the purely algebraic choice we will give.

Let M be the submodule of $H^*(Z_p)_{\mathrm{loc}}$ which consists of the groups in degrees ≥ -1. It takes part in the following short exact sequence.

$$0 \to H^*(Z_p) \to M \xrightarrow{\mathrm{res}} F_p \to 0.$$

We take the class of this extension as our element

$$E \in \mathrm{Ext}_A^{1,1}(F_p, H^*(Z_p)).$$

This settles the functor β.

§11. THE CASE $U = Z_p$

We sketch the proof of the following.

THEOREM 11.1. (9.2) *and* (1.6) *are true for* $U = Z_p$.

The proof is based upon the following diagram.

$$\begin{tikzcd}
{} & \\
\mathrm{Ext}^s_A(H^*(V)\otimes H^*(Z_p)_{\mathrm{loc}}, M) \arrow[u] & \bigoplus_{k=1}^{p^n} \mathrm{Ext}^s_A(H^*(W_k), M) \arrow[l, "\{res_k\}"', "\cong"] \arrow[d, "\{X_k\}"] \\
\mathrm{Ext}^s_A\left(H^*(V)\otimes \dfrac{H^*(Z_p)_{\mathrm{loc}}}{H^*(Z_p)}, M\right) \arrow[u, "1\otimes j"] & \mathrm{Ext}^s_A(H^*(V), M\otimes H^*(Z_p)) \arrow[l, "D"', "\cong"] \\
\mathrm{Ext}^{s-1}_A(H^*(V)\otimes H^*(Z_p), M) \arrow[u, "\Delta"] \arrow[ur, "X_0"'] & \\
{} \arrow[u] &
\end{tikzcd}$$

Diagram 11.2

In this diagram, the vertical exact sequence on the left is obtained from the short exact sequence

$$0 \to H^*(Z_p) \to H^*(Z_p)_{\mathrm{loc}} \xrightarrow{j} \frac{H^*(Z_p)_{\mathrm{loc}}}{H^*(Z_p)} \to 0$$

by applying $H^*(V)\otimes\_\_$ and then applying Ext_A. The indices X are those which arise in (1.6); they are as described in §9. If $U = Z_p$, then p^n of the indices X correspond to the homomorphisms $\theta_k : Z_p \to V$; we assign them the numbers $k = 1, 2, \ldots, p^n$. There is one more index, namely the (Z_p, V)-set $V \times Z_p$; we assign it the number $k = 0$. The arrows labelled X_k, X_0 are induced as described in §9.

The map

$$\mathrm{res}_k : H^*(V)\otimes H^*(Z_p)_{\mathrm{loc}} \to H^*(W_k)$$

is a residue of the sort used in (1.5). In fact, the group W_k corresponding to X_k, although isomorphic to V, should be considered as a quotient of $V\times Z_p$ via the map $V\times Z_p \to W_k$ which carries (v, z) to $v + \theta_k z$; therefore $H^*(W_k)$ is a subalgebra of $H^*(V)\otimes H^*(Z_p)$, and we can take residues of formal Laurent series with coefficients in $H^*(W_k)$. The map $\{\mathrm{res}_k\}$ in (11.2) is iso by (1.5) plus (1.2).

The map marked D (for "duality") is provided as follows.

Let L, M, P be left A-modules; let P^*be the dual of P, made into a left A-module in the usual way so that the evaluation map $P^*\otimes P \xrightarrow{ev} F_p$ is A-linear. We assume that M is bounded below, and that P is bounded above and finite-dimensional over F_p in each degree.

LEMMA 11.3. *Then the natural transformation*

$$Ext^{**}_A(L, M\otimes P^*) \to Ext^{**}_A(L\otimes P, M)$$

is iso.

The natural transformation carries an element $L \xrightarrow{f} M\otimes P^*$ of the Ext category to the composite

$$L\otimes P \xrightarrow{f\otimes 1} M\otimes P^*\otimes P \xrightarrow{1\otimes ev} M.$$

The proof of (11.3) is easy, and we omit it.

To apply (11.3), we replace the map $P^* \otimes P \xrightarrow{ev} F_p$ by a map

$$H^*(Z_p) \otimes \frac{H^*(Z_p)_{\mathrm{loc}}}{H^*(Z_p)} \to F_p$$

obtained as follows. Consider

$$H^*(Z_p)_{\mathrm{loc}} \otimes H^*(Z_p)_{\mathrm{loc}} \xrightarrow{\bar{\Delta}^*} H^*(Z_p)_{\mathrm{loc}} \xrightarrow{\mathrm{res}} F_p;$$

this is an A-map (of degree $+1$) which is also a dual pairing, and in which $H^*(Z_p)$ annihilates $H^*(Z_p)$. Here we define $\bar{\Delta}: Z_p \to Z_p \times Z_p$ by $\bar{\Delta}(z) = (-z, z)$. The sign serves to get some details correct when we check that diagram (11.2) commutes (up to a fixed sign for the triangle). It takes several lemmas to prove that (11.2) commutes; we omit them for brevity. The result (11.1) then follows by diagram-chasing.

§12. PROOF OF (1.6)

We will sketch the proof of (9.2).

First we remark that if β preserves one function-object from U to V, then it preserves all function-objects from U to V. (If we have two function-objects $\{W'_i, w'_i\}$, $\{{}''_j \cdot w''_j\}$ for the same source and target, then one can be thrown onto the other by an invertible matrix of maps $W'_i \to W''_j$ of suitable degrees; β carries an invertible matrix to an invertible matrix.)

Secondly we show how to make new function-objects from old. Suppose given a monoidal category $\mathscr{C}$ and three objects F, G, H in $\mathscr{C}$. Suppose given a function-object

$$\{W_j, W_j \otimes G \xrightarrow{w_j} H\}$$

from G to H, and suppose that for each W_j we have a function-object

$$\{V_{ij}, V_{ij} \otimes F \xrightarrow{v_{ij}} W_j\}$$

from F to W_j. Then we can form the morphism

$$V_{ij} \otimes F \otimes G \xrightarrow{v_{ij} \otimes 1} W_j \otimes G \xrightarrow{w_j} H.$$

LEMMA 12.1. *$\{V_{ij}, (v_{ij} \otimes 1) w_j\}$ is a function-object from $F \otimes G$ to H.*

The proof is easy.

We will call this construction of a function-object from $F \otimes G$ to H the "product construction".

LEMMA 12.2. *Suppose that a functor β preserves the function-object $\{W_j, w_j\}$ from G to H and also preserves the function-object $\{V_{ij}, v_{ij}\}$ from F to W_j for all j. Then it preserves the function-object $\{V_{ij}, (v_{ij} \otimes 1) w_j\}$ from $F \otimes G$ to H given by the product construction.*

The proof is easy.

Proof of (9.2). Consider a function-object in $\mathscr{A}^{gr}$ from U to V. If U is of rank 0 or 1 the result is trivial or true by (11.1); so we proceed by induction over the rank of U. Suppose $U = U' \times U''$ where U' and U'' are of less rank. Then by (9.1) there is a function-object $\{W_j, w_j\}$ from U'' to V and there is also a function-object $\{V_{ij}, v_{ij}\}$ from U' to W_j. By the inductive hypothesis β preserves these function-objects; so by (12.2) it preserves the function-object from $U' \times U''$ to V given by the product construction. Therefore β preserves any other function-object from $U' \times U''$ to V. This completes the induction and proves (9.2), which finishes the sketch proof of (1.6).

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