The Segal conjecture for elementary abelian p-groups, I.

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§1. Introduction. We have confirmed by calculation that Segal's conjecture about the cohomotopy of the classifying space BG is true when G is an elementary abelian p-group. Our methods of calculation depend on the Adams spectral sequence, and so most of the work goes into computing Ext groups taken over the mod p Steenrod algebra A . In this paper we will calculate sufficiently many groups Ext_A relevant to Segal's conjecture for elementary abelian p-groups.

We see reason to generalise Segal's original question, so as to study the groups

$$[\underset{\sim}{T} \wedge \underset{\sim}{B}G_1 , \underset{\sim}{B}G_2] ;$$

here we explain the notation as follows. G_1 and G_2 are finite groups. $\underset{\sim}{B}G$ is the spectrum-level analogue of the classifying space BG . That is, we define

$$\underset{\sim}{B}G = \underset{\sim}{\Sigma}^{\infty}(BG \sqcup P) ;$$

here the functor $\_\_ \sqcup P$ adjoins a disjoint base-point, and the functor $\underset{\sim}{\Sigma}^{\infty}$ passes from a space with base-point to the corresponding suspension spectrum. The object $\underset{\sim}{T}$ is a spectrum which we choose to use as a "test object". The notation $[\underset{\sim}{X},\underset{\sim}{Y}]$ means homotopy classes of maps from $\underset{\sim}{X}$ to $\underset{\sim}{Y}$ in the category of spectra.

To relate this to Segal's original conjecture [16,1], we substitute $\underset{\sim}{T} = \underset{\sim}{S}^m$ and $G_2 = 1$; then $BG_2 \sqcup P$ becomes S^0 , and $[\underset{\sim}{T} \wedge \underset{\sim}{B}G_1 , \underset{\sim}{B}G_2]$ becomes the cohomotopy group $\pi^{m}(BG_1)$.

The study of $[\underset{\sim}{T} \wedge \underset{\sim}{B}G_1 , \underset{\sim}{B}G_2]$ leads to a corresponding problem about Ext groups, namely to compute

$$\mathrm{Ext}_A^{**}(H^*(G_2) , M \otimes H^*(G_1)) .$$

Here we write $H^*(G)$ as an abbreviation for $H^*(\underset{\sim}{B}G ; F_p) = H^*(BG ; F_p)$ since all our cohomology groups will have coefficients F_p , and since the cohomology of a group G is defined to be the cohomology of its classifying space BG . The letter M stands for a suitable A-module, which in the applications becomes the cohomology of $\underset{\sim}{T}$.

Of course, we only "compute" these Ext groups in the sense that we reduce them to other Ext groups. More precisely, we reduce to Ext groups of the form

$$\mathrm{Ext}_A^{**}(H^*(G_3) , M) .$$

This is analogous to reducing the homotopy problem

$$[\underset{\sim}{T} \wedge \underset{\sim}{B}G_1 , \underset{\sim}{B}G_2]$$

to problems of the form

$$[\underset{\sim}{T} , \underset{\sim}{B}G_3] .$$

For the homotopy problem this is more or less the best one can expect, and is in line with the original results of W.H. Lin [6] for the case $\underset{\sim}{T} = \underset{\sim}{S}^m$, $G_1 = Z_2$, $G_2 = 1$. (In this case G_3 takes the two values 1 , Z_2 .)

We can now explain our main result. Let U,V be elementary abelian p-groups. (We regard elementary abelian p-groups as vector spaces over F_p ; we use the letter V and the adjacent letters U,W for such vector-spaces.) In §11 we shall associate to U and V a finite set of indices X . We shall also associate to each index X an integer s(X) and an elementary abelian p-group W(X) . We shall then introduce a homomorphism

$$\bigoplus_X \mathrm{Ext}_A^{s-s(X),t-s(X)} (H^*(W(X)), M) \xrightarrow{\omega}$$
$$\longrightarrow \mathrm{Ext}_A^{s,t} (H^*(V), M \otimes H^*(U)) .$$

<u>Theorem 1.1</u>. If U and V are elementary abelian p-groups, and if the module M is bounded below and finite-dimensional over F_p in each degree, then the map

$$\bigoplus_X \mathrm{Ext}_A^{s-s(X),t-s(X)} (H^*(W(X)), M) \xrightarrow{\omega}$$
$$\longrightarrow \mathrm{Ext}_A^{s,t} (H^*(V), M \otimes H^*(U))$$

is an isomorphism.

This theorem answers the purpose stated above, and we pause to comment.

First, Theorem 1.1 is an explicit result, adapted for calculation; it follows immediately from two more conceptual results, which will be stated in §11 as Theorems 11.1 and 11.2. These conceptual results involve categorical considerations, applied in the correct categories. Roughly speaking, in the homotopy problem $[T \wedge BG_1, BG_2]$ is a representable functor of T , and the representing object is a function-spectrum; we introduce and exploit corresponding considerations in algebra. We would like to draw the reader's attention to these categorical considerations, and in particular to a construct which we call the "Burnside category"; we hope it may be of wider use. We therefore urge the reader to study at least the beginnings of §9, §10 and §11. For the moment, we omit the details needed to make the statement of (1.1) complete.

Secondly, it is essential for the applications that the map ω, which we prove to be iso in (1.1), is the same as one which arises geometrically. More precisely, in the applications we have an Adams spectral sequence for computing

$$[\underset{\sim}{T} \wedge \underset{\sim}{B}U , \underset{\sim}{B}V] .$$

We also have for each index X an Adams spectral sequence for computing

$$[\underset{\sim}{T} , \underset{\sim}{B}W(X)] ;$$

by summing these (after suitable regrading) we obtain a spectral sequence for computing

$$\bigoplus_X [\underset{\sim}{T} , \underset{\sim}{B}W(X)] .$$

By suitable constructions, involving transfer, we set up a comparison map between these spectral sequences; the source is the spectral sequence for computing $\bigoplus_X [\underset{\sim}{T} , \underset{\sim}{B}W(X)]$, and the target is the spectral sequence for computing $[\underset{\sim}{T} \wedge \underset{\sim}{B}U , \underset{\sim}{B}V]$. In particular, the map of E_2-terms is an instance of the map ω proved to be iso in (1.1). It follows that the comparison map of spectral sequence is iso. The fact that ω arises geometrically is thus important to our overall strategy; for the purposes of the present paper we need not prove it here.

Thirdly, the proof of Theorem 1.1 flows by a simple and inevitable induction over the rank of U. We sketch the step from "rank 1" to "rank 2". Obviously, if you can compute $\mathrm{Ext}_A^{**}(H^*(V) , M \otimes H^*(Z_p))$ for all M, then you can substitute $M = L \otimes H^*(Z_p)$; since $H^*(Z_p) \otimes H^*(Z_p) = H^*(Z_p \times Z_p)$, you can compute $\mathrm{Ext}_A^{**}(H^*(V) , L \otimes H^*(Z_p \times Z_p))$ in terms of groups $\mathrm{Ext}_A^{**}(H^*(W(X) , L \otimes H^*(Z_p))$, which you can compute by the same token. The proper proof will be presented in §13; here we make only three points. (a) This proof requires book-keeping; the categorical considerations are there to keep the books straight. (b) As is usual with inductions, the proof depends on formulating the inductive hypothesis in the correct generality; it is essential to consider the algebraic analogue of

$$[\underset{\sim}{T} \wedge \underset{\sim}{B}G_1 , \underset{\sim}{B}G_2]$$

rather than the algebraic analogue of the special case $\underset{\sim}{T} = \underset{\sim}{S}^m$, $G_2 = 1$. (c) The proof depends on a prior treatment of the special case $U = Z_p$, to start the induction. This case will be proved in §12, by deducing it from Theorem 1.3, which we will state as soon as we can.

First we must explain the language of "Tor-equivalences". We say that a map $\theta: L \longrightarrow M$ of A-modules is a "Tor-equivalence" if the induced map

$$\theta_*: \mathrm{Tor}^A_{**}(F_p,L) \longrightarrow \mathrm{Tor}^A_{**}(F_p,M)$$

is iso. The point of this definition emerges from the following result.

<u>Proposition 1.2</u>. If $\theta: L \longrightarrow M$ is a Tor-equivalence, then the induced map

$$\theta_*: \mathrm{Tor}^A_{**}(K,L) \longrightarrow \mathrm{Tor}^A_{**}(K,M)$$

is iso for any (right) A-module K which is bounded above; the induced map

$$\theta^*: \mathrm{Ext}_A^{**}(L,N) \longleftarrow \mathrm{Ext}_A^{**}(M,N)$$

is iso for any (left) A-module N which is bounded below and finite-dimensional over F_p in each degree.

The hypotheses of boundedness are essential. The proof, which is easy, will be given in §3.

We now introduce the principle that "localisation makes life easier".

First let us recall something about the structure of $H^*(V)$. We may identify $H^1(V)$ with V^*, the dual of V. The Bockstein boundary $\beta: H^1(V) \longrightarrow H^2(V)$ is mono; let its image be $\beta V^* \subset H^2(V)$. Let S be a subset of $\beta V^* \subset H^2(V)$. We write $H^*(V)_S$ for the result of localising so as to invert all the non-zero elements of $S \subset \beta V^* \subset H^2(V)$. Inverting s inverts λs for $s \neq 0$, so we may as well assume that S is closed under the formation of multiples λs $(\lambda \in F_p)$. The ring $H^*(V)_S$ is actually an algebra over the mod p Steenrod algebra A.

We assume $S > 0$ and suppose given a non-zero element $x \in S$. We write $\langle x\rangle$ for the subspace of βV^* generated by x. In §7 we shall introduce a homomorphism

$$H^*(V)_S \xrightarrow{\{res_W\}} \bigoplus_W H^*(W)_{S\cap\beta W^*} ;$$

here W runs over certain quotients of V, so that W^* runs over certain subspaces of V^*; more precisely, βW^* runs over complements for $\langle x\rangle$ in βV^*, that is, subspaces of βV^* such that $\beta V^* = \langle x\rangle \oplus \beta W^*$. If V is of rank n and x is given there are p^{n-1} choices for W.

<u>Theorem 1.3</u>. The map

$$H^*(V)_S \xrightarrow{\{res_W\}} \bigoplus_W H^*(W)_{S\cap\beta W^*}$$

is a Tor-equivalence.

Theorem 1.3 enables one to reduce the calculation of Ext groups for localised algebras $H^*(V)_S$ to the unlocalised case. In fact, if on the right we have an algebra $H^*(W)_{S \cap \beta W^*}$ with $S \cap \beta W^*$ non-zero, then we may choose a non-zero element $x_2 \in S \cap \beta W^*$ and apply the theorem again, and so on by induction. This process must stop in at most n steps, where n is the rank of V, since each step reduces the rank by 1.

Theorem 1.3 is proved in §7 by downwards induction over S (the more your module is localised, the easier it is to deal with it). We deduce the special case $U = Z_p$ of Theorem 1.1 from the case of Theorem 1.3 which we reach at the end of the induction, that is, the case in which S is smallest, $S = \langle x \rangle$. The case which we need in order to start the induction is that in which S is largest, $S = \beta V^*$. (In this case we write $H^*(V)_{loc}$ instead of $H^*(V)_{\beta V^*}$ to indicate localisation so as to invert all the non-zero elements of $\beta V^* \subset H^2(V)$.) This case of (1.3) will be proved in §7, by deducing it from the next result, Theorem 1.4.

For each A-module we can introduce the quotient $F_p \otimes_A M$, which has trivial Steenrod operations.

<u>Theorem 1.4</u> (a) The quotient map

$$H^*(V)_{loc} \longrightarrow F_p \otimes_A H^*(V)_{loc}$$

is a Tor-equivalence.

(b) $F_p \otimes_A H^*(V)_{loc}$ is zero except in degree $-n$, where n is the rank of V.

(c) In degree $-n$, the rank of $F_p \otimes_A H^*(V)_{loc}$ is $p^{\frac{1}{2}n(n-1)}$.

(d) The representation of $GL(V) = Aut(V)$ afforded by $F_p \otimes_A H^*(V)_{loc}$ is the Steinberg representation [17].

The case $n = 1$, $p = 2$ of (1.4) is due to [7], while the case $n = 1$, $p > 2$ is due to [3a]. Thus Theorem 1.4 generalises a step already known to be relevant.

Our proof of Theorem 1.4 is based upon the "Singer construction" [14, 15, 5]. We will deal with this more fully in §2 and §3; for the moment we need only explain three points. First, the Singer construction gives a functor $T(M)$ from A-modules to A-modules, which comes provided with a natural transformation $\varepsilon: T(M) \longrightarrow M$. Secondly, the Singer construction allows one to reduce the calculation of Ext groups for a larger module, namely $T(M)$, to the calculation of Ext for a smaller module, namely M .

<u>Theorem 1.5</u>. The map $\varepsilon: T(M) \longrightarrow M$ of Singer's construction is a Tor-equivalence.

This reduction theorem was originally found by the second and third authors independently.

Thirdly, there is a relation between $H^*(V)_{loc}$ and the iterated Singer construction

$$T^n(F_p) = T(T(\dots T(F_p) \dots)) ,$$

where n is the rank of V .

<u>Theorem 1.6</u>. There is an isomorphism of A-algebras

$$T^n(F_p) \cong H^*(V)_{loc}^{Syl(V)} .$$

Here $Syl(V)$ means a Sylow subgroup of $GL(V) = Aut(V)$; this group acts on $H^*(V)$ and on $H^*(V)_{loc}$ in the obvious way. We write M^G for the subobject of elements in M fixed under G , as usual.

The case $p = 2$ of (1.6) is due to Singer [15], while the case $p > 2$ is modelled on a result of Li and Singer [5]. More precisely, Li and Singer prove the corresponding result for the subalgebra of invariants $H^*(V)_{loc}^{Bor(V)}$, where $Bor(V)$ is a Borel subgroup of $GL(V)$. At this stage we should explain that for $p > 2$ our version of the "Singer construction" is not quite the same as that of Li and Singer [5]. Theorem 1.5 is true for both versions; but for the purposes of our proof, a reduction theorem like (1.5) grows more useful as $T(M)$ grows larger. Our version of $T(M)$ is (roughly speaking) $(p-1)$ times as large as that of Li and Singer [5], and our subalgebra of invariants is (roughly speaking) $(p-1)^n$ times as large as theirs; this allows us to get closer to $H^*(V)_{loc}$. Of course, for $p = 2$, this point disappears.

After our results for the case $p = 2$ were known to interested parties, G. Carlsson conceived a remarkable argument, which shows that the Segal conjecture for finite groups in general can be deduced from the Segal conjecture for elementary abelian p-groups [2, 3]. By the private communication of manuscripts, we assured Carlsson that our methods worked as well for p odd as for $p = 2$, and that we stood ready to prove almost any result he might reasonably require as input for his argument, subject to two provisos. First, the only groups G to be considered were to be elementary abelian p-groups, and secondly, the results required were to lie in ordinary stable homotopy rather than in equivariant homotopy.

Since that time, there has appeared a growing prospect that an optimised form of Carlsson's argument may require as input only the calculation of Ext groups, rather than results in homotopy. More precisely, it seems that as a minimal input, Theorem 1.4 should suffice. It therefore seems reasonable to accelerate the publication of our results on Ext groups.

Priddy and Wilkerson [13] have shown how the deduction of Theorem 1.4 from (1.5) and (1.6) may be illuminated by their observation that $H^*(V)_{loc}$ is projective as a module over $F_p[GL(V)]$. However, we will present our original argument, which is elementary.

Mitchell [10] and his collaborators [4, 11] have shown that the A-modules $H^*(U)$, $H^*(V)$ can be decomposed into various summands. Let N be a typical summand of $H^*(U)$, and let L be a typical summand of $H^*(V)$; then

$$\mathrm{Ext}_A^{**}(L\,,\ M \otimes N)$$

occurs as a direct summand of

$$\mathrm{Ext}_A^{**}(H^*(V)\,,\ M \otimes H^*(U)) .$$

Since Theorem 1.1 computes the latter, it would seem regrettable if one could not identify the former. We leave this question for a later paper.

The body of this paper is organised as follows.

§2 and §3 deal with the Singer construction. §2 should enable the reader to understand the rest of the paper; §3 gives the proofs. §4 locates the subalgebra of invariants $H^*(V)_{loc}^{Syl(V)}$, and also provides a chain of A-submodules between $H^*(V)_{loc}^{Syl(V)}$ and $H^*(V)_{loc}$, for use in proving homological results later. §5 identifies the subalgebra of invariants with the iterated Singer construction, and so proves (1.6). §6 completes the proof of Theorem 1.4 (a) - (c); the proof of part (d), which is not essential to our purpose, is postponed to §8. §7 proves Theorem 1.3; the idea is to take information about objects which are more localised and deduce information about objects which are less localised.

At this point we must say some more about the proof of Theorem 1.1. We have said that it involves book-keeping. The art of book-keeping is to establish a correspondence between entries in a ledger, where the information is easy to find, and certain aspects of the real world, where things may be harder. The analogue of the real world, for us, is a category $\mathcal{E}$ where we keep our unknown Ext groups. The analogue of the ledger is a category A^{gr} which is open to inspection. The analogue of the correspondence between the two (which is vital) is a certain functor β from A^{gr} to $\mathcal{E}$.

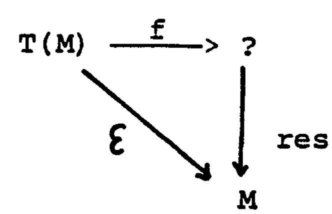
In §9 we shall set up the category A^{gr}, which corresponds to our ledger. In §10 we set up the functor β. This does take some work; but the effort is justified, because the final proof in §13, which is so short and sweet, depends totally on the functorial properties of β. §11 gives the conceptual restatement of Theorem 1.1.

It remains only to take the special case $U = Z_p$ and reconcile the statements of the bookkeeping system with the facts proved in §7. This is done in §12.

We are grateful to W.M. Singer for keeping us informed of his work, and similarly to G. Carlsson and to Priddy and Wilkerson. We are grateful to the Sloan Foundation, to the University of Aarhus, and to the University of Chicago for enabling us to meet in spite of our usual geographical separation.

§2. <u>The Singer Construction: Statements</u>. In this section we will state some facts about the Singer construction. The statements we need to use later will be proved in §3. Since our construction is not exactly the same as that of Li and Singer [5], we must clearly add something to their work; we will try to make our account self-contained. (Of course, for $p = 2$, [15] suffices.) We treat the Singer construction as a matter of pure algebra; we realise the value and interest of the topological interpretation, but we neglect that aspect for brevity.

The Singer construction accepts as input an A-module M, which may be an A-algebra. The Singer construction delivers as output a diagram of the following form, which we must explain below.



If M is an A-module, then this is a diagram of A-modules and A-maps. If M is an A-algebra, then the objects are A-algebras, and f is a map of A-algebras, but ε and res are not.

The map ε is needed for the statement of Theorem 1.5. The map f is needed for the proof of Theorem 1.6.

Additively, the Singer construction $T(M)$ is isomorphic to the tensor product $L \otimes M$ of M with a fixed object L. (However, the A-module structure on $T(M)$ is not given by the usual "diagonal" formula.) Just as one assigns to each cohomology theory K^* the "coefficient groups" $K^*(P)$, so to each functor T from A-modules to A-modules one assigns the "coefficient groups" $T(F_p)$. In our case $T(F_p)$ is $L \otimes F_p$, that is, L ; thus L plays the role of a "coefficient ring" for our construction. It is usual to write T for this coefficient ring, and to write $T(M) = T \otimes M$.

This coefficient ring T may be identified with $H^*(Z_p)_{loc}$, the case $n = 1$ of the algebra considered in (1.4). We will describe this algebra explicitly to fix notation. Let e be a generator in $H^1(Z_p)$, and let $x = \beta e \in H^2(Z_p)$. Then $H^*(Z_p)$ is a free module on two generators 1 , e over the polynomial algebra $F_p[x]$. (We state matters in this way to avoid making too much distinction between cases; for $p = 2$ we have $e^2 = x$ and for $p > 2$ we have $e^2 = 0$.) Similarly, $H^*(Z_p)_{loc}$ is a free module on two generators 1 , e over the algebra of finite Laurent series $F_p[x,x^{-1}]$.

We next describe the object "?" in the top right-hand corner of our diagram. This is a completed tensor product $T \hat{\otimes} M$. We topologise $T \otimes M$ so that a typical neighbourhood of zero is

$$\left(\sum_{r \le -N} T^r \right) \otimes M .$$

$T \hat{\otimes} M$ is of course the completion of $T \otimes M$; a typical element of $T \hat{\otimes} M$ is a "downward-going formal Laurent series"

$$\sum_{r \le R} x^r \otimes m'_r + \sum_{r \le R} e x^r \otimes m''_r .$$

We place a A-module structure on $T \hat{\otimes} M$ in the obvious way: we take the usual (diagonal) action on $T \otimes M = H^*(Z_p)_{loc} \otimes M$ and pass to the completion. (We repeat that this is not the action we intend on $T(M) = T \otimes M$.)

Our diagram now looks as follows.

$$\begin{array}{ccc} T(M) = T \otimes M & \xrightarrow{\ f\ } & T \hat{\otimes} M \\ & \searrow^{\varepsilon} & \downarrow \text{res} \\ & & M \end{array}$$

In §3 we will construct the maps f and res by means which are more or less conceptual. Here we will merely record (for later use) the explicit formulae which say what f and res do.

(2.1) f is a map of modules over $T = H^*(Z_p)_{loc}$.

It is therefore sufficient to give f on elements of the form $1 \otimes m$.

(2.2) $f(1 \otimes m) =$

$$= \sum_{k \geq 0} (-1)^k x^{-k(p-1)} \otimes P^k m + \sum_{k \geq 0} (-1)^{k+1} ex^{-k(p-1)-1} \otimes \beta P^k m .$$

The signs in this formula are there for good and sufficient reason.

(2.3) $\text{res}\left(\sum_{r \leq R} x^r \otimes m'_r + \sum_{r \leq R} ex^r \otimes m''_r\right) = m''_{-1}$.

It is reasonable to think of this map as a "residue", since it "takes the coefficient of ex^{-1} ", and for $p = 2$ this becomes the coefficient of e^{-1} .

(2.4)

$$\varepsilon(ex^r \otimes m) = \begin{cases} (-1)^k P^k m & \text{if } r = k(p-1)-1 , k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon(x^r \otimes m) = \begin{cases} (-1)^{k+1} \beta P^k m & \text{if } r = k(p-1) , k \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Perhaps we should point out explicitly that res and ε are maps of degree +1 .

We turn to the multiplicative properties of our diagram.

(2.5) The obvious action of $T = H^*(Z_p)_{loc}$ on $T(M) = T \otimes M$ is an A-action; that is, it satisfies the Cartan formula.

We now consider the special case in which M is an A-algebra. In this case we have an obvious product in $T \otimes M = H^*(Z_p)_{loc} \otimes M$.

(2.6) If M is an A-algebra then the obvious product map for $T(M) = T \otimes M$ is an A-map; that is, this product satisfies the Cartan formula.

The product in $T \hat{\otimes} M$ is obtained by passing to the completion from the product in $T \otimes M$; this product makes $T \hat{\otimes} M$ an A-algebra. These are the products with respect to which f is a map of algebras, as we have stated.

This completes our list of properties of the Singer construction.

§3. The Singer Construction: Proofs. In this section we will prove those facts about the Singer construction which we need to use. Our plan is as follows. We will begin with a construction which is conceptual and a priori; then we will transform it into the form described in §2. In other words, we do not start with $T(M)$; we define T in terms of T' , and T' in terms of T'' , these being similar constructions which approximate to T . We start with T'' ; here the description makes the A-module structure transparent, but it does not make any multiplicative structure transparent. In fact, the gradings need to be changed by one unit before the multiplicative structure can work correctly. So we next introduce T' ; this is isomorphic to T'' , but now the structure of $T'(M)$ as a module over the coefficient ring T' becomes transparent. We then identify the coefficient ring T' with a more familiar object, namely the localisation $H^*(\Sigma_p)_{loc}$ of the cohomology of the symmetric group Σ_p . To make our construction "bigger" than that of Li and Singer [5] we must replace $H^*(\Sigma_p)_{loc}$ by $H^*(Z_p)_{loc}$, and in this way we reach T .

As in [7] we use the dual A_* of the mod p Steenrod algebra A [9]. This dual has exterior generators $\tau_0, \tau_1, \ldots$ and polynomial generators $\xi_1, \xi_2, \ldots$. (For $p = 2$ it has only polynomial generators $\zeta_1, \zeta_2, \ldots$; one should interpret τ_r as ζ_{r+1} and ξ_r as ζ_r^2 .)

We have to use the usual finite subalgebras of the Steenrod algebra. We write $A_*(n)$ for the quotient $A_*/I(n)$, where the ideal $I(n)$ is generated by the τ_r with $r > n$ and the $\xi_r^{p^s}$ with $r + s \geq n + 1$. (If $p = 2$, $I(n)$ is generated by the $\zeta_r^{2^s}$ with $r + s \geq n + 2$.) The quotient $A_*(n)$ is dual to a sub-Hopf-algebra $A(n)$ of A. The subalgebra $A(-1)$ is to be interpreted as the ground field F_p ; the subalgebra $A(0)$ is the exterior algebra generated by β. Otherwise $A(n)$ is generated by β and the P^k with $k < p^n$. (If $p = 2$, $A(n)$ is generated by the Sq^k with $k < 2^{n+1}$.)

We also introduce a localised quotient

$$B_*(n) = (A_*/J(n))[\xi_1^{-1}] \quad (n \geq 0)$$

where the ideal $J(n)$ is generated by the τ_r with $r > n$ and the $\xi_r^{p^s}$ with $r \geq 2$, $r + s \geq n + 1$. (If $p = 2$, $J(n)$ is generated by the $\zeta_r^{2^s}$ with $r \geq 2$, $r + s \geq n + 2$.) The object $A_*/J(n)$ is a left comodule over $A_*(n)$ and a right comodule over $A_*(n-1)$. Multiplication by $\xi_1^{p^n}$ preserves both comodule structures. Since $B_*(n)$ may be regarded as the direct limit of $A_*/J(n)$ under multiplication by $\xi_1^{p^n}$, it becomes a left comodule over $A_*(n)$ and a right comodule over $A_*(n-1)$. It is also an algebra, and is finite-dimensional over F_p in each degree.

We define $B(n)$ to be the dual of $B_*(n)$. This object is a binmodule; it is a left module over $A(n)$ and a right module over $A(n-1)$.

For example, $B_*(0)$ has a base consisting of the elements ξ_1^k and $\tau_0 \xi_1^k$ for $k \in Z$. We take the dual base in $B(0)$ and call its elements P^k and βP^k for $k \in Z$.

Since we have canonical maps $A_* \longrightarrow B_*(n+1) \longrightarrow B_*(n)$, we have canonical maps $B(n) \longrightarrow B(n+1) \longrightarrow A$ preserving all the relevant structure. The element written P^k in $B(0)$ maps to P^k in A if $k \geq 0$, to 0 if $k < 0$; similarly for βP^k.

Lemma 3.1. (i) $B(n)$ is free as a left module over $A(n)$; the elements P^k with $k \equiv 0 \bmod p^n$ may be taken as a base; the left-primitive subobject of $B_*(n)$ is $F_p[\xi_1^{p^n}, \xi_1^{-p^n}]$.

(ii) $B(n)$ is free as a right module over $A(n-1)$; the elements P^k , βP^k for $k \in Z$ may be taken as a base. Equivalently, the map

$$B(0) \otimes A(n-1) \longrightarrow B(n)$$

is iso.

Proof. (i) It is clear that as a left comodule over $A_*(n)$, $B_*(n)$ is a direct sum of copies of $A_*(n)$ shifted by multiplication with the powers $\xi_1^{rp^n}$, $r \in Z$.

(ii) First we will show that in A , the elements P^k , βP^k with k sufficiently large (say $k \geq k_o$) are linearly independent under right multiplication by $A(n-1)$. In fact, A has a base consisting of the admissible monomials

$$m = \beta^{\varepsilon_1} P^{k_1} \beta^{\varepsilon_2} P^{k_2} \ldots \beta^{\varepsilon_r} P^{k_r} .$$

The subalgebra $A(n-1)$ is finite-dimensional over F_p , and therefore all its elements may be written as F_p-linear combinations of a finite number of admissible monomials, say $m_1, m_2, \ldots, m_s$. We have only to arrange that $k_o \geq \varepsilon_1 + pk_1$ for all the monomials m_t of this finite set; then the products

$$P^k m_t \, , \; \beta P^k m_t \quad \text{with} \quad k \geq k_o$$

will be distinct admissible monomials, and therefore linearly independent. This proves that in A , the elements P^k , βP^k with $k \geq k_o$ are linearly independent under right multiplication by $A(n-1)$.

Using the canonical map $B(n) \longrightarrow A$, we see that the same result holds also in $B(n)$.

We will now deduce that all the elements P^k , βP^k in $B(n)$ are linearly independent under right multiplication by $A(n-1)$. In fact, multiplication by $\xi_1^{-rp^n}$ gives a linear map $B_*(n) \longrightarrow B_*(n)$ which is a map of bicomodules. Its dual is a linear map $B(n) \longrightarrow B(n)$ which is a map of bimodules. Suppose we had any linear relation over $A(n-1)$ between the elements P^k , βP^k in $B(n)$; by applying this map for a suitable value of r , we could shift the relation up until it involved only elements P^k , βP^k with $k \geq k_o$.

Thus we see that the map

$$B(0) \otimes A(n-1) \longrightarrow B(n)$$

is mono. On the other hand, the objects $B(0) \otimes A(n-1)$ and $B(n)$ have the same dimension over F_p in each degree, namely

$$2^n \, p^{\frac{1}{2}n(n-1)} .$$

So if the map

$$B(0) \otimes A(n-1) \longrightarrow B(n)$$

is mono, it is iso.. This proves (3.1).

We can now define our first approximation to T . If M is an $A(n-1)$-module, we define

$$T''(M) = B(n) \otimes_{A(n-1)} M ;$$

this is an $A(n)$-module. If M is an $A(n)$-module, then the canonical map

$$B(n) \otimes_{A(n-1)} M \longrightarrow B(n+1) \otimes_{A(n)} M$$

is iso, since both groups are isomorphic to $B(0) \otimes M$ by (3.1) (ii). Thus the definition is essentially independent of n. If M is an A-module, we can interpret $T''(M)$ as the attained limit $\underset{n}{\underline{\mathrm{Lim}}_{>}} (B(n) \otimes_{A(n-1)} M)$; in this case $T''(M)$ is an A-module.

If M is an A-module, the map

$$B(n) \otimes_{A(n-1)} M \longrightarrow A \otimes_{A(n-1)} M \longrightarrow M$$

passes to the limit to give a map of A-modules

$$\varepsilon'' : T''(M) \longrightarrow M .$$

The explicit formulae for ε'' are as follows.

$$\varepsilon''(P^k \otimes m) = \begin{cases} P^k m & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 . \end{cases}$$

(3.2)

$$\varepsilon''(\beta P^k \otimes m) = \begin{cases} \beta P^k m & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 . \end{cases}$$

We introduce a further structure map. By dualising the product map of $B_*(n)$ we obtain a coproduct map

$$B(n) \xrightarrow{\psi} B(n) \hat{\otimes} B(n) .$$

(Here and in general we topologise tensor products $L \otimes M$ so that $\ell \otimes m$ is small if either ℓ or m is small. We topologise each factor $B(n)$ so that elements of large negative degree are small.) The coproduct map ψ preserves the left action of $A(n)$ and the right action of $A(n-1)$. (Here the actions on

$B(n) \hat{\otimes} B(n)$ are given by the usual diagonal formulae plus continuity.) The coproduct ψ induces a map

$$T''(M \otimes N) \xrightarrow{\Psi} T''(M) \hat{\otimes} T''(N) ;$$

this is an $A(n)$-map or an A-map according to the case. The explicit formulae for Ψ are as follows.

$$\Psi(P^k \otimes m \otimes n) = \sum_{i+j=k} (P^i \otimes m) \otimes (P^j \otimes n)$$

(3.3) $\Psi(\beta P^k \otimes m \otimes n) =$

$$= \sum_{i+j=k} (\beta P^i \otimes m) \otimes (P^j \otimes n) + \sum_{i+j=k} (-1)^{\deg m} (P^i \otimes m) \otimes (\beta P^j \otimes n)$$

We define the map f'' to be the following composite.

$$\begin{array}{ccc} T''(M) = T''(F_p \otimes M) & \xrightarrow{\Psi} & T''(F_p) \hat{\otimes} T''(M) \\ & & \downarrow 1 \hat{\otimes} \varepsilon'' \\ & & T''(F_p) \hat{\otimes} M \end{array}$$

By writing ε'' we have assumed that M is an A-module; then f'' is an A-map. The explicit formulae for f'' are as follows.

$$f''(P^k \otimes m) = \sum_{i+j=k, j \geq 0} (P^i \otimes 1) \otimes P^j m$$

(3.4) $f''(\beta P^k \otimes m) =$

$$= \sum_{i+j=k, j \geq 0} (\beta P^i \otimes 1) \otimes P^j m + \sum_{i+j=k, j \geq 0} (P^i \otimes 1) \otimes \beta P^j m .$$

The map f'' is mono; this may be seen by filtering source and target so as to give

$$f''(P^k \otimes m) \equiv (P^k \otimes 1) \otimes m$$

$$f''(\beta P^k \otimes m) \equiv (\beta P^k \otimes 1) \otimes m$$

modulo less significant terms.

We must now supply a coefficient ring. Let $T'(n)$ be the right-primitive subobject of $B_*(n)$; the reader may have noticed that Lemma 3.1 does not mention it explicitly. However, the maps

$$\cdots \longrightarrow B_*(n+1) \longrightarrow B_*(n) \longrightarrow \cdots \longrightarrow B_*(0)$$

induce

$$\cdots \longrightarrow T'(n+1) \longrightarrow T'(n) \longrightarrow \cdots \longrightarrow T'(0)$$

and all these maps are iso, since each $T'(n)$ is dual to a quotient $B(n) \otimes_{A(n-1)} F_p$ with base P^k , βP^k for $k \in Z$. Let us write T' for the (attained) limit $\varprojlim_n T'(n)$; this is an algebra. It contains an algebra $F_p[\xi,\xi^{-1}]$ of finite Laurent series on one generator ξ , which maps to ξ_1 in $B_*(0)$ and (for example) to $\xi_1 - \xi_1^{-p}\xi_2$ in $B_*(2)$. As a module over $F_p[\xi,\xi^{-1}]$, T' is free on two generators $1,\tau$ which map to $1,\tau_0$ in $B_*(0)$. Similarly if $p = 2$; we should then interpret τ as ζ and ξ as ζ^2 .

Multiplication by $t' \in T'$ gives a linear map $B_*(n) \longrightarrow B_*(n)$ which is a map of right comodules. Its dual is a map of right modules

$$B(n) \xrightarrow{t'} B(n)$$

which we think of as "cap product with t' ." If we use cohomological degrees then ξ must be given degree $-2(p-1)$ since "cap product with ξ " lowers degree by $2(p-1)$; similarly, τ must be given degree -1 . Cap product with t' defines

$$B(n) \otimes_{A(n-1)} M \xrightarrow{t'} B(n) \otimes_{A(n-1)} M$$

and passes to the limit to give

$$T''(M) \xrightarrow{t'} T''(M) \ .$$

In $B_*(n)$ we have the associative law $t'(xy) = (t'x)y$; dualising, we see that the following diagram is commutative.

$$\begin{array}{ccc} B(n) & \xrightarrow{\psi} & B(n) \hat{\otimes} B(n) \\ {\scriptstyle t'}\downarrow & & \downarrow{\scriptstyle t' \hat{\otimes} 1} \\ B(n) & \xrightarrow{\psi} & B(n) \hat{\otimes} B(n) \end{array}$$

It follows that we have a similar diagram for

$$T''(M \otimes N) \xrightarrow{\psi} T''(M) \hat{\otimes} T''(N) \ .$$

In particular, the map

$$T''(M) \xrightarrow{f''} T''(F_p) \hat{\otimes} M$$

is a map of modules over T' .

We now rewrite $T''(M)$ in the form $T' \otimes M$. We define a map

$$T' \otimes M \xrightarrow{\theta} T''(M)$$

by

$$\theta(t' \otimes m) = (-1)^{\deg t'} t'(\beta \otimes m) \ .$$

Here the sign arises because θ is of degree +1 , and we wish θ to be a map of T'-modules with the usual sign conventions. As for the presence of β , it is sufficient for the moment to note that $T''(F_p)$ is a free T'-module on one generator β , but it is not a free T'-module on the generator 1 (at least if $p > 2$). Anyway, the map θ is an isomorphism of T'-modules. We now define

$$T'(M) = T' \otimes M$$

and give it an action of $A(n)$ or A , as the case may be, by using θ to pull back the action of $A(n)$ or A on $T''(M)$. Of course, since θ is of degree 1 , this introduces the usual signs.

By rewriting the source and target of f'' , we obtain a map

$$T'(M) = T' \otimes M \xrightarrow{f'} T' \hat{\otimes} M \, .$$

This map f' is a map of modules over T' , and also an A-map (provided of course that on the right, the A-action on T' is that which it gets as $T'(F_p)$.) Otherwise the explicit formulae for f' are as follows.

$$f'(1 \otimes m) =$$

$$(3.5) \quad = \sum_{j \geq 0} (\xi^j \otimes 1) \otimes P^j m - \sum_{j \geq 0} (\tau\xi^j \otimes 1) \otimes \beta P^j m \qquad (p > 2)$$

$$= \sum_{j \geq 0} (\zeta^j \otimes 1) \otimes Sq^j m \qquad (p = 2).$$

The minus sign arises from the sign in the definition of θ . The map f' , like f'' , is mono.

Now that we have a coefficient ring T' with the correct products, we have an external product

$$(T' \otimes M) \otimes (T' \otimes N) \longrightarrow T' \otimes (M \otimes N)$$

given by

$$(t_1' \otimes m)(t_2' \otimes n) = (-1)^{(\deg m)(\deg t_2')} t_1' t_2' \otimes m \otimes n \, .$$

Similarly, we have an external product

$$(T' \hat{\otimes} M) \otimes (T' \hat{\otimes} N) \longrightarrow T' \hat{\otimes} (M \otimes N) \, .$$

We can check that the map f' given by (3.5) preserves external products. In particular, if M is an A-algebra, then we get internal products in $T' \otimes M$ and in $T' \hat{\otimes} M$, and the map f' preserves these products. The shift of degrees by 1 was essential for all this.

We prepare to replace T' with a more familiar object. We recall that $H^*(Z_p)$ contains $H^*(\Sigma_p)$. The latter is a free module on two generators 1, ex^{p-2} over the polynomial subalgebra $F_p[x^{p-1}]$. It is natural to define $H^*(\Sigma_p)_{loc}$ by localising so as to invert x^{p-1}. Thus $H^*(Z_p)_{loc}$ contains $H^*(\Sigma_p)_{loc}$; the latter is a free module on two generators 1, ex^{-1} over the Laurent subalgebra $F_p[x^{p-1}, x^{-(p-1)}]$.

<u>Lemma 3.6</u>. There is an isomorphism of algebras

$$T' \xrightarrow{\phi} H^*(\Sigma_p)_{loc}$$

which is also an isomorphism of A-modules from $T'(F_p)$ to $H^*(\Sigma_p)_{loc}$. The explicit formulae are

$$\phi(\xi) = -x^{-(p-1)} \, , \quad \phi(\tau) = ex^{-1} \, .$$

<u>Proof</u>. We have the following $A(n)$-map of degree -1.

$$B(n) \longrightarrow A \xrightarrow{\gamma} H^*(\Sigma_p)_{loc} \, .$$

Here γ is defined by

$$\gamma(a) = (-1)^{\deg(a)} a(ex^{-1}) ;$$

we introduce the sign because we want an $A(n)$-map of degree -1 with the usual signs. We claim that in sufficiently high degrees, this map factors through $B(n) \otimes_{A(n-1)} F_p$. In fact, $A(n-1)$ is bounded above, say by degree d . So for any element $a \in A(n-1)$ of positive degree, $a(ex^{-1})$ lies in $H^*(\Sigma_p)$ in the range of degrees from 0 to $d-1$. Now $H^*(\Sigma_p)$ satisfies the unstable axiom, so we have

$$P^k a(ex^{-1}) = 0 \ , \ \beta P^k a(ex^{-1}) = 0$$

for k sufficiently large - more precisely for $k > \frac{1}{2}(d-1)$. So the $A(n)$-map factors as stated.

The $A(n)$-map we have used is given by the following formulae.

$$P^k \longmapsto \begin{cases} (-1)^k \, ex^{k(p-1)-1} & (k \geq 0) \\ 0 & (k < 0) \end{cases}$$

$$\beta P^k \longmapsto \begin{cases} (-1)^{k+1} \, x^{k(p-1)} & (k \geq 0) \\ 0 & (k < 0) \ . \end{cases}$$

Here the sign $(-1)^k$ comes from the following calculation.

$$\left(\sum_{k=o}^{\infty} P^k\right) x = x(1 + x^{p-1})$$

so

$$\left(\sum_{k=o}^{\infty} P^k\right) x^{-1} = x^{-1}(1 + x^{p-1})^{-1}$$

$$= x^{-1}(1 - x^{p-1} + x^{2(p-1)} - x^{3(p-1)} \dots) \ .$$

We have thus shown that in sufficiently high degrees we have a composite

$$T'(F_p) \xrightarrow{\theta} T''(F_p) \longrightarrow H^*(\Sigma_p)_{loc}$$

which is an A-map and is given by the following formulae.

$$\xi^{-k} \longrightarrow (-1)^{k+1} x^{k(p-1)}$$
$$\tau\xi^{-k} \longrightarrow (-1)^{k+1} ex^{k(p-1)-1} .$$

Up to a sign -1, this is the map ϕ in the enunciation, so ϕ is an $A(n)$-map in sufficiently high degrees.

But now we can use again the device of shifting elements into sufficiently high degrees. The map ϕ carries the action of $\xi^{-p^n} \in T'$ (on $T'(F_p)$) into the action of $(-x^{p-1})^{p^n}$ (on $H^*(\Sigma_p)_{loc}$). The action of ξ^{-p^n} on $T'(F_p)$ is an $A(n)$-map, and the action of $(-x^{p-1})^{p^n}$ on $H^*(\Sigma_p)_{loc}$ is an $A(n)$-map. These actions can be used to shift elements into arbitrarily high degrees; therefore ϕ is an $A(n)$-map in all degrees. This proves (3.6).

We now use Lemma 3.6 to identify T' with $H^*(\Sigma_p)_{loc}$. Of course this changes nothing; everything has the same properties as before. The explicit formula for f' now reads as follows.

$$f'(1 \otimes m) =$$
(3.7)
$$= \sum_{k \geq 0} (-1)^k x^{-k(p-1)} \otimes P^k m + \sum_{k \geq 0} (-1)^{k+1} ex^{-k(p-1)-1} \otimes \beta P^k m .$$

At this point we have obtained the formula (2.2).

Lemma 3.8. The external products

$$T'(M) \otimes T'(N) \longrightarrow T'(M \otimes N)$$

satisfy the Cartan formula.

Proof. There are two possible interpretations of this lemma. First we assume that M and N are A-modules, so that the Cartan formula is asserted for all $a \in A$. In this case we have the following commutative diagram.

$$\begin{array}{ccc} T'(M) \otimes T'(N) & \longrightarrow & T'(M \otimes N) \\ {\scriptstyle f' \otimes f'} \downarrow & & \downarrow {\scriptstyle f'} \\ (T' \hat{\otimes} M) \otimes (T' \hat{\otimes} N) & \longrightarrow & T' \hat{\otimes} (M \otimes N) \end{array}$$

Since f' is mono and the Cartan formula holds for the external products in $T' \hat{\otimes} (M \otimes N)$, it must hold for the external products in $T'(M \otimes N)$ also.

Secondly we assume that M and N are merely $A(n-1)$-modules, so that the Cartan formula is asserted for all $a \in A(n)$. In this case we embed the $A(n-1)$-modules M,N in the A-modules $\overline{M} = A \otimes_{A(n-1)} M$, $\overline{N} = A \otimes_{A(n-1)} N$. The functor $T'(M) = T' \otimes M$ preserves exactness, so the map

$$T'(M \otimes N) \longrightarrow T'(\overline{M} \otimes \overline{N})$$

is mono. The required Cartan formula holds for the external products in $T'(\overline{M} \otimes \overline{N})$, so it must hold in $T'(M \otimes N)$. This proves (3.8).

In particular (taking $M = F_p$) we see that the action of T' on $T'(N)$ satisfies the Cartan formula.

We can now proceed to our final construction $T(M)$. We define $T = H^*(Z_p)_{loc}$, and we define

$$T(M) = T \otimes_{T'} T'(M) = T \otimes M .$$

Given Lemma 3.8, $T(M)$ automatically receives an action of $A(n)$ or A as the case may be, extending the action of $A(n)$ or A on $T'(M)$, so that the products yz $(y \in T , z \in T(M))$ satisfy the Cartan formula.

Assuming M is an A-module, we have an embedding

$$T' \hat{\otimes} M \longrightarrow T \hat{\otimes} M ,$$

and $T \hat{\otimes} M$ is a module over T ; therefore f' extends automatically to a map

$$T(M) = T \otimes_{T'} T'(M) \xrightarrow{f} T \hat{\otimes} M$$

of modules over T . This map f is automatically an A-map. If M is an A-algebra then the obvious product on $T(M) = T \otimes_{T'} T'(M) = T \otimes M$ makes it an A-algebra, and f is a map of algebras. This completes our description of the map f .

We turn to the map res . As an A-module, $T = H^*(Z_p)_{loc}$ splits as a direct sum of submodules indexed over the residue classes mod $(p-1)$; the r^{th} summand consists of the groups in degrees congruent to $2r$, $2r-1$ mod $2(p-1)$. (Clearly β preserves these summands, and the operations P^k are of degree congruent to 0 mod $2(p-1)$.) We therefore have an A-map

$$T = H^*(Z_p)_{loc} \longrightarrow H^*(\Sigma_p)_{loc}$$

which projects onto the summand in degrees congruent to 0 , -1 mod $2(p-1)$. We may compose this with the following A-map.

$$H^*(\Sigma_p)_{loc} \xleftarrow[\cong]{\phi} T'(F_p) \xrightarrow[\cong]{\theta} T''(F_p) \xrightarrow{\varepsilon''} F_p \ .$$

Clearly the result is an A-map of degree +1 , and it annihilates the groups in degrees other than -1 . In this degree it carries ex^{-1} to -1 . (The sign comes from that in the definition of θ .) However, we still get an A-map if we change the sign of our map. Tensoring with the identity map of M and passing to the completion, we obtain an A-map

$$T \hat{\otimes} M \xrightarrow{res} M \ ;$$

it is given by the formula (2.3).

Finally, we can define $\varepsilon: T(M) \longrightarrow M$ to be the composite res. f , and it is given by the formula (2.4).

We have now justified everything about the structure of the Singer construction stated in §2. We turn to the reduction theorem, (1.5).

Lemma 3.9. (i) If M is A-free then T(M) is A-flat.

(ii) If M is A-free then the map

$$F_p \otimes_A T(M) \xrightarrow{1\otimes\varepsilon} F_p \otimes_A M$$

is iso.

Proof. (i) If M is A-free then it is A(n-1)-free. If M is free over A(n-1) then $B(n) \otimes_{A(n-1)} M$ is a direct sum of copies of B(n) , so it is free over A(n) by (3.1)(i). This shows that T'(M) is free over A(n) . Over A(n) , $T(M) = T \otimes_{T'} T'(M)$ is a direct sum of (p-1) copies of T'(M) , because multiplication by x^{p^n} gives a shift map commuting with A(n) . Therefore T(M) is free over A(n) ; this holds for all n . So

$$\mathrm{Tor}^{A}_{s,t}(K\ ,\ T(M)) = \underset{n}{\underrightarrow{\mathrm{Lim}}}\ \mathrm{Tor}^{A(n)}_{s,t}(K\ ,\ T(M))$$

$$= 0 \quad \text{for} \quad s > 0\ .$$

Thus $T(M)$ is A-flat.

(ii) It is sufficient to prove the special case $M = A$, for the general case follows by passing to direct sums. By (3.1)(i), $B(n) \otimes_{A(n-1)} A(n-1)$ is $A(n)$-free on generators P^{kp^n}, $k \in Z$. Thus $T'(A(n-1))$ is $A(n)$-free on generators $\tau\xi^{-kp^n}$, or alternatively $ex^{k(p-1)p^n-1}$, $k \in Z$. Using the shift map x^{p^n}, as above, $T(A(n-1))$ is $A(n)$-free on generators ex^{kp^n-1}, $k \in Z$. Thus $F_p \otimes_{A(n)} T(A(n-1))$ has a base consisting of the elements ex^{kp^n-1}, $k \in Z$. Passing to the direct limit, $F_p \otimes_A T(A)$ has a base consisting of the single element ex^{-1}. Therefore the map

$$F_p \otimes_A T(A) \xrightarrow{1\otimes\varepsilon} F_p \otimes_A A = F_p$$

is iso. This proves the lemma.

<u>Proof of Theorem 1.5</u>. Take a free resolution of M, say

$$\cdots \longrightarrow C_s \longrightarrow C_{s-1} \longrightarrow \cdots \qquad \cdots \longrightarrow C_o \longrightarrow M\ .$$

We obtain the following ladder diagram.

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & T(C_s) & \longrightarrow & T(C_{s-1}) & \longrightarrow \cdots & & \cdots \longrightarrow & T(C_o) & \longrightarrow & T(M) \\
& \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} \\
\cdots \longrightarrow & C_s & \longrightarrow & C_{s-1} & \longrightarrow \cdots & & \cdots \longrightarrow & C_o & \longrightarrow & M
\end{array}$$

Since T preserves exactness and carries free modules to flat ones, the upper row is a flat resolution of $T(M)$. We can compute Tor from a flat resolution. When we apply $F_p \otimes_A$ —— , we get the following ladder diagram.

$$\begin{array}{ccccccc} \cdots \longrightarrow & F_p \otimes_A T(C_s) & \longrightarrow & F_p \otimes_A T(C_{s-1}) & \longrightarrow \cdots \\ & \downarrow {\scriptstyle 1 \otimes \varepsilon} & & \downarrow {\scriptstyle 1 \otimes \varepsilon} & \\ \cdots \longrightarrow & F_p \otimes_A C_s & \longrightarrow & F_p \otimes_A C_{s-1} & \longrightarrow \cdots \end{array}$$

Here the vertical arrows are iso. Therefore the induced map

$$\varepsilon_*: \mathrm{Tor}^A_{**}(F_p , T(M)) \longrightarrow \mathrm{Tor}^A_{**}(F_p , M)$$

is iso. This proves Theorem 1.5.

<u>Proof of Proposition 1.2</u>. Suppose that $\theta: L \longrightarrow M$ is a Tor-equivalence; that is, the induced map

$$\theta_*: \mathrm{Tor}^A_{**}(F_p , L) \longrightarrow \mathrm{Tor}^A_{**}(F_p , M)$$

is iso. By an obvious 5-lemma argument, we see that

$$\theta_*: \mathrm{Tor}^A_{**}(K,L) \longrightarrow \mathrm{Tor}^A_{**}(K,M)$$

is iso for any A-module K which is finite-dimensional over F_p. If K is bounded above, then K is a direct limit of submodules which are finite-dimensional over F_p, and Tor commutes with direct limits; therefore

$$\theta_*: \mathrm{Tor}^A_{**}(K,L) \longrightarrow \mathrm{Tor}^A_{**}(K,M)$$

is iso when K is bounded above.

Passing to duals, we see that the map

$$(\theta_*)^*: (\mathrm{Tor}^A_{**}(K,L))^* \longleftarrow (\mathrm{Tor}^A_{**}(K,M))^*$$

is iso. Since we have a natural isomorphism

$$(\mathrm{Tor}^{A}_{**}(K,L))^* \cong \mathrm{Ext}_A^{**}(L,K^*) ,$$

this shows that

$$\theta^*: \mathrm{Ext}_A^{*} (L,K^*) \longleftarrow \mathrm{Ext}_A^{**}(M,K^*)$$

is iso. If N is bounded below and finite-dimensional over F_p in each degree, then we can write N as the dual K^* of a (right) A-module K which is bounded above; therefore

$$\theta^*: \mathrm{Ext}_A^{**}(L,N) \longleftarrow \mathrm{Ext}_A^{**}(M,N)$$

is iso. This proves Proposition 1.2.

§4. <u>Algebras of Invariants</u>. In this section we shall prove two results on $H^*(V)_{loc}$ which are needed in the proof of Theorem 1.4. The first result identifies the subalgebra of $H^*(V)_{loc}$ invariant under a Sylow subgroup of $GL(V)$. The second result allows us to ascend from this subalgebra to the whole of $H^*(V)_{loc}$ by a chain of A-submodules. Of course, the basic results on algebras of invariants are due to Mui [12]. However, we need our proof of the first result in order to prove the second.

Let us recall the structure of $H^*(V)$. We have

$$H^1(V) = \mathrm{Hom}(V,F_p) = V^* ,$$

the dual of V . Let $e_1,e_2,\ldots,e_n$ be a base in $H^1(V) = V^*$. Let $x_r = \beta e_r \in H^2(V)$. Then $H^*(V)$ contains a polynomial subalgebra $F_p[x_1,x_2,\ldots,x_n]$; this may be written more shortly and invariantly as $S[\beta V^*]$, where βV^* is the image of $H^1(V) = V^*$ under $\beta: H^1(V) \longrightarrow H^2(V)$. The whole algebra $H^*(V)$ is a free module over $S[\beta V^*]$, having as a base the 2^n monomials

$$e_1^{i_1} e_2^{i_2} \ldots e_n^{i_n}$$

where each i_r is either 0 or 1 . (We state matters in this way to avoid making too much distinction between cases; for $p = 2$ we have $e_r^2 = x_r$ and for $p > 2$ we have $e_r^2 = 0$.) Similarly, $H^*(V)_{loc}$ is a free module on the same 2^n generators over the subalgebra $S[\beta V^*]_{loc}$.

We regard $GL(V)$ as a matrix group by using the base $e_1,e_2,\ldots,e_n$ in V^* , or equivalently the base $x_1,x_2,\ldots,x_n$ in βV^* . Our preferred Borel subgroup Bor (V) is the group of upper triangular matrices. Our preferred Sylow subgroup

Syl(V) is the group of upper uni-triangular matrices.

We will now introduce the following elements invariant under Syl(V) .

$$f_r = \begin{vmatrix} x_1^{p^{r-2}} & \cdots & x_r^{p^{r-2}} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ x_1^p & \cdots & x_r^p \\ x_1 & \cdots & x_r \\ e_1 & \cdots & e_r \end{vmatrix}$$

$$y_r = \begin{vmatrix} x_1^{p^{r-1}} & \cdots & x_r^{p^{r-1}} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ x_1^{p^2} & \cdots & x_r^{p^2} \\ x_1^p & \cdots & x_r^p \\ x_1 & \cdots & x_r \end{vmatrix}$$

We pause to comment. These elements and their constructions go back to Mui [12]. The element f_r is displayed as a determinant, and determinants belong to commutative algebra; but our algebra $H^*(V)$ is not commutative in general. This causes no trouble; in the expansion of f_r each product involves only one factor e_j , which commutes with all the other factors x_k .

The elements f_r and y_r are obviously invariant under Syl(V) ; an element $g \in$ Syl(V) has the effect of adding to each column of f_r or y_r a linear combination of the preceding columns, and this does not change the determinant.

If $p = 2$ we have $f_r^2 = y_r$; if $p > 2$ we have $f_r^2 = 0$.

The determinant y_r is a product of factors which are non-zero elements of βV^* ; thus y_r is invertible in $H^*(V)_{loc}$.

Theorem 4.1. $H^*(V)_{loc}^{Syl(V)}$ is a free module on the 2^n generators

$$f_1^{i_1} f_2^{i_2} \ldots f_n^{i_n} \quad \text{(where each } i_r \text{ is } 0 \text{ or } 1)$$

over the algebra of finite Laurent series

$$F_p[y_1, y_1^{-1}] \otimes F_p[y_2, y_2^{-1}] \otimes \ldots \otimes F_p[y_n, y_n^{-1}] .$$

The localisation required may be accomplished by inverting

$$\prod y \mid y \in \beta V^* , y \neq 0 ;$$

this element is fixed under $GL(V)$; therefore it makes no difference whether we localise before or after passing to a subalgebra of fixed elements.

We shall prove Theorem 4.1 by determining $H^*(V)_{loc}^G$ inductively for a chain of subgroups G which increase from 1 to $Syl(V)$. We will next explain what subgroups we mean. The quickest way is to give an explicit description in terms of matrices.

We will call a set G of matrices a "good subgroup" if it can be obtained in the following way. (It may not be instantly obvious that the subset we shall define is a subgroup, but we shall show that it is.) Take a function

$$q: \{1,2,\ldots,n\} \longrightarrow \{0,1,2,\ldots,n-1\}$$

which has $q(r) \leq r-1$ and is non-decreasing, so that $r \leq s$ implies $q(r) \leq q(s)$. Then the condition

$$a_{ij} = \delta_{ij} \quad \text{for} \quad i > q(j)$$

defines a subset G of the matrices A, namely, those which agree with the identity matrix below a certain stepwise boundary line. For example, with $q(r) = 0$ we obtain $G = 1$; with $q(r) = r-1$ we obtain $G = \mathrm{Syl}(V)$.

We will state the properties of good subgroups. We need one piece of notation: we define U_r to be the subspace of $U = \beta V^*$ generated by $x_1, x_2, \ldots, x_r$. Thus $\mathrm{Bor}(V)$ is the subgroup of elements in $GL(V) = GL(U)$ which preserve the "flag"

$$0 = U_o \subset U_1 \subset U_2 \subset \ldots \subset U_n = U .$$

<u>Lemma 4.2</u> (a) For each function $q(r)$, the subset G defined above is a normal subgroup of $\mathrm{Bor}(V)$, and hence normal in any subgroup G' such that $G \subset G' \subset \mathrm{Bor}(V)$.

(b) When G acts on $U = \beta V^*$, each orbit is a coset of some U_q; more precisely, each orbit which is in U_r but not in U_{r-1} is a coset of $U_{q(r)}$.

(c) Each good subgroup $G > 1$ contains a good subgroup F such that $G/F \cong Z_p$.

(d) Conversely, each good subgroup $F < \mathrm{Syl}(V)$ is contained in a good subgroup G such that $G/F \cong Z_p$.

(e) If $F \subset G$ are good subgroups with $G/F \cong Z_p$, then F differs from G only by the imposition of one extra condition $a_{qr} = 0$ for some pair (q,r) with $q < r$. A generator g for $G/F \cong Z_p$ is given by the elementary matrix which agrees with the identity matrix except for $a_{qr} = 1$.

<u>Proof</u>. (a) We have an obvious homomorphism

$$\mathrm{Bor}(V) \longrightarrow GL(U_r/U_q) \quad (q < r)$$

which assigns to each matrix $A \in \mathrm{Bor}(V)$ the induced automorphism of U_r/U_q. Let $K(q,r)$ be the kernel of this homomorphism; a matrix $A \in \mathrm{Bor}(V)$ lies in $K(q,r)$ if

$$a_{ij} = \delta_{ij} \quad \text{for } q < i \leq r\,, \quad q < j \leq r\,.$$

The subset G defined above is

$$G = \bigcap_{1 \leq r \leq n} K(q(r),r)\;.$$

Since G is an intersection of kernels, it is a normal subgroup. This proves part (a).

Parts (b) to (e) follow immediately from our description of the matrices in G.

We will call a pair $F \subset G$ of good subgroups a "good pair" if $G/F = Z_p$, as in (4.2)(c)-(e). For the purpose of proving Theorem 4.1, we can descend directly from $H^*(V)^F_{loc}$ to $H^*(V)^G_{loc}$; $G/F = Z_p$ acts on $H^*(V)^F_{loc}$, with fixed subalgebra $H^*(V)^G_{loc}$. However, to ascend again in §6 we need a chain of A-submodules between $H^*(V)^G_{loc}$ and $H^*(V)^F_{loc}$. For this purpose we introduce the following considerations.

Suppose given a group Z_p with generator g acting on an algebra R over F_p. Then we can form

$$M_j = \mathrm{Ker}((g-1)^j : R \longrightarrow R)\;.$$

In the applications R is an A-algebra and g is an A-map, so M_j is an A-submodule. We have

$$(g-1)^0 = 1 \quad \text{and} \quad (g-1)^p = 0\;,$$

so we have a chain

$$0 = M_o \subset M_1 \subset M_2 \subset \dots \subset M_p = R$$

where $M_1 = R^{Z_p}$. We will say that the extension $R^{Z_p} \subset R$ is "good" if the map

$$(g-1)^{j-1} : M_j/M_{j-1} \longrightarrow M_1/M_o = R^{Z_p}$$

is iso for $1 \leq j \leq p$. In the applications, this ensures that the A-module R is filtered with p subquotients each A-isomorphic to R^{Z_p}.

Proposition 4.3. If $F \subset G$ is a good pair of good subgroups, then the extension

$$H^*(V)^G_{loc} \subset H^*(V)^F_{loc}$$

is good.

We record two easy lemmas about good extensions. First, let $R^{Z_p} \subset R$ be an extension; suppose that by inverting elements in R^{Z_p} we obtain an extension

$$R^{Z_p}_{loc} \subset R_{loc} .$$

(As before, it does not matter whether we localise before or after passing to a subalgebra of fixed elements.)

Lemma 4.4. If the extension $R^{Z_p} \subset R$ is good, then so is the extension $R^{Z_p}_{loc} \subset R_{loc}$.

This follows immediately from the principle that localisation preserves exactness.

Secondly, suppose given a pair of algebras $R \supset S$, with Z_p acting on both; and suppose that R is free as an S-module, on generators $b_1, b_2, \dots, b_m$ fixed under Z_p.

Lemma 4.5. If the extension $S^{Z_p} \subset S$ is good, then so is the extension $R^{Z_p} \subset R$.

In fact, the module $M_j(R)$ for R is the direct sum of m copies of the module $M_j(S)$ for S, obtained by multiplying with $b_1, b_2, \ldots, b_m$. The same goes for M_j/M_{j-1}. So if

$$(g-1)^{j-1} : M_j/M_{j-1} \longrightarrow M_1/M_o$$

is iso for S, then it is iso for R.

Next we will explain the basic lemma which we use to determine fixed subalgebras and to obtain good extensions.

Let R be an algebra over F_p. Let the group Z_p, with generator g, act on the polynomial algebra $R[x]$ so that g fixes R and

$$g(x) = x + r$$

where r is a constant in R.

Lemma 4.6. If r is invertible in R then $R[x]^{Z_p} \subset R[x]$ is a good extension and $R[x]^{Z_p}$ is a polynomial algebra $R[y]$, where

$$y = \prod_{o \le i < p} (g^i x) = x^p - r^{p-1} x .$$

Proof. The element

$$y = \prod_{o \le i < p} (g^i x) = x(x+r)(x+2r)\ldots(x+(p-1)r)$$

is equal to $x^p - r^{p-1}x$, and is clearly fixed under g; thus

$$R[y] \subset R[x]^{Z_p} .$$

It is clear that $R[x]$ is a free module over $R[y] = R[x^p - r^{p-1}x]$ on generators $1, x, x^2, \ldots, x^{p-1}$. The matrix of $(g-1)$ with respect to this base has the following form.

$$\begin{bmatrix} 0 & r & r^2 & r^3 & & & \\ 0 & 0 & 2r & 3r^2 & & & \\ 0 & 0 & 0 & 3r & & & \\ 0 & 0 & 0 & 0 & \cdots & & \\ & & & & \cdots & 0 & (p-1)r \\ & & & & & 0 & 0 \end{bmatrix}$$

Here the terms $r, 2r, 3r, \ldots, (p-1)r$ are invertible. It follows that $M_j = \mathrm{Ker}(g-1)^j$ is exactly the $R[y]$-submodule generated by $1, x, x^2, \ldots, x^{j-1}$. In particular, $R[x]^{Z_p} = M_1 = R[y]$. It follows next that M_j/M_{j-1} is free over $R[y]$ on one generator x^{j-1}. The map

$$g-1 : M_{j+1}/M_j \longrightarrow M_j/M_{j-1}$$

carries the generator x^j to $jr\, x^{j-1}$, and is iso since jr is invertible (for $1 \le j \le p-1$). This proves the lemma.

To apply this lemma, we work in the "polynomial part" $S[\beta V^*]$ of $H^*(V)$. Next we must explain the elements which we shall use as generators in our description of $S[\beta V^*]^G_{loc}$.

Suppose given a vector space U containing a subspace T. For each coset C of T in U, let $\pi(C)$ be the product (in $S[U]$) of the elements in C.

<u>Lemma 4.7</u>. $\pi(C + D) = \pi(C) + \pi(D)$.

<u>Proof</u>. We proceed by induction over the dimension of T . The result is trivially true if $T = 0$, so we assume it true for T' of codimension 1 in T . A coset C of T breaks up into cosets of T' of the form

$$E , E + F , E + 2F , \dots , E + (p-1)F$$

where F is a generator for T/T' . So we have

$$\pi(C) = \pi'(E)\ \pi'(E+F)\ \pi'(E+2F) \dots \pi'(E+(p-1)F)$$

$$= \pi'(E)[\pi'(E) + \pi'(F)][\pi'(E) + 2\pi'(F)] \dots [\pi'(E) + (p-1)\pi'(F)]$$

(by the inductive hypothesis)

$$= \pi'(E)^p - \pi'(F)^{p-1}\,\pi'(E) .$$

By the inductive hypothesis, this is an additive function of E . This proves the lemma.

We can now formulate a result about fixed subalgebras in $S[\beta V^*]_{loc} = S[U]_{loc}$. Let G be a good matrix group, acting on $S[U]$ where U is a vector-space with a given basis $x_1, x_2, \dots, x_n$. Let U_r be the subspace of U spanned by $x_1, x_2, \dots, x_r$, as above. For each $r \in \{1,2,\dots,n\}$ we choose a G-orbit C_r in U_r which is not in U_{r-1} .

<u>Proposition 4.8</u> (a) For each such choice, the subalgebra of invariant elements $S[U]^G_{loc}$ is

$$F_p[\pi(C_1) , \pi(C_2) , \dots , \pi(C_n)]_{loc} .$$

(b) If $F \subset G$ is a good pair of good subgroups, then the extension

$$S[U]^G_{loc} \subset S[U]^F_{loc}$$

is good.

Proof. We shall prove part (a) by induction. The result is trivially true when $n = 0$ or $n = 1$, so we may assume as an inductive hypothesis that the result is true for $n' < n$; in particular, we have the corresponding result for U_r if $r < n$.

The result is also true when $G = 1$. We now argue by induction over the order of G. If $G > 1$ then by (4.2)(c) we may consider a good pair $F \subset G$. We assume as our inductive hypothesis that part (a) holds for F.

Let $C_1, C_2, \ldots, C_n$ be chosen as above for G, so that C_r is a G-orbit in U_r which is not in U_{r-1}. Since $F \subset G$, each G-orbit is a union of F-orbits; choose an F-orbit D_r in C_r. Our second inductive hypothesis is thus that

$$S[U]^F_{loc} = F_p[\pi(D_1), \pi(D_2), \ldots, \pi(D_n)]_{loc} .$$

By (4.2)(e), F differs from G only by the imposition of one extra condition $a_{qr} = 0$, where $q < r$. It follows that we have $C_s = D_s$ except for $s = r$; the G-orbit C_r, which is a coset of U_q, decomposes into p F-orbits, which are cosets of U_{q-1}, and are $D_r, gD_r, g^2D_r, \ldots, g^{p-1}D_r$.

We can now apply Lemma 4.6, taking

$$R = F_p[\pi(D_1), \ldots, \pi(D_{r-1})]_{loc}[\pi(D_{r+1}), \ldots, \pi(D_n)]$$

and $x = \pi(D_r)$. Of course $G/F = Z_p$ acts on $R[x]$, fixing R. We have

$$g(D_r) = D_r + E$$

where the coset E is some generator for U_q/U_{q-1} . Thus

$$\begin{aligned} g(\pi D_r) &= \pi(gD_r) \\ &= \pi(D_r + E) \\ &= \pi(D_r) + \pi(E) \qquad \text{by (4.7)} . \end{aligned}$$

Now E is invariant under $Syl(U_q)$, since elements of $Syl(U_q)$ fix U_q/U_{q-1} . By the inductive hypothesis for U_q , $\pi(E)$ lies in

$$F_p[\pi(D_1) , \pi(D_2) , \ldots , \pi(D_q)]_{loc}$$

where $q \le r-1$. Moreover, $\pi(E)$ is invertible in this algebra (being a product of non-zero elements in U_q) . By Lemma 4.6, our extension is good and the fixed subalgebra is $R[y]$ where

$$\begin{aligned} y &= \prod_{o \le i < p} g^i(\pi D_r) \\ &= \prod_{o \le i < p} \pi(g^i D_r) \\ &= \pi(C_r) . \end{aligned}$$

This shows that the subalgebra of

$$F_p[\pi(D_1) , \ldots , \pi(D_{r-1})]_{loc} [\pi(D_r) , \ldots , \pi(D_n)]$$

fixed under G/F is

$$F_p[\pi(C_1) , \ldots , \pi(C_{r-1})]_{loc} [\pi(C_r) , \ldots , \pi(C_n)] ,$$

and that this extension is good.

Performing the remaining localisation, we see that the subalgebra of

$$S[U]^F_{loc} = F_p[\pi(D_1)\ ,\ \pi(D_2)\ ,\ \ldots\ ,\ \pi(D_n)]_{loc}$$

fixed under G/F is

$$F_p[\pi(C_1)\ ,\ \pi(C_2)\ ,\ \ldots\ ,\ \pi(C_n)]_{loc}\ .$$

This identifies $S[U]^G_{loc}$, which completes the induction and proves part (a) of the proposition.

Now that the induction is complete, the argument applies to any good pair of good subgroups $F \subset G$. The argument shows that the subalgebra of

$$F_p[\pi(D_1)\ ,\ \ldots\ ,\ \pi(D_{r-1})]_{loc}\ [\pi(D_r)\ ,\ \ldots\ ,\ \pi(D_n)]$$

fixed under G/F is

$$F_p[\pi(C_1)\ ,\ \ldots\ ,\ \pi(C_{r-1})]_{loc}\ [\pi(C_r)\ ,\ \ldots\ ,\ \pi(C_n)]$$

and that this extension is good. Part (b) of the conclusion now follows by using Lemma 4.4.

<u>Corollary 4.9</u>. $S[\beta V^*]^{Syl(V)}_{loc}$ is the algebra of finite Laurent series

$$F_p[y_1,y_1^{-1}] \otimes F_p[y_2,y_2^{-1}] \otimes \ldots \otimes F_p[y_n,y_n^{-1}]$$

where $y_1,y_2,\ldots,y_n$ are as in (4.1).

<u>Proof</u>. By (4.8)(a), $S[\beta V^*]^{Syl(V)}_{loc}$ is

$$F_p[z_1\ ,\ z_2\ ,\ \ldots\ ,\ z_n]_{loc}$$

where $z_r = \pi(C_r)$ and we may for example take C_r to be the coset of U_{r-1} which contains x_r. The localisation required is to invert $z_1\ z_2 \ldots z_n$. So $S[\beta V^*]^{Syl(V)}_{loc}$ is the algebra of finite Laurent series

$$F_p[z_1, z_1^{-1}] \otimes F_p[z_2, z_2^{-1}] \otimes \ldots \otimes F_p[z_n, z_n^{-1}] .$$

The relation between the y's and z's is

$$y_r = (-1)^{\frac{1}{2}r(r-1)} z_1 z_2 \ldots z_r$$

$$z_r = (-1)^{r-1} y_r / y_{r-1}$$

(where we interpret y_{-1} as 1) . The result follows.

We will now climb up from the polynomial part $S[\beta V^*]$ to the whole algebra $H^*(V)$.

<u>Lemma 4.10</u>. $H^*(V)_{loc}$ is a free module over $S[\beta V^*]_{loc}$ on 2^n generators

$$f_1^{i_1} f_2^{i_2} \ldots f_n^{i_n}$$

where each i_r is 0 or 1 .

<u>Proof</u>. Let $S = S[\beta V^*]_{loc}$. By its definition, the generator f_r in (4.1) is already written as an S-linear combination of $e_1, e_2, \ldots, e_r$. Moreover, the coefficient of e_r in f_r is y_{r-1} , which is invertible in S .

We can obtain a corresponding result for monomials if we order the monomials correctly. We order them first by their degree $i_1 + i_2 + \ldots + i_n$; the monomials of a given degree we order lexicographically, regarding i_n as the most significant digit and i_1 as the least significant digit. Then $f_1^{i_1} f_2^{i_2} \ldots f_n^{i_n}$ is an invertible coefficient times $e_1^{i_1} e_2^{i_2} \ldots e_n^{i_n}$, plus a linear combination of lower monomials.

(This statement remains true for $p = 2$.) The lemma follows.

Using (4.10), and the fact that the 2^n generators

$$f_1^{i_1} f_2^{i_2} \cdots f_n^{i_n}$$

are fixed under any subgroup $G \subset Syl(V)$, it follows that $H^*(V)^G_{loc}$ is a free module over $S[\beta V^*]^G_{loc}$ on the same 2^n generators

$$f_1^{i_1} f_2^{i_2} \cdots f_n^{i_n} .$$

Thus Theorem 4.1 follows from (4.9).

In view of the last paragraph, Proposition 4.3 follows from (4.8)(b) by using (4.5).

§5. <u>Proof of Theorem 1.6</u>. In this section we will prove Theorem 1.6. We do this by induction. If V is of rank n over F_p, let W be of rank $(n-1)$.

<u>Theorem 5.1</u>. There is an isomorphism of A-algebras

$$T(H^*(W)_{loc}^{Syl(W)}) \cong H^*(V)_{loc}^{Syl(V)} .$$

The isomorphism in this theorem is induced by the map

$$T \otimes H^*(W)_{loc} \xrightarrow{f} T \hat{\otimes} H^*(W)_{loc}$$

of §2. We will exhibit an embedding of $H^*(V)_{loc}$ in $T \hat{\otimes} H^*(W)_{loc}$. In fact, we identify the generators e,x for $T = H^*(Z_p)_{loc}$ with the generators e_1, x_1 in $H^*(V)_{loc}$. We regard W^* as the subspace of V^* spanned by $e_2, e_3, \ldots, e_n$; it follows that we regard βW^* as the subspace of βV^* spanned by $x_2, x_3, \ldots, x_n$. This certainly identifies $H^*(V)$ with a subalgebra of $T \hat{\otimes} H^*(W)_{loc}$; we must see that this embedding extends to the localisation. In fact, for any element m of degree 2 in $H^*(W)_{loc}$ the element $x_1 + m$ is invertible in $T \hat{\otimes} H^*(W)_{loc}$; an inverse is provided by

$$x_1^{-1} - x_1^{-2}m + x_1^{-3}m^2 - \ldots .$$

This gives the embedding.

To give formulae for f, we must calculate the Steenrod operations on the generators f_r, y_r for $H^*(W)_{loc}^{Syl(W)}$.

Lemma 5.2. (i) We have $P^k y_r = 0$ unless $k = \frac{p^r - p^j}{p-1}$ for some j such that $0 \le j \le r$. In this case we have

$$P^k y_r = \begin{vmatrix} x_2^{p^r} & \cdots & x_{r+1}^{p^r} \\ \vdots & & \vdots \\ x_2^{p^{j+1}} & & x_{r+1}^{p^{j+1}} \\ x_2^{p^{j-1}} & & x_{r+1}^{p^{j-1}} \\ \vdots & & \vdots \\ x_2 & \cdots & x_{r+1} \end{vmatrix} .$$

(ii) We have $\beta P^k y_r = 0$.

(iii) We have $P^k f_r = 0$ unless $k = \frac{p^{r-1} - p^j}{p-1}$ for some j such that $0 \le j \le r-1$. In this case we have

$$P^k f_r = \begin{vmatrix} x_2^{p^{r-1}} & \cdots & x_{r+1}^{p^{r-1}} \\ \vdots & & \vdots \\ x_2^{p^{j+1}} & & x_{r+1}^{p^{j+1}} \\ x_2^{p^{j-1}} & & x_{r+1}^{p^{j-1}} \\ \vdots & & \vdots \\ x_2 & & x_{r+1} \\ e_2 & \cdots & e_{r+1} \end{vmatrix} .$$

(iv) We have $\beta P^k f_r = 0$ unless $k = \frac{p^{r-1}-1}{p-1}$. In this case we have

$$\beta P^k f_r = y_r .$$

Proof. Since $P = \sum_{k=o}^{\infty} P^k$ is a homomorphism of algebras, we have

$$P(y_r) = \begin{vmatrix} x_2^{p^{r-1}} + x_2^{p^r} & , & \cdots & , & x_{r+1}^{p^{r-1}} + x_{r+1}^{p^r} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ x_2^p + x_2^{p^2} & , & \cdots & , & x_{r+1}^p + x_{r+1}^{p^2} \\ x_2 + x_2^p & , & \cdots & , & x_{r+1} + x_{r+1}^p \end{vmatrix} .$$

This determinant can be written as a sum of 2^r determinants; in each row we must choose whether to take the first summand in each entry or the second summand in each. Most of the resulting determinants are zero because they have two rows equal. The remainder are those given in the enunciation.

The proof of part (iii) is similar. Parts (ii) and (iv) follow.

Lemma 5.3. The map

$$T \otimes H^*(W)_{loc} \xrightarrow{f} T \hat{\otimes} H^*(W)_{loc}$$

has

$$f(x_1 \otimes 1) = y_1$$

$$f(e_1 \otimes 1) = f_1$$

and

$$f(x_1^{p^r} \otimes y_r) = y_{r+1}$$

$$f(x_1^{p^{r-1}} \otimes f_r) = f_{r+1}$$

for $r \geq 1$.

<u>Proof</u>. The map f is given by (2.1), (2.2). Using the Steenrod operations given by (5.2), we calculate as follows.

$$f(x_1^{p^r} \otimes y_r) = \sum_{0 \leq j \leq r} (-1)^{r-j} x_1^{p^j} \begin{vmatrix} x_2^{p^r} & \cdots & x_{r+1}^{p^r} \\ \vdots & & \vdots \\ x_2^{p^{j+1}} & & x_{r+1}^{p^{j+1}} \\ x_2^{p^{j-1}} & & x_{r+1}^{p^{j-1}} \\ \vdots & & \vdots \\ x_2 & \cdots & x_{r+1} \end{vmatrix}$$

$$= y_{r+1} \ .$$

$$f(x_1^{p^{r-1}} \otimes f_r) = \sum_{0 \le j \le r-1} (-1)^{r-1-j} x_1^{p^j} \begin{vmatrix} x_2^{p^{r-1}} & \cdots & x_{r+1}^{p^{r-1}} \\ \vdots & & \vdots \\ x_2^{p^{j+1}} & & x_{r+1}^{p^{j+1}} \\ x_2^{p^{j-1}} & & x_{r+1}^{p^{j-1}} \\ \vdots & & \vdots \\ x_2 & & x_{r+1} \\ e_2 & \cdots & e_{r+1} \end{vmatrix}$$

$$+ (-1)^r e_1 \begin{vmatrix} x_2^{p^{r-1}} & \cdots & x_{r+1}^{p^{r-1}} \\ \vdots & & \vdots \\ x_2^{p} & & x_{r+1}^{p} \\ x_2 & \cdots & x_r \end{vmatrix}$$

$$= f_{r+1} .$$

This proves the lemma.

<u>Proof of Theorem 5.1</u>. The subalgebras $H^*(W)_{loc}^{Syl(W)}$, $H^*(V)_{loc}^{Syl(V)}$ are identified by Theorem 4.1. We see from (5.3) that the map

$$T \otimes H^*(W)_{loc} \xrightarrow{f} T \hat{\otimes} H^*(W)_{loc}$$

carries the algebra of finite Laurent series

$$F_p[x_1, x_1^{-1}] \otimes F_p[y_1, y_1^{-1}] \otimes \ldots \otimes F_p[y_{n-1}, y_{n-1}^{-1}]$$

(on the left) isomorphically onto the algebra of finite Laurent series

$$\mathbb{F}_p[y_1,y_1^{-1}] \otimes \mathbb{F}_p[y_2,y_2^{-1}] \otimes \dots \otimes \mathbb{F}_p[y_n,y_n^{-1}]$$

(on the right). On the left, $T \otimes H^*(W)^{\mathrm{Syl}(W)}_{\mathrm{loc}}$ is free over the subalgebra just described, on the 2^n generators

$$e_1^{i_1} f_1^{j_1} f_2^{j_2} \dots f_{n-1}^{j_{n-1}} .$$

On the right, $H^*(V)^{\mathrm{Syl}(V)}_{\mathrm{loc}}$ is free over the subalgebra just described on the 2^n generators

$$f_1^{k_1} f_2^{k_2} \dots f_n^{k_n} .$$

We see from (5.3) that f carries the one set of 2^n generators to the other, up to invertible factors in the subalgebra just described. This proves Theorem 5.1.

Theorem 1.6 follows immediately by induction over n .

6. Proof of Theorem 1.4 (a)-(c). In this section we will prove most of Theorem 1.4. As the proof is by induction, we must formulate the inductive hypothesis. Let V be of rank n, and let $G \subset Syl(V)$ be a good subgroup, as in §4.

Theorem 6.1 (a) The quotient map

$$H^*(V)^G_{loc} \xrightarrow{q} F_p \otimes_A H^*(V)^G_{loc}$$

is a Tor-equivalence.

(b) $F_p \otimes_A H^*(V)^G_{loc}$ is zero except in degree $-n$, where it is of rank $|Syl(V) : G|$.

Proof. We first remark that this is true for the case $G = Syl(V)$. In this case we have

$$H^*(V)^{Syl(V)}_{loc} \cong T^n(F_p)$$

by (1.6). By (1.5), we have n Tor-equivalences

$$T^nF_p \xrightarrow{\varepsilon} T^{n-1}F_p \longrightarrow \cdots \longrightarrow TF_p \xrightarrow{\varepsilon} F_p ,$$

each of degree $+1$. Thus we have a Tor-equivalence (of degree n)

$$H^*(V)^{Syl(V)}_{loc} \xrightarrow{\phi} F_p .$$

Consider the following diagram.

$$\begin{array}{ccc} H^*(V)^{Syl(V)}_{loc} & \xrightarrow{\phi} & F_p \\ \Big\downarrow q & & \Big\downarrow q' \\ F_p \otimes_A H^*(V)^{Syl(V)}_{loc} & \xrightarrow{1\otimes\phi} & F_p \otimes_A F_p = F_p \end{array}$$

Since ϕ is a Tor-equivalence, $1 \otimes \phi$ is iso; also q' is trivially iso. Thus q is a Tor-equivalence.

We now proceed by downwards induction over G. Let F be the good subgroup for which we wish to prove the result. By (4.2) (d) we have a good pair of good subgroups $F \subset G$, and we suppose as our inductive hypothesis that the result is true for G. By (4.3) we have a filtration

$$0 = M_o \subset M_1 \subset M_2 \subset \ldots \subset M_p = H^*(V)^F_{loc}$$

of $H^*(V)^F_{loc}$ by A-submodules, in which each subquotient M_j/M_{j-1} is isomorphic to $H^*(V)^G_{loc}$. Suppose, as the hypothesis of a subsidiary induction over j, that the quotient map

$$M_j \xrightarrow{q} F_p \otimes_A M_j$$

is a Tor-equivalence, and that $F_p \otimes_A M_j$ is zero except in degree $-n$, where it is of rank $j|Syl(V) : G|$. (This is trivially true for $j = 0$.) Consider the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_j & \longrightarrow & M_{j+1} & \longrightarrow & H^*(V)^G_{loc} & \longrightarrow & 0 \\
 & & \downarrow q_j & & \downarrow q_{j+1} & & \downarrow q & & \\
Tor^A_{1*}(F_p, H^*(V)^G_{loc}) & \longrightarrow & F_p \otimes_A M_j & \longrightarrow & F_p \otimes_A M_{j+1} & \longrightarrow & F_p \otimes_A H^*(V)^G_{loc} & \longrightarrow & 0
\end{array}$$

By the hypothesis of the subsidiary induction, $F_p \otimes_A M_j$ is zero except in degree $-n$. By the main inductive hypothesis,

$$\begin{aligned} Tor^A_{1,-n}(F_p, H^*(V)^G_{loc}) &\cong \bigoplus_{|Syl(V):G|} Tor^A_{1,o}(F_p, F_p) \\ &= 0 . \end{aligned}$$

So the lower sequence is short exact. We see that $F_p \otimes_A M_{j+1}$ is zero except in degree $-n$, where it is of rank $(j+1) \mid Syl(V):G \mid$. Now we use the Five Lemma; by our inductive hypotheses, q_j and q are Tor-equivalences, and therefore q_{j+1} is a Tor-equivalence. This completes the subsidiary induction, which runs up to $j = p$ and proves the required result for $H^*(V)^F_{loc}$. This completes the main induction, and proves Theorem 6.1.

The special case $G = 1$ of (6.1) proves parts (a), (b) and (c) of Theorem 1.4. The proof of part (d) will be given in §8. In §7, references to Theorem 1.4 will refer only to the parts (a), (b), (c) already proved.

§7. Delocalisation. In this section we shall prove Theorem 1.3. The theme of our argument is that we take information about objects which are more localised, and deduce information about objects which are less localised.

Our first task is to define the map which appears in (1.3). This map is defined by using residues. The basic definition of the residue was given in §2. If M is an A-module, then the completed tensor product

$$H^*(Z_p)_{loc} \hat{\otimes} M$$

has as its elements the "downward-going formal Laurent series"

$$\sum_{r \le R} x^r \otimes m'_r + \sum_{r \le R} ex^r \otimes m''_r ,$$

where $e \in H^1(Z_p)$ and $x = \beta e \in H^2(Z_p)$ are the generators. The residue map

$$H^*(Z_p)_{loc} \hat{\otimes} M \xrightarrow{\text{res}} M$$

is defined by

$$\text{res}\left(\sum_{r \le R} x^r \otimes m'_r + \sum_{r \le R} ex^r \otimes m''_r\right) = m''_{-1} .$$

This is an A-map of degree +1 .

In practice we do not usually need the whole of $H^*(Z_p)_{loc} \hat{\otimes} M$. Suppose for example that we have in play an A-algebra R and that M is an (A,R)-module. Then $H^*(Z_p) \otimes M$ is an $(A, H^*(Z_p) \otimes R)$-module. We define

$$(H^*(Z_p) \otimes M)_{loc}$$

by localising so as to invert $x + r$ for each r of degree 2 in R . Then we have an embedding

$$(H^*(Z_p) \otimes M)_{loc} \subset H^*(Z_p)_{loc} \hat{\otimes} M .$$

For this it is sufficient to see that $x + r$ acts invertibly on $H^*(Z_p)_{loc} \hat{\otimes} M$; the inverse is given by

$$x^{-1} - rx^{-2} + r^2x^{-3} - \dots .$$

Thus we have a residue map

$$(H^*(Z_p) \otimes M)_{loc} \xrightarrow{res} M .$$

If we have several residue maps in play, then it is important that the notation should indicate the domain of coefficients M ; one cannot take the coefficients of a series without knowing what counts as a coefficient. It is not so important to display the variable e or x , because in fact the residue does not depend on that. If we replace e by λe and x by λx for some $\lambda \neq 0$, then we replace ex^{-1} by ex^{-1} ; if we replace e by $e + r$ for some r of degree 1 in R , and x by $x + \beta r$, then we easily check that we get the same residue. (Heuristically, our "res" is the sum of the residues at all the finite poles $z = -r$.)

Later on we have to consider the special case in which M comes as a tensor product, $M = N \otimes P$, and R acts on $N \otimes P$ by acting on N . In that case P plays a dummy role, in the following sense.

<u>Lemma 7.1</u>. There is a natural isomorphism

$$(H^*(Z_p) \otimes N)_{loc} \otimes P \xrightarrow{\cong} (H^*(Z_p) \otimes N \otimes P)_{loc} ;$$

it makes the following diagram commute.

$$(H^*(Z_p) \otimes N)_{loc} \otimes P \xrightarrow{\cong} (H^*(Z_p) \otimes N \otimes P)_{loc}$$

$$\text{res} \otimes 1 \searrow \qquad \swarrow \text{res}$$

$$N \otimes P$$

The proof is easy.

We proceed to the applications. We would like to suppose given a non-zero element x, as in (1.3). However, in this section, it will be convenient to simplify the notation by identifying the subspace $\beta V^* \subset H^2(V)$ with V^* (dropping the symbol β). So we suppose given a non-zero element $x \in V^*$, and choose a direct-sum splitting

$$V^* = \langle x \rangle \oplus W^* .$$

This corresponds to a decomposition

$$H^*(V) = H^*(Z_p) \otimes H^*(W)$$

(where $\langle x \rangle = Z_p^*$). We can apply the work above by taking

$$R = H^*(W)$$

$$M = H^*(W)_{S \cap W^*} ,$$

where S is a subset of V^*. By inverting elements $x + r$ we surely invert all the elements of S which are not in W^*, so we obtain the following residue map.

$$H^*(V)_S \xrightarrow{\text{res}_W} H^*(W)_{S \cap W^*} .$$

Using these maps res_W as components, we obtain the map

$$H^*(V)_S \xrightarrow{\{\text{res}_W\}} \bigoplus_W H^*(W)_{S \cap W^*}$$

of Theorem 1.3.

Theorem 1.3 asserts that the map $\{res_W\}$ is a Tor-equivalence, and to prove it, we begin with the special case $S = V^*$.

We take a base $e_1, e_2, \ldots, e_n$ in $H^1(V)$ and set $x_r = \beta e_r$, so that $x_1, x_2, \ldots, x_n$ is a base in V^* . Without loss of generality we may suppose that x_1 is the element x in (1.3). We regard $GL(V)$ as a matrix group by using these bases. Our preferred Sylow group $Syl(V)$ is the group of upper uni-triangular matrices. We let g run over $Syl(V)$.

<u>Lemma 7.2</u>. Under the map

$$H^*(V)_{V^*} \xrightarrow{res_W} H^*(W)_{W^*}$$

the element $g(e_1 x_1^{-1} e_2 x_2^{-1} \ldots e_n x_n^{-1})$ maps to zero unless all the elements $gx_2, gx_3, \ldots, gx_n$ lie in W^* , in which case it maps to $g(e_2 x_2^{-1} e_3 x_3^{-1} \ldots e_n x_n^{-1})$.

<u>Proof</u>. We have

$$g(e_1 x_1^{-1} e_2 x_2^{-1} \ldots e_n x_n^{-1}) = e_1 x_1^{-1} g(e_2 x_2^{-1} \ldots e_n x_n^{-1}) .$$

Suppose that after the initial term $e_1 x_1^{-1}$, just d of $gx_2, gx_3, \ldots, gx_n$ do not lie in W^* . In this case the expansion of

$$e_1 x_1^{-1} g(e_2 x_2^{-1} \ldots e_n x_n^{-1})$$

in the form

$$\sum_r x_1^r \otimes m'_r + \sum_r e_1 x_1^r \otimes m''_r$$

only contains terms with $r \leq -d-1$. So the residue is zero unless $d = 0$. If $d = 0$, then $g(e_2 x_2^{-1} \ldots e_n x_n^{-1})$ lies in $H^*(W)_{W^*}$ and counts as a constant in computing the residue, which becomes $g(e_2 x_2^{-1} \ldots e_n x_n^{-1})$.

With suitable interpretation the proof applies also for $p = 2$; the expansion of $e_1^{-1} g(e_2^{-1} e_3^{-1} \dots e_n^{-1})$ in the form

$$\sum_r e_1^r \otimes m_r$$

only contains terms with $r \leq -d-1$. This proves (7.2).

Lemma 7.3 (a) The $p^{\frac{1}{2}n(n-1)}$ elements

$$1 \otimes g(e_1 x_1^{-1} e_2 x_2^{-1} \dots e_n x_n^{-1})$$

(where g runs over $\mathrm{Syl}(V)$) form a base for $F_p \otimes H^*(V)_{V^*}$.

(b) The map

$$F_p \otimes_A H^*(V)_{V^*} \xrightarrow{\{1 \otimes \mathrm{res}_W\}} \bigoplus_W F_p \otimes_A H^*(W)_{W^*}$$

(where W^* runs over complements for $\langle x_1 \rangle$ in V^*) is iso.

Proof. For $n = 1$ both parts are contained in Theorem 1.4. We proceed by induction and assume that part (a) is true for spaces of dimension $(n-1)$. By (7.2) the $p^{\frac{1}{2}n(n-1)}$ elements

$$1 \otimes g(e_1 x_1^{-1} e_2 x_2^{-1} \dots e_n x_n^{-1})$$

in $F_p \otimes H^*(V)_{V^*}$ map to elements in $\bigoplus_W F_p \otimes_A H^*(W)_{W^*}$ which form a base there by the inductive hypothesis. Therefore the elements

$$1 \otimes g(e_1 x_1^{-1} e_2 x_2^{-1} \dots e_n x_n^{-1})$$

are linearly independent in $F_p \otimes_A H^*(V)_{V^*}$.

But this group is of rank $p^{\frac{1}{2}n(n-1)}$ by Theorem 1.4, so they form a base in $F_p \otimes_A H^*(V)_{V^*}$, and the map $\{1 \otimes \mathrm{res}_W\}$ is iso. This proves (7.3).

Corollary 7.4. Theorem 1.3 is true for $S = V^*$.

Proof. Consider the following diagram.

$$\begin{array}{ccc} H^*(V)_{V^*} & \xrightarrow{\{res_W\}} & \bigoplus_W H^*(W)_{W^*} \\ \downarrow q & & \downarrow \{q_W\} \\ F_p \otimes_A H^*(V)_{V^*} & \xrightarrow{\{1 \otimes res_W\}} & \bigoplus_W F_p \otimes_A H^*(W)_{W^*} \end{array}$$

The lower horizontal arrow is an isomorphism by (7.3). The two vertical arrows are Tor-equivalences by Theorem 1.4. Therefore the upper horizontal arrow is a Tor-equivalence. This proves (7.4).

For technical reasons, we pause to interpolate a homological lemma.

Lemma 7.5. If an A-map $\theta: L \longrightarrow M$ is a Tor-equivalence, and if K is an A-module which is bounded above, then

$$\theta \otimes 1: L \otimes K \longrightarrow M \otimes K$$

is a Tor-equivalence.

Proof. If K is finite-dimensional over F_p this follows by an obvious induction over the rank of K , using the Five Lemma. If K is bounded above then K is a direct limit of A-submodules finite-dimensional over F_p ; since Tor commutes with direct limits, the result follows.

As we have said, we shall prove Theorem 1.3 by downward induction over S . We must now study the effect of changing S ; so we suppose given two subsets $S \subset T \subset V^*$, with $x \in S$ as before. Let W^* run over complements for $\langle x \rangle$ in V^* ;

for each such complement we can consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*(V)_S & \longrightarrow & H^*(V)_T & \longrightarrow & \dfrac{H^*(V)_T}{H^*(V)_S} & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle res_W} & & \Big\downarrow {\scriptstyle res_W} & & \Big\downarrow {\scriptstyle \rho_W} & & \\ 0 & \longrightarrow & H^*(W)_{S\cap W^*} & \longrightarrow & H^*(W)_{T\cap W^*} & \longrightarrow & \dfrac{H^*(W)_{T\cap W^*}}{H^*(W)_{S\cap W^*}} & \longrightarrow & 0 \end{array}$$

Since the square is commutative and the rows are exact, there is just one map ρ_W which makes the diagram commute.

We now suppose that T contains just one more line than S, say

$$T = S \cup \langle y\rangle , \quad y \not\subset S .$$

We suppose that V is of rank n, and we suppose as an inductive hypothesis that Theorem 1.3 is true for spaces $\overline{V}$ of rank $n-1$.

<u>Proposition 7.6.</u> Under these assumptions, the map

$$\frac{H^*(V)_T}{H^*(V)_S} \xrightarrow{\{\rho_W\}} \bigoplus_W \frac{H^*(W)_{T\cap W^*}}{H^*(W)_{S\cap W^*}}$$

is a Tor-equivalence.

We begin with some remarks. We have said that W^* should run over all complements for $\langle x\rangle$ in V^*. However, in (7.6) it is sufficient to run W^* over complements for $\langle x\rangle$ which contain $\langle y\rangle$; for if $\langle y\rangle$ is not contained in W^*, then $T \cap W^* = S \cap W^*$ and the summand

$$\frac{H^*(W)_{T\cap W^*}}{H^*(W)_{S\cap W^*}}$$

is zero. So we can restrict attention to the p^{n-2} complements for $\langle x\rangle$ which do contain $\langle y\rangle$.

<u>Lemma 7.7</u>. In (7.6), the truth or falsity of the conclusion depends only on the image of S in the quotient space $V^*/\langle y\rangle$.

<u>Proof</u>. We will study the effect of replacing S by S', where S' is the set of elements $s \in \lambda y$ with $s \in S$, $\lambda \in F_p$. Thus S' is the largest set with the same image in $V^*/\langle y\rangle$ as S. Of course we take $T' = S' \cup \langle y\rangle$.

By passing to the quotient from the commutative diagram

$$\begin{array}{ccc} H^*(V)_T & \xrightarrow{res_W} & H^*(W)_{T\cap W^*} \\ \downarrow & & \downarrow \\ H^*(V)_{T'} & \xrightarrow{res_W} & H^*(W)_{T'\cap W^*} \end{array}$$

we obtain the following commutative diagram.

$$\begin{array}{ccc} \dfrac{H^*(V)_T}{H^*(V)_S} & \xrightarrow{\{\rho_W\}} & \dfrac{H^*(W)_{T\cap W^*}}{H^*(W)_{S\cap W^*}} \\ \downarrow & & \downarrow \\ \dfrac{H^*(V)_{T'}}{H^*(V)_{S'}} & \xrightarrow{\{\rho_W\}} & \dfrac{H^*(W)_{T'\cap W^*}}{H^*(W)_{S'\cap W^*}} \end{array}$$

We will show that the two vertical arrows are iso. It will follow that the upper horizontal arrow is a Tor-equivalence if and only if the lower one is. This will complete the proof, because if S_1, S_2 are two choices for S with the same image in $V^*/\langle y \rangle$, then they have the same S'.

For this purpose we will show that the elements $s' \in S'$ act invertibly on

$$H^*(V)_T \; / \; H^*(V)_S \; .$$

In fact, the series

$$s^{-1} - \lambda y \, s^{-2} + \lambda^2 \, y^2 \, s^{-3} - \ldots$$

provides an inverse for $s + \lambda y$; on any particular element of

$$H^*(V)_T \; / \; H^*(V)_S$$

this series converges after a finite number of terms, because for any $z \in H^*(V)_T$ there is a power y^m of y such that $y^m z \in H^*(V)_S$.

In the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*(V)_S & \longrightarrow & H^*(V)_T & \longrightarrow & \dfrac{H^*(V)_T}{H^*(V)_S} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^*(V)_{S'} & \longrightarrow & H^*(V)_{T'} & \longrightarrow & \dfrac{H^*(V)_{T'}}{H^*(V)_{S'}} & \longrightarrow & 0 \end{array}$$

the lower row is obtained from the upper row by localising so as to invert S' ; therefore the right-hand vertical arrow is iso, because localisation does not change a module on which S' acts invertibly.

Since we need only consider subspaces W^* containing $\langle y \rangle$, precisely the same considerations show that the map

$$\frac{H^*(W)_{T\cap W^*}}{H^*(W)_{S\cap W^*}} \longrightarrow \frac{H^*(W)_{T'\cap W^*}}{H^*(W)_{S'\cap W^*}}$$

is iso. This proves Lemma 7.7.

<u>Proof of Proposition 7.6</u>. Choose a complement $\bar{V}^*$ for $\langle y\rangle$ in V^*, such that $\langle x\rangle \subset \bar{V}^*$. Then $\bar{V}^*$ provides one representative for each coset in $V^*/\langle y\rangle$, and so Lemma 7.7 allows us to suppose that $S \subset \bar{V}^*$. In this way we have cleaned up the position of S.

As we have said, we need only consider the p^{n-2} complements W^* for $\langle x\rangle$ in V^* which contain $\langle y\rangle$. These are in (1-1) correspondence with the p^{n-2} complements $\bar{W}^*$ for $\langle x\rangle$ in $\bar{V}^*$; the correspondence is

$$W^* \longmapsto W^* \cap \bar{V}^*$$

$$\bar{W}^* \longmapsto \bar{W}^* \oplus \langle y\rangle .$$

The diagram which defines

$$\frac{H^*(V)_T}{H^*(V)_S} \xrightarrow{\rho_W} \frac{H^*(W)_{T\cap W^*}}{H^*(W)_{S\cap W^*}}$$

can now be rewritten as follows by using Lemma 7.1.

$$\begin{array}{ccccccccc}
0 \to & H^*(\bar{V})_S \otimes H^*(Z_p) & \longrightarrow & H^*(\bar{V})_S \otimes H^*(Z_p)_{loc} & \longrightarrow & H^*(\bar{V})_S \otimes \dfrac{H^*(Z_p)_{loc}}{H^*(Z_p)} & \to 0 \\
& \downarrow {\scriptstyle res_{\bar{W}} \otimes 1} & & \downarrow {\scriptstyle res_{\bar{W}} \otimes 1} & & \downarrow {\scriptstyle res_{\bar{W}} \otimes 1} & \\
0 \to & H^*(\bar{W})_{S\cap\bar{W}^*} \otimes H^*(Z_p) & \to & H^*(\bar{W})_{S\cap\bar{W}^*} \otimes H^*(Z_p)_{loc} & \to & H^*(\bar{W})_{S\cap\bar{W}^*} \otimes \dfrac{H^*(Z_p)_{loc}}{H^*(Z_p)} & \to 0
\end{array}$$

Here $\langle y \rangle = (Z_p)^*$. Since $\overline{V}$ is of rank n-1 , our main inductive hypothesis, which was stated as part of the assumptions for (7.6), says that the map

$$H^*(\overline{V})_* \xrightarrow{\{res_{\overline{W}}\}} \bigoplus_{\overline{W}} H^*(\overline{W})_{S\cap\overline{W}^*}$$

is a Tor-equivalence. Since $H^*(Z_p)_{loc}/H^*(Z_p)$ is bounded above, Lemma 7.5 shows that the map

$$H^*(\overline{V})_S \otimes \frac{H^*(Z_p)_{loc}}{H^*(Z_p)} \xrightarrow{\{res_{\overline{W}} \otimes 1\}} \bigoplus_{\overline{W}} H^*(\overline{W})_{S\cap\overline{W}^*} \otimes \frac{H^*(Z_p)_{loc}}{H^*(Z_p)}$$

is a Tor-equivalence. This shows that

$$\frac{H^*(V)_T}{H^*(V)_S} \xrightarrow{\{\rho_W\}} \bigoplus_W \frac{H^*(W)_{T\cap W^*}}{H^*(W)_{S\cap W^*}}$$

is a Tor-equivalence, which proves Proposition 7.6.

<u>Proof of Theorem 1.3</u>. We prove this result by induction over n . For n = 1 there is only one way to localise, and the result is true by (7.4). We therefore assume the result true in dimension (n-1) . We now proceed by downward induction over S . Corollary 7.4 shows that the result is true for $S = V^*$; for the inductive step, we must assume that T contains just one more line than S , say $T = S \cup \langle y \rangle$ as in (7.6), and assume that the result holds for T . We now have the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^*(V)_S & \longrightarrow & H^*(V)_T & \longrightarrow & \dfrac{H^*(V)_T}{H^*(V)_S} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \{res_W\}} & & \Big\downarrow{\scriptstyle \{res_W\}} & & \Big\downarrow{\scriptstyle \{\rho_W\}} & & \\
0 & \longrightarrow & \bigoplus_W H^*(W)_{S\cap W^*} & \longrightarrow & \bigoplus_W H^*(W)_{T\cap W^*} & \longrightarrow & \bigoplus_W \dfrac{H^*(W)_{T\cap W^*}}{H^*(W)_{S\cap W^*}} & \longrightarrow & 0
\end{array}$$

Here the middle vertical arrow is a Tor-equivalence by the inductive hypothesis, and the right-hand vertical arrow is a Tor-equivalence by Proposition 7.6. So the left-hand vertical arrow must be a Tor-equivalence, by the Five Lemma applied to the obvious ladder diagram of Tor groups. This completes the induction and proves Theorem 1.3.

Corollary 7.8. The map

$$H^*(V)_{<x>} \xrightarrow{\{res_W\}} \bigoplus_W H^*(W)$$

is a Tor-equivalence.

This follows immediately from Theorem 1.3; if $S = <x>$ then $S \cap W^* = 0$ for each W.

§8. The Steinberg representation. In this section we will prove that $F_p \otimes_A H^*(V)_{loc}$ affords the (mod p) Steinberg representation of $GL(V)$. We will also comment on related results.

First we explain the definition which we use for the Steinberg representation, in terms of the Tits building. The Tits building TB (for V^*) is a certain finite simplicial complex. For each subspace of V^* (other than the trivial subspaces 0 and V^*) it has a vertex. For each flag

$$0 < V_1^* < V_2^* < \dots < V_r^* < V^*$$

it has a simplex whose vertices are the ones corresponding to $V_1^*, V_2^*, \dots, V_r^*$; the dimension of this simplex is thus $(r-1)$. In particular, the maximum dimension of a simplex is $(n-2)$. By "the homology of the Tits building" we mean $\tilde{H}_{n-2}(TB)$, the integral homology of TB in the top dimension $(n-2)$, taken reduced if $n-2 = 0$.

The Tits building for V^* is isomorphic to that for V, because there is a (1-1) correspondence between linear subspaces in V and linear subspaces in V^*, given by passing to annihilators.

Our object is to prove the following result.

Proposition 8.1. There is a canonical isomorphism

$$F_p \otimes_A H^*(V)_{loc} \cong F_p \otimes_Z \tilde{H}_{n-2}(TB) .$$

We shall subdivide the proof by introducing an alternative construction of the Steinberg module. For this we present a Z-module $M = M(V^*)$ by generators and relations, as follows. We take one generator

$$m(x_1, x_2, \dots, x_n)$$

for each base $(x_1, x_2, \ldots, x_n)$ of V^*. We prescribe the following relations.

(i) m is antisymmetric in its arguments, that is,

$$m(x_{\rho 1}, x_{\rho 2}, \ldots, x_{\rho n}) = \varepsilon(\rho)\, m(x_1, x_2, \ldots, x_n)$$

for each permutation ρ.

(ii) If λ is a non-zero scalar, then

$$m(\lambda x_1, x_2, \ldots, x_n) = m(x_1, x_2, \ldots, x_n).$$

(iii) Suppose that V^* comes as the direct sum $V^* = X^* \oplus Y^*$ of a subspace X^* of dimension 2 and a subspace Y^* of dimension $n-2$. Suppose that any two of x_1, x_2, x_3 form a base for X^*, while $y_3, y_4, \ldots, y_n$ form a base for Y^*. Then

$$m(x_1, x_2, y_3, y_4, \ldots, y_n) + m(x_2, x_3, y_3, y_4, \ldots, y_n) + m(x_3, x_1, y_3, y_4, \ldots, y_n) = 0.$$

It is clear how $\mathrm{Aut}(V) = GL(V)$ acts on M.

Our object is now to prove the following two results.

<u>Proposition 8.2</u>. $M = M(V^*)$ is canonically isomorphic to $\tilde{H}_{n-2}(TB)$, the homology of the Tits building.

<u>Proposition 8.3</u>. There is a canonical map from $F_p \otimes_Z M$ to $H^*(V)_{loc}$ such that the composite

$$F_p \otimes_Z M \longrightarrow H^*(V)_{loc} \longrightarrow F_p \otimes_A H^*(V)_{loc}$$

is an isomorphism.

Proposition 8.3 shows that the quotient map

$$H^*(V)_{loc} \longrightarrow F_p \otimes_A H^*(V)_{loc}$$

has a canonical splitting. Thus, for example, the result of Priddy and Wilkerson [13], that $H^*(V)_{loc}$ is projective over $F_p[GL(V)]$, implies that the quotient $F_p \otimes_A H^*(V)_{loc}$ is also projective over $F_p[GL(V)]$.

We will begin by giving the canonical map

$$F_p \otimes_Z M \longrightarrow H^*(V)_{loc} ,$$

for this will motivate and explain the construction of M. We give the map on the generators and check that it preserves the relations. Let $(x_1, x_2, \ldots, x_n)$ be a base for V^*, and let $e_r \in H^1(V)$ be such that $\beta e_r = x_r \in H^2(V)$. Then we send the generator

$$m(x_1, x_2, \ldots, x_n)$$

to

$$e_1 x_1^{-1} e_2 x_2^{-1} \ldots e_n x_n^{-1} \in H^*(V)_{loc} .$$

It is clear that the relations (i) and (ii) are preserved, so we turn to relation (iii). Let x_1, x_2 and x_3 be as assumed, so that any two of them form a base for X^*. By relation (ii) we may assume that the linear dependence between x_1, x_2 and x_3 has the form

$$x_1 + x_2 + x_3 = 0$$

$$e_1 + e_2 + e_3 = 0 .$$

Dividing the first equation by $x_1 x_2 x_3$ we find

$$x_2^{-1} x_3^{-1} + x_3^{-1} x_1^{-1} + x_1^{-1} x_2^{-1} = 0 ;$$

multiplying the second by e_3, e_1 and e_2 we find

$$e_2 e_3 = e_3 e_1 = e_1 e_2 .$$

Thus

$$e_2 x_2^{-1} e_3 x_3^{-1} + e_3 x_3^{-1} e_1 x_1^{-1} + e_1 x_1^{-1} e_2 x_2^{-1} = 0 .$$

This sets up the canonical map

$$F_p \otimes_Z M \longrightarrow H^*(V)_{loc} .$$

We will now analyse the structure of M, and give a base for it. Let $(x_1 , x_2 , \ldots , x_n)$ be one base for V^*, and let g run over the (corresponding) upper uni-triangular matrices.

<u>Proposition 8.4</u>. Then the generators

$$m(gx_1 , gx_2 , \ldots , gx_n)$$

form a Z-base for M.

We will begin by showing that the generators

$$m(gx_1 , gx_2 , \ldots , gx_n)$$

span M. For this we need a lemma. Let W^* be a subspace of dimension $(n-1)$ in V^*.

<u>Lemma 8.5</u>. M is spanned by generators

$$m(y_1 , y_2 , \ldots , y_n)$$

in which all but one of the y_r lie in W^*.

Proof. Consider a generator

$$m(y_1 , y_2 , \dots , y_n)$$

in which two of the y_r do not lie in W^* ; by relation (i), we may assume that y_1 and y_2 do not lie in W^* . Since V^*/W^* is of dimension 1 , there is a linear dependence between y_1 , y_2 and some element $w \in W^*$. Then relation (iii) allows us to replace

$$m(y_1 , y_2 , y_3 , \dots , y_n)$$

by

$$- m(y_2 , w , y_3 , \dots , y_n)$$

$$- m(w , y_1 , y_3 , \dots , y_n) .$$

In this way we can reduce the number of the y_r which do not lie in W^* , and the result follows by induction.

We may re-express (8.5) as follows. For each y_n which is in V^* but not in W^* we have a map

$$M(W^*) \longrightarrow M(V^*)$$

which carries

$$m(y_1 , y_2 , \dots , y_{n-1})$$

to

$$m(y_1 , y_2 , \dots , y_{n-1} , y_n)$$

(for this clearly preserves the relations (i), (ii), (iii).) Replacing y_n by λy_n (for $\lambda \neq 0$) gives the same map. We may now run y_n over the p^{n-1} representatives $w + x_n$, where w runs over W^* and x_n is a fixed element which is in V^* but not in W^*. We thus obtain a map

Printer $\bigoplus_1^{p^{n-1}}$

$$\bigoplus_1^{p^{n-1}} M(W^*) \xrightarrow{\phi} M(V^*) ,$$

and Lemma 8.5 shows that this map ϕ is epi.

Consider the generators

$$m(gx_1 , gx_2 , \ldots , gx_n)$$

named in (8.4). The statement that they span M follows immediately by induction over n; the result for W^* implies that for V^*.

At this stage we can already deduce (8.3). In fact, we have shown that $F_p \otimes_Z M$ is spanned by the elements

$$gm(x_1 , x_2 , \ldots , x_n) ;$$

the images of these elements in $F_p \otimes_Z H^*(V)_{loc}$ are

$$g(e_1 x_1^{-1} e_2 x_2^{-1} \ldots e_n x_n^{-1}) ,$$

and these images form a base in $F_p \otimes_A H^*(V)_{loc}$ according to the work in §7.

We still have to show that the generators

$$m(gx_1 , gx_2 , \ldots , gx_n)$$

are linearly independent over Z. The way to do this is suggested by the last paragraph; we will define "residues" which work over Z.

For each base $x_1 , x_2 , \ldots , x_n$ in V^* and each maximal flag F there is at most one permutation ρ such that

$$x_{\rho 1}, x_{\rho 2}, \ldots, x_{\rho n}$$

is a base adapted to the flag F. We will define a homomorphism

$$\theta_F : M = M(V^*) \longrightarrow Z$$

which carries $m(x_1, x_2, \ldots, x_n)$ to $\varepsilon(\rho) = \pm 1$ if there is such a permutation ρ, to 0 otherwise. We just have to check that θ_F preserves the relations used to define M. For the relations (i) and (ii) this is clear, so it remains to consider a typical relation (iii). Let $x_1, x_2, x_3 \in X^*$ be as assumed in relation (iii).

Suppose that the maximal flag F is

$$0 = V_o^* < V_1^* < V_2^* < \ldots < V_n^* = V^* .$$

Consider the resulting filtration

$$0 = V_o^* \cap X^* \subset V_1^* \cap X^* \subset \ldots \subset V_n^* \cap X^* = X^*$$

of the subspace X^*. Suppose that $V_r^* \cap X^*$ has dimension

$$\begin{array}{ll} 0 & \text{for } 0 \le r < i \\ 1 & \text{for } i \le r < j \\ 2 & \text{for } j \le r \le n . \end{array}$$

We may divide cases as follows.

(a) None of the three given elements x_1, x_2, x_3 lies in $V_i^* \cap X^*$ (although some non-zero linear combination of them does so). In this case the homomorphism θ_F is zero on all three of

$$m(x_1, x_2, y_3, \ldots, y_n)$$

$$m(x_2, x_3, y_3, \ldots, y_n)$$

$$m(x_3, x_1, y_3, \ldots, y_n) .$$

(b) Just one of x_1, x_2, x_3 lies in $V_i^* \cap X^*$, say x_3. In this case both x_1 and x_2 map to generators for V_j^*/V_{j-1}^*. The homomorphism θ_F is zero on

$$m(x_1, x_2, y_3, \ldots, y_n) ;$$

it takes equal and opposite values on

$$m(x_2, x_3, y_3, \ldots, y_n)$$

$$m(x_3, x_1, y_3, \ldots, y_n) .$$

This sets up the homomorphism θ_F.

We can now complete the proof of (8.4). With the notation of (8.4), consider the maximal flag F determined by the base

$$(gx_n, gx_{n-1}, \ldots, gx_2, gx_1) .$$

The corresponding homomorphism θ_F maps the generator

$$(gx_1, gx_2, \ldots, gx_{n-1}, gx_n)$$

to ± 1, and all the other generators

$$(g'x_1, g'x_2, \ldots, g'x_{n-1}, g'x_n)$$

to zero. Therefore the generators named in (8.4) are linearly independent; this completes the proof of (8.4).

The same considerations lead to the proof of (8.2). By using the homomorphisms θ_F as components, we obtain a map

$$M = M(V^*) \xrightarrow{\{\theta_F\}} \bigoplus_F Z = C_{n-2}(TB) .$$

Since the subgroup of boundaries is zero, $\tilde{H}_{n-2}(TB)$ is the subgroup of cycles $\tilde{Z}_{n-2}(TB)$. We will show first that $\{\theta_F\}$ maps into this subgroup $\tilde{Z}_{n-2}(TB)$, and secondly that it maps onto $\tilde{Z}_{n-2}(TB)$.

First we wish to show that $d\{\theta_F\} = 0$, where d is interpreted as the augmentation if $n = 2$. We consider the component of $d\{\theta_F\}$ corresponding to a typical simplex of dimension $(n-3)$ in TB , or equivalently to a flag

$$0 < \ldots < V^*_{i-1} < V^*_{i+1} < \ldots < V^*$$

in which there is no subspace of dimension i . This simplex is a face of just $(p+1)$ simplexes of dimension $(n-2)$, corresponding to the maximal flags

$$0 < \ldots < V^*_{i-1} < U^* < V^*_{i+1} < \ldots < V^* .$$

The incidence numbers are the same in each case, so we wish to prove

$$\Sigma_F \; \theta_F = 0$$

where the sum runs over these $(p+1)$ maximal flags F .

Consider the value of this sum $\Sigma_F \; \theta_F$ on a typical generator

$$m(x_1 , x_2 , \ldots , x_n) .$$

All the homomorphisms θ_F will be zero on this generator unless the question can be reduced, by permuting the x's , to the case

$$m(y_1 , y_2 , \ldots , y_n)$$

where $y_1, y_2, \ldots, y_r$ form a base for V_r^* provided $r \neq i$. In this case we get just two non-zero values

$$\theta_F\, m(y_1, y_2, \ldots, y_n)$$

corresponding to the flags with U^* spanned by

$$(y_1, \ldots, y_{i-1}, y_i), \quad (y_1, \ldots, y_{i-1}, y_{i+1})$$

respectively. These two values cancel. Thus $d\{\theta_F\} = 0$ as claimed.

We turn to the proof that $\{\theta_F\}$ maps onto $\tilde{Z}_{n-2}(TB)$. Let us say that two maximal flags

$$F_o: 0 = V_o^* < V_1^* < \ldots \qquad < V_n^* = V^*$$
$$G: 0 = W_o^* < W_1^* < \ldots \qquad < W_n^* = V^*$$

are complementary if $V_r^* \cap W_{n-r}^* = 0$ for each r. Let us fix a maximal flag F_o. In proving (8.4), we have seen that elements $\mu \in M = M(V^*)$ can be found on which the homomorphisms θ_G take assigned values $\theta_G(\mu)$ for G complementary to F_o. So it will be sufficient to prove the following.

run on

<u>Lemma 8.6</u>. Let $\{c_F\} \in \bigoplus_F Z$ be a cycle such that $c_G = 0$ whenever G is complementary to a fixed maximal flag F_o. Then $c_F = 0$ for all F.

Space

The proof is by induction over n, so we assume the result true for spaces of dimension $(n-1)$. We assume that $\{c_F\}$ is a cycle and that $c_G = 0$ whenever G is complementary to F_o.

We now remark that the single condition

$$V_1^* \cap W_{n-1}^* = 0$$

is sufficient ~~to ensure~~ ensure $c_G = 0$. In fact, by restricting attention to the maximal flags G with a fixed value of the final space W^*_{n-1} , we obtain a corresponding problem with V^* replaced by W^*_{n-1} and V^*_i replaced by $V^*_{i+1} \cap W^*_{n-1}$; and by the inductive hypothesis, our data ensure that $c_G = 0$ for all such G .

We will now show, by a subsidiary induction downwards over j , that the condition

$$V^*_1 \cap W^*_j = 0$$

is sufficient to ensure $c_G = 0$. Suppose as an inductive hypothesis that the condition $V^*_1 \cap W^*_j = 0$ is sufficient for some $j < n$. Consider a maximal flag

$$G:\ 0 = W^*_o < \ldots < W^*_{j-1} < W^*_j < W^*_{j+1} < \ldots < W^*_n = V^*$$

in which

$$V^*_1 \cap W^*_{j-1} = 0$$

but

$$V^*_1 \cap W^*_j \neq 0 .$$

Consider the flags

$$H:\ 0 = W^*_o < \ldots < W^*_{j-1} < U^* < W^*_{j+1} < \ldots < W^*_n = V^* .$$

There are just (p+1) of them, and we have

$$\sum_H c_H = 0$$

since c is a cycle. For just p of the flags H we have

$$V^*_1 \cap U^* = 0$$

and therefore $c_H = 0$ by the inductive hypothesis. The remaining one flag is G, and the cycle condition gives $c_G = 0$. This completes the induction over j. At the end of the induction over j we conclude that the condition

$$V_1^* \cap 0 = 0$$

is sufficient; this proves that $c_G = 0$ for all G. This completes the induction over n and proves (8.6). This completes the proof of (8.2) and (therefore) of (8.1).

For completeness we add that these considerations lead also to a canonical map

$$\tilde{H}^{n-2}(TB) \longrightarrow M$$

which becomes an isomorphism upon localising at p. Since $\tilde{H}^{n-2}(TB)$ and $M = \tilde{H}_{n-2}(TB)$ enjoy a pairing (the Kronecker product of $\tilde{H}^{n-2}$ and $\tilde{H}_{n-2}$), we see that $\tilde{H}^{n-2}(TB)$ carries a bilinear product which is non-singular in the sense that its determinant is prime to p. This product is symmetric; equivalently, the map

$$\tilde{H}^{n-2}(TB) \longrightarrow \tilde{H}_{n-2}(TB)$$

is self-dual. Since these considerations are not essential to our purpose, we will not spend more space on them.

§9. The Burnside category and its associated graded category.

Before we can prove Theorem 1.1, we must certainly define the map ω which appears in it. For this purpose we require the categorical considerations we hinted at in §1, and we will begin by giving some motivation.

In §1 we introduced the homotopy-theoretic problem of studying

$$[T \wedge BG_1 , BG_2] .$$

Let us consider the special case $T = S^0$, so that the problem is to study

$$[BG_1 , BG_2] .$$

The most reasonable approach is to follow the ideas which Segal proposed for the special case $G_2 = 1$. The first step should be to define an algebraic construct $A(G_1 , G_2)$ and a homomorphism

$$A(G_1 , G_2) \xrightarrow{\alpha} [BG_1 , BG_2] .$$

Here the construct $A(G_1 , G_2)$ should play the same role as the usual Burnside ring does in the special case $G_2 = 1$; it should be the closest approximation to $[BG_1 , BG_2]$ that can be constructed by algebraic means (without using analytic methods such as completion).

We should expect to give the groups $A(G_1 , G_2)$ any further structure which we find in the groups $[BG_1 , BG_2]$. To begin with, we can form composites

$$BG_1 \longrightarrow BG_2 \longrightarrow BG_3 ;$$

in other words, we have a category S in which the objects are the finite groups $G_1, G_2, \ldots$ and the hom-set from G_1 to G_2 is the set of stable maps $[\underset{\sim}{B}G_1 , \underset{\sim}{B}G_2]$. Therefore we should expect to make the groups $A(G_1 , G_2)$ into the hom-sets of a category A . Similarly, we should expect to introduce further structure into the category A , copying what one can do with smash-products of spectra.

We will in fact set up such a category A ; we call it the Burnside category, because our definition is modelled on the usual definition of the Burnside ring. In the special case $G_1 = G_2$, our construct $A(G , G)$ already appears in the work of C.M Witten [18] , for the same reason and purpose.

It is essential to our overall strategy that one can define a functor $\alpha: A \longrightarrow S$, that is, a set of homomorphisms

$$A(G_1 , G_2) \xrightarrow{\alpha} [\underset{\sim}{B}G_1 , \underset{\sim}{B}G_2]$$

which preserve the structure. This is done using transfer. It is not needed for the algebraic purposes of the present paper, and so we omit it.

Our remarks about the special case $\underset{\sim}{T} = \underset{\sim}{S}^0$ apply with suitable modifications to the general case also; however, it is necessary to discuss the special case first in order to provide the correct categorical setting for a discussion of the general case.

Our first business in this section is to give details of the Burnside category A . Once this is done, our second business is to restrict to the full subcategory of elementary abelian p-groups and introduce a filtration. It is important to our overall strategy that the algebraic filtration of a morphism $f \in A(U,V)$ is in fact the Adams filtration of the resulting map ~~of spectra~~

$$\underset{\sim}{B}U \xrightarrow{\alpha f} \underset{\sim}{B}V \;;$$

for the purposes of the present paper we do not need to prove it.

Using the filtration, we can pass to an associated graded category A^{gr}.

After that we shall define a functor β from A^{gr} to an Ext category E. The idea is that if $f \in A(U,V)$, then

$$\beta f \in \mathrm{Ext}_A^{**}(H^*(V)\, ,\ H^*(U))$$

gives the position of

$$\underset{\sim}{B}U \xrightarrow{\alpha f} \underset{\sim}{B}V$$

in the Adams spectral sequence for computing $[\underset{\sim}{B}U\, ,\ \underset{\sim}{B}V]$; more formally, if f is of filtration s in $A(U,V)$, then

$$\beta f \in \mathrm{Ext}_A^{s,s}(H^*(V)\, ,\ H^*(U))$$

is a permanent cycle in the Adams spectral sequence

$$\mathrm{Ext}_A^{**}(H^*(V)\, ,\ H^*(U)) \Longrightarrow [\underset{\sim}{B}U\, ,\ \underset{\sim}{B}V]\, ,$$

and αf, βf have the same image in E_∞.

We cannot expect a statement of this form to define βx uniquely except for $s = 0,1$; in general, differentials might cause some permanent cycles in E_2 to map to zero in E_∞.

For the purposes of the present paper, we need not prove any assertions about the Adams spectral sequence; we need the functor β for algebraic book-keeping. By using β we can introduce and manipulate elements of the Ext category; thus we can define the map ω in Theorem 1.1 and establish its properties.

We postpone further discussion of the functor β to §10 and of the bookkeeping to §11, and proceed to business.

The objects of the Burnside category A will be the finite groups $G,H,\dots$. We wish to describe the hom-set of morphisms from G to H in A . We consider finite sets X which come provided with an action of G on the left of X and an action of H on the right of X , so that these two actions commute and the action of H on the right of X is free. Such sets X we call "(G,H)-sets". We take the (G,H)-sets and classify them into isomorphism classes. The operation of disjoint union passes to isomorphism classes, and turns the set of isomorphism classes into a commutative monoid. This monoid is a free commutative monoid; we obtain a base by considering the isomorphism classes of (G,H)-sets X which are irreducible under disjoint union. (It is equivalent to say that the action of G on X/H is transitive.) We define $A(G,H)$ to be the Grothendieck group or universal group associated to this monoid. This is a free abelian group; we obtain a base by considering the same irreducibles as before.

For example, if $H = 1$, then a $(G,1)$-set is essentially just a G-set, and so $A(G,1)$ reduces to the usual group $A(G)$.

We define the set of morphisms in A from G to H to be $A(G,H)$. We have to define the composition product

$$A(G,H) \otimes A(H,K) \longrightarrow A(G,K)$$

(where the notation reveals that we shall compose morphisms from left to right).

Let X be a (G,H)-set and Y an (H,K)-set; then $X \times_H Y$ is a (G,K)-set. This operation passes to isomorphism classes and is biadditive with respect to the disjoint union; so it defines a product as stated. This product is associative and has units $1_G \in A(G,G)$ is the class of G, considered as a (G,G)-set with the obvious left and right actions. This makes A into a category.

We also define particular morphisms in A. For each homomorphism $\theta: G \longrightarrow H$, we introduce an element $\theta_* \in A(G,H)$; this is the class of H, with G acting on its left via θ and H acting on its right. For each monomorphism $\phi: H \longrightarrow G$, we introduce an element $\phi^* \in A(G,H)$; this is the class of G, with G acting on its left and H acting on its right via ϕ. This action of H is free because we assume that ϕ is mono.

We can now give more motivation for the category A. A functor T defined on A provides a functor on the usual category of finite groups: on objects G we take $T(G)$ and on morphisms $\theta: G \longrightarrow H$ we take $T(\theta_*)$. But beyond this we get homomorphisms $T(\phi^*)$, which correspond to the possibility of "induction". (For example, the "homology of groups" is such a functor T, essentially because it factors as a composite of two functors: the functor α from A to spectra, and the homology-functor from spectra to graded groups.) If T is a functor defined on A, then the homomorphisms $T(\theta_*)$ and $T(\phi^*)$ satisfy all the usual axioms for "induction" and "restriction", including the double coset formula. However, we do not have to state these axioms explicitly; they are implicit in the structure of the category A. We regard the category A as the place where one can do "universal" calculations with induction and restriction subject to the usual axioms.

We now proceed to make A into a monoidal category [8]. The product on objects is the cartesian product $G \times H$ of groups. (In the ordinary category of groups and homomorphism this is a categorical product; it is no longer a categorical product in A .) The product on morphisms is defined as follows. Let X_1 be a (G_1, H_1)-set and let X_2 be a (G_2, H_2)-set; then $X_1 \times X_2$ is a $(G_1 \times G_2, H_1 \times H_2)$-set. This construction passes to isomorphism classes and is biadditive with respect to disjoint union; so it defines a product

$$A(G_1, H_1) \otimes A(G_2, H_2) \longrightarrow A(G_1 \times G_2, H_1 \times H_2)$$

as required.

These products in A are associative, up to the canonical isomorphism

$$(G_1 \times G_2) \times G_3 \longleftrightarrow G_1 \times (G_2 \times G_3) ;$$

and they have the trivial group 1 as unit, up to the canonical isomorphisms

$$1 \times G \longleftrightarrow G \longleftrightarrow G \times 1 .$$

These canonical isomorphisms are coherent, and we shall run no risk if we neglect them.

However, it is usually inadvisable to neglect the switch map; this is of course the isomorphism

$$G_1 \times G_2 \xrightarrow{\tau} G_2 \times G_1$$

given by $\tau(g_1, g_2) = (g_2, g_1)$.

We should list the basic formal properties of the things we have mentioned.

<u>Lemma 9.1</u> (i) $1_G \times 1_H = 1_{G \times H}$.

(ii) Suppose given

$$a \in A(G_1\ ,\ G_2)\ ,\quad c \in A(G_2\ ,\ G_3)$$

$$b \in A(H_1\ ,\ H_2)\ ,\quad d \in A(H_2\ ,\ H_3)\ ;$$

then

$$(a \times b)(c \times d) = (ac) \times (bd)\ .$$

(iii) $\tau_*(f \times g) = (g \times f)\tau_*$.

(iv) We have

$$1_* = 1\ ,\ (\alpha\beta)_* = \alpha_*\beta_*\ ,\ (\alpha \times \beta)_* = \alpha_* \times \beta_*$$

where α,β are homomorphisms of groups.

(v) We have

$$1^* = 1\ ,\ (\alpha\beta)^* = \beta^*\alpha^*\ ,\ (\alpha \times \beta)^* = \alpha^* \times \beta^*$$

if α,β are mono.

(vi) If $\theta : G \longrightarrow H$ is an isomorphism then

$$\theta_*\theta^* = 1_G\ ,\ \theta^*\theta_* = 1_H\ .$$

Parts (i), (ii) are basic formal properties of a monoidal category. Part (vi) says that although there are two ways to interpret an isomorphism of groups as an equivalence in A , these two ways agree.

We pass on to results which give more specific information about the structure of A .

Lemma 9.2. Each irreducible (G,H)-set X has

$$[X] = \phi^*\theta_*$$

for some monomorphism $G \xleftarrow{\phi} K$ and some homomorphism $K \xrightarrow{\theta} H$; these are determined by $[X]$ up to an isomorphism of K .

<u>Sketch proof</u>. Pick a particular element $x_o \in X$. Let $K \subset G \times H$ be the subgroup of pairs (g,h) such that $gx_o = x_o h$; then the obvious map $\phi: K \longrightarrow G$ is mono because the action of H on X is free. We define a map $G \times_K H \longrightarrow X$ by

$$(g,h) \longrightarrow gx_o h \;;$$

this map is injective, and surjective since we assume X irreducible.

Conversely, if $[X] = \phi^* \theta_*$ then X is up to isomorphism $G \times_K H$ and K is (essentially) the stabiliser in $G \times H$ of some point $x \in X$.

For the next lemma, we suppose given a diagram

$$\begin{array}{ccc} & & V \\ & & \downarrow \phi \\ T & \xrightarrow{\theta} & U \end{array}$$

in which ϕ is mono. We form the following pullback diagram.

$$\begin{array}{ccc} S & \xrightarrow{\chi} & V \\ \psi \downarrow & & \downarrow \phi \\ T & \overset{\theta}{\longrightarrow} & U \end{array}$$

It follows that ψ is mono.

Lemma 9.3. If U is abelian then

$$\theta_* \phi^* = \left(\frac{\text{index } \phi}{\text{index } \psi}\right) \psi^* \chi_* \quad \text{in} \quad A(T,V)$$

where index $\phi = |U: \phi(V)|$ and similarly for ψ.

Proof. $\theta_* \phi^*$ is the class of the (T,V)-set U, with T acting on the left via θ and V acting on the right via ϕ. Since U is abelian, all the (T,V)-orbits are isomorphic to any one of them, say the one containing the identity element. This one is isomorphic to $T \times_S V$, and the isomorphism class of that is $\psi^* \chi_*$. Since $|U|$ elements fall into orbits of size $(|T|\ |V|)/|S|$, the number of orbits is

$$(|U|\ |S|)/(|V|\ |T|) = (\text{index } \phi)/(\text{index } \psi) .$$

We pass on to results about a particular element in A. Corresponding to the injection $i: 1 \longrightarrow Z_p$ we have an element $i^* \in A(Z_p, 1)$.

Lemma 9.4. This element has the following properties.

(a) If $\theta: Z_p \longrightarrow Z_p$ is an automorphism, then

$$\theta_* \, i^* = i^* .$$

(b) $\tau_* \, (1 \times i^*) = (i^* \times 1)$.

(c) Let $\theta: Z_p \times Z_p \longrightarrow Z_p \times Z_p$ be the homomorphism

$$\theta(x,y) = (x + \lambda y, y)$$

(for some fixed $\lambda \in F_p$). Then

$$\theta_*(1 \times i^*) = (1 \times i^*) .$$

(d) $i_* i^* = p \in A(1,1)$.

Proof. All four parts can be viewed as instances of (9.3) (using (9.1)(v) as needed).

We will now move towards our associated graded category. First we take the full subcategory of the Burnside category in which the objects are elementary abelian p-groups. Next we shall define a filtration on its hom-sets $A(U,V)$.

If X is an irreducible (U,V)-set, we define $s(X)$ by

$$p^{s(x)} = |X/V| .$$

Clearly this depends only on the isomorphism class of X . By (9.2) we can write $[X]$ in the form $\phi^*\theta_*$; then

$$p^{s(x)} = \text{index } \phi .$$

We define the filtration subgroup

$$F_s A(U,V) \subset A(U,V)$$

to be the subgroup generated by the elements $p^\lambda[X]$, where X runs over the irreducible (U,V)-sets and λ, X satisfy

$$\lambda + s(X) \geq s .$$

Lemma 9.5. Composition and cross product preserve this filtration. More precisely, if X is an irreducible (U,V)-set and Y is an irreducible (V,W)-set then $[X][Y] = p^\lambda[Z]$ where Z is an irreducible (U,W)-set with

$$\lambda + s(Z) = s(X) + s(Y) ;$$

similarly for the cross product, with $\lambda = 0$.

<u>Proof</u>. Write [X] , [Y] in the form $\phi_1^*\theta_{1*}$, $\phi_2^*\theta_{2*}$. By (9.3) we have

$$\theta_{1*}\phi_2^* = \left(\frac{\text{index } \phi_2}{\text{index } \psi}\right) \psi^*\chi_*$$

where ψ and χ come from a suitable pullback diagram. Thus

$$[X][Y] = p^\lambda[Z]$$

where

$$p^\lambda = \frac{\text{index } \phi_2}{\text{index } \psi} \quad \text{and} \quad [Z] = (\psi\phi_1)^*(\chi\theta_2)_* ;$$

thus

$$\begin{aligned} p^{\lambda+s(Z)} &= \frac{\text{index } \phi_2}{\text{index } \psi} \quad (\text{index } \psi\phi_1) \\ &= (\text{index } \phi_1)(\text{index } \phi_2) \\ &= p^{s(X) + s(Y)} . \end{aligned}$$

The assertion about the cross product is easy to verify.

We can now define the associated graded category A^{gr} . The objects of A^{gr} are to be the elementary abelian p-groups U,V,W,... . The hom-set $A^{gr}(U,V)$ from U to V is to be a graded vector-space over F_p , whose s^{th} component is

$$F_sA(U,V)/F_{s+1}A(U,V) .$$

Lemma 9.5 shows that composition and cross product pass to the quotient and give operations in A^{gr} .

§10. Construction of the functor β. In this section we will set up the functor β promised in §9. We have already defined the source category A^{gr} ; we must begin by defining the target category E .

The objects of E are A-modules $L, M, N, \ldots$ which are bounded below and finite-dimensional over F_p in each degree. The hom-set $E(L,M)$ from L to M in E is the bigraded Ext group $\mathrm{Ext}_A^{**}(M,L)$. (Thus E is the opposite of the usual Ext category; this makes some formulae look better. In particular, cohomology is a covariant functor with values in E .) Composition in E is given by the usual Yoneda product.

We make E into a monoidal category. On objects the monoidal operation is the usual tensor product $L \otimes M$. On morphisms it is the usual external tensor product in Ext groups.

We need to name a particular element in E . In §7 we considered the A-module $H^*(Z_p)_{loc}$. Let E be the submodule of $H^*(Z_p)_{loc}$ which consists of the groups in degrees ≥ -1 . It takes part in the following short exact sequence.

$$0 \longrightarrow H^*(Z_p) \longrightarrow E \xrightarrow{res} F_p \longrightarrow 0 .$$

Here the map res is of degree +1 ; thus the class of this extension is an element

$$[E] \in \mathrm{Ext}_A^{1,1}(F_p, H^*(Z_p)) .$$

To reassure the reader, we remark that E does indeed give (up to a sign) the position in the appropriate Adams spectral sequence of the map

$$\alpha(i^*): \underset{\sim}{B}Z_p \longrightarrow \underset{\sim}{B}1 = \underset{\sim}{S}^0 .$$

(The map $\alpha(i^*)$ is the "transfer" corresponding to the covering map $B1 = \widetilde{BZ_p} \longrightarrow BZ_p$.) For our present purposes we do not need to prove this.

In case the reader worries about signs, we remark that the results of this paper remain true if [E] is replaced by λ[E] for any non-zero scalar $\lambda \in F_p$. For definiteness we stick to [E] .

Proposition 10.1. There is a functor

$$\beta: A^{gr} \longrightarrow E$$

with the following properties.

(a) β is given on objects by $\beta(V) = H^*(V)$.

(b) For each morphism $\theta: U \longrightarrow V$ we have

$$\beta(\theta_*) = H^*(\theta): H^*(V) \longrightarrow H^*(U) .$$

(c) β is additive and preserves the monoidal structure.

(d) For the injection $i: 1 \longrightarrow Z_p$ we have

$$\beta(i^*) = [E] \in \mathrm{Ext}_A^{**}(H^*(1), H^*(Z_p)) .$$

All the rest of this section will be devoted to proving Proposition 10.1. In effect we will show that the category A^{gr} can be presented by generators and relations, giving as generators the morphism θ_* (of filtration 0) subject to their obvious formal properties, and in addition one generator i^* (of filtration 1) subject to the relations (9.4)(a)-(c). If so, then the functor β must exist provided that the element [E] satisfies the corresponding relations in E . For convenience, however, we do not present matters in this way; we just build up the construction and properties of β on successively larger parts of A .

The functor β is given on objects by (10.1)(a). It is given on morphisms of the form θ_* by (10.1)(b); if $\theta: U \longrightarrow V$ is a homomorphism of groups then

$$\beta(\theta_*) = H^*(\theta) \in \mathrm{Ext}_A^{0,0}(H^*(V), H^*(U)) .$$

Lemma 10.2. This satisfies

$$\beta(1_*) = 1$$

$$\beta((\theta_1\theta_2)_*) = (\beta(\theta_1)_*)(\beta(\theta_2)_*)$$

$$\beta((\theta_1 \times \theta_2)_*) = (\beta(\theta_1)_*) \otimes (\beta(\theta_2)_*) .$$

This is clear.

Before going any further, we must check that the element [E] satisfies relations corresponding to those in (9.4).

Lemma 10.3.

(a) If $\theta: Z_p \longrightarrow Z_p$ is an automorphism, then

$$(\beta\theta_*)[E] = [E] .$$

(b) Let $\tau: Z_p \times Z_p \longrightarrow Z_p \times Z_p$ be the switch map, $\tau(u,v) = (v,u)$. Then

$$(\beta\tau_*)(1 \otimes [E]) = ([E] \otimes 1) .$$

(c) Let $\theta: Z_p \times Z_p \longrightarrow Z_p \times Z_p$ be the homomorphism $\theta(u,v) = (u + \lambda v, v)$ (for some fixed $\lambda \in F_p$) . Then

$$(\beta\theta_*)(1 \otimes [E]) = (1 \otimes [E]) .$$

The reason one can predict this is as follows. The relations hold in A by (9.4). Applying α, we see that the corresponding relations hold in homotopy, and so in E_∞ of the Adams spectral sequence. But more precisely they hold in $E_\infty^{1,1}$, and $E_\infty^{1,1} \subset E_2^{1,1}$ as we have said: so they hold in $\mathrm{Ext}_A^{1,1}$.

As the previous paragraph uses considerations we have not given in detail, we need a purely algebraic proof, and this is easy. Indeed, we dismiss parts (a) and (b) as trivial, and give the proof for part (c). It is sufficient to construct a diagram of extensions of the following form.

$$\begin{array}{ccccc} H^*(Z_p) \otimes H^*(Z_p) & \longrightarrow & H^*(Z_p) \otimes E & \xrightarrow{1\otimes \mathrm{res}} & H^*(Z_p) \otimes F_p \\ \downarrow {\scriptstyle H^*(\theta)} & & \downarrow {\scriptstyle \phi} & & \downarrow {\scriptstyle 1} \\ H^*(Z_p) \otimes H^*(Z_p) & \longrightarrow & H^*(Z_p) \otimes E & \xrightarrow{1\otimes \mathrm{res}} & H^*(Z_p) \otimes F_p \end{array}$$

Let us write e_1, x_1, e_2, x_2 for the cohomology generators in the two copies of $H^*(Z_p)$; then we construct ϕ by copying the formulae for $H^*(\theta)$, setting

$$\phi|E = 1$$

$$\phi\,(e_1) = e_1 + \lambda e_2$$

$$\phi\,(x_1) = x_1 + \lambda x_2 \,.$$

It remains to see that the right-hand square commutes. We have

$$(1 \otimes \mathrm{res})\phi(e_1^{i_1} x_1^{j_1} e_2 x_2^{-1})$$

$$= (1 \otimes \mathrm{res})(e_1 + \lambda e_2)^{i_1}(x_1 + \lambda x_2)^{j_1} e_2 x_2^{-1}$$

$$= e_1^{i_1} x_1^{j_1} \,,$$

since the remaining terms give no residue. A similar proof works for $p = 2$.

We proceed with our construction. Because of the formal properties of the object 1 , the hom-sets $A^{gr}(U,V)$ are graded modules over the polynomial ring

$$A^{gr}(1,1) = F_p[p] .$$

They are even free modules; the canonical base consists of the elements $[X]$, where X runs over the irreducible (U,V)-sets. By (9.4) we have

$$p = i_* i^* ;$$

so we are committed to the definition

$$\beta[p] = \beta(i_*).[E] \in \mathrm{Ext}_A^{1,1}(F_p, F_p) .$$

(This extension is the obvious one, namely the class of

$$F_p \longrightarrow \tilde{H}^*(S^0 \cup_p e^1) \longrightarrow F_p .)$$

The hom-sets $E(U,V)$ are graded modules over $F_p[\beta[p]]$; we shall make β preserve this module structure. It will thus be sufficient to prescribe β on the irreducibles $[X]$.

We now embark on our programme of enlarging the class of morphisms on which β is defined. We begin by considering monomorphisms of the special form

$$I = x_1 \times x_2 \times \dots \times x_{d+e} : Z_p^d \longrightarrow Z_p^{d+e}$$

where just d of the factors x_j are $1: Z_p \longrightarrow Z_p$ and just e of the factors x_j are $i: 1 \longrightarrow Z_p$. For such an I the construction is forced; we must define

$$\beta(I^*) = \beta x_1^* \otimes \beta x_2^* \otimes \ldots \otimes \beta x_{d+e}^* ,$$

where $\beta 1^* = 1$ and $\beta i^* = [E]$.

Lemma 10.4. If $I_1 : Z_p^{d_1} \longrightarrow Z_p^{d_1+e_1}$ and $I_2 : Z_p^{d_2} \longrightarrow Z_p^{d_2+e_2}$ are monomorphisms of this special form, then

$$\beta((I_1 \times I_2)^*) = \beta(I_1^*) \otimes \beta(I_2^*) .$$

This is clear.

Lemma 10.5. Suppose given a commutative diagram

$$\begin{array}{ccc} Z_p^d & \xrightarrow[\cong]{\chi} & Z_p^d \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle J} \\ Z_p^{d+e} & \xrightarrow[\cong]{\theta} & Z_p^{d+e} \end{array}$$

in which χ, θ are isomorphisms and I, J are monomorphisms of the special form considered. Then

$$(\beta\theta_*)(\beta J^*) = (\beta I^*)(\beta\chi_*) .$$

The proof is in several steps.

Step 1. The result is true if $\chi = 1$ and $I = J = 1^d \times i^e = 1 \times 1 \times \ldots \times 1 \times i \times i \times \ldots \times i$, where d factors 1 come first and e factors i come second.

In this case θ is an isomorphism of the vector space Z_p^{d+e} which leaves the subspace Z_p^d fixed; by the theory of "elementary operations" in linear algebra, we can write θ as a composite of elementary automorphisms, and so it is sufficient to consider the cases in which θ is an elementary automorphism.

(i) Suppose first that θ takes the j^{th} coordinate of Z_p^{d+e} for some $j > d$ and multiplies it by some non-zero scalar, leaving the other coordinates as they are. Then the result follows from (10.3)(a) by passing to tensor products.

(ii) By taking the relation (10.3)(b) and composing with βi^*, we see that

$$(\beta\tau_*)(\beta i^* \otimes \beta i^*) = (\beta i^* \otimes \beta i^*) .$$

Passing to tensor products, we see that the result is true when θ interchanges coordinates j and $j+1$ for $j > d$, leaving the other coordinates as they were. By composition we see that the result is true when θ permutes the last e coordinates in any manner.

(iii) By taking the relation (10.3)(c) and passing to products, we see that the result is true when θ takes the $(j-1)^{th}$ coordinate and adds to it a scalar multiple of the j^{th} coordinate, provided $j > d$. Let us write θ' for such an elementary automorphism. Consider now the more general case in which θ takes the i^{th} coordinate and adds to it a scalar multiple of the j^{th} coordinate, where $i \neq j$, $j > d$. Then we can write

$$\theta = (\rho^{-1} \times \sigma^{-1})\theta'(\rho \times \sigma)$$

where ρ and σ permute the first d and the last e coordinates respectively. Then we have

$$(\beta\theta_*)(1^d \otimes (\beta i^*)^e)$$

$$= (\beta\rho_*^{-1} \otimes \beta\sigma_*^{-1})(\beta\theta_*')(\beta\rho_* \otimes \beta\sigma_*)(1^d \otimes (\beta i^*)^e) \qquad (10.2)$$

$$= (\beta\rho_*^{-1} \otimes \beta\sigma_*^{-1})(\beta\theta_*')(1 \otimes \beta\sigma_*)(1^d \otimes (\beta i^*)^e)(\beta\rho_*)$$

$$= (\beta\rho_*^{-1} \otimes \beta\sigma_*^{-1})(\beta\theta_*')(1^d \otimes (\beta i^*)^e)(\beta\rho_*)$$

(paragraph (ii) above)

$$= (\beta\rho_*^{-1} \otimes \beta\sigma_*^{-1})(1^d \otimes (\beta i^*)^e)(\beta\rho_*)$$

(special case above)

$$= (1 \otimes \beta\sigma_*^{-1})(1^d \otimes (\beta i^*)^e)$$

$$= (1^d \otimes (\beta i^*)^e)$$

(paragraph (ii) above).

This covers sufficiently many elementary automorphisms, and completes Step 1.

<u>Step 2</u>. As in Step 1, consider the special case $K = 1^d \times i^e$. Then for any χ there is some θ for which we have both the data $\chi K = K\theta$ and the conclusion

$$(\beta\theta_*)(\beta K^*) = (\beta K^*)(\beta\chi_*) .$$

In fact, we have only to take $\theta = \chi \times 1^e$. We have

$$\beta((\chi \times 1^e)_*)(1^d \otimes (\beta i^*)^e)$$

$$= ((\beta\chi_*) \otimes 1^e)(1^d \otimes (\beta i^*)^e) \qquad (10.2)$$

$$= (\beta\chi_*) \otimes (\beta i^*)^e$$

$$= (1^d \otimes (\beta i^*)^e)(\beta\chi_*) .$$

<u>Step 3</u>. For any I and J with the same d,e there is some θ for which we have the data $I\theta = J$ and the conclusion

$$(\beta\theta_*)(\beta J^*) = \beta I^* .$$

We can get from any monomorphism I of the special form considered to any other by a sequence of steps, each of which interchanges two consecutive factors of which one is 1 and the other is i. We will proceed by induction over the number of steps. Suppose that for some I,J we have found θ so that

$$I\theta = J \quad \text{and} \quad (\beta\theta_*)(\beta J^*) = \beta I^* .$$

Suppose further that K is obtained from J by interchanging two consecutive factors of which one is 1 and the other is i, and let ρ be the homomorphism which interchanges the corresponding factors of Z_p^{d+e}. Then clearly we have $J\rho = K$, and by passing to tensor products from (10.3)(b) we get

$$(\beta J^*) = (\beta\rho_*)(\beta K^*) .$$

So for I and K we have

$$I\theta\rho = K$$

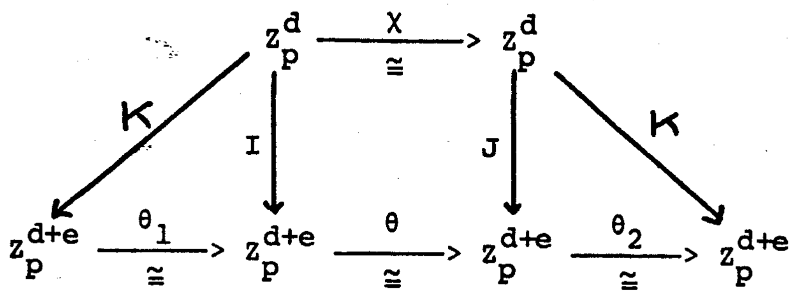
and

$$\begin{aligned} (\beta(\theta\rho)_*)(\beta K^*) &= (\beta\theta_*)(\beta\rho_*)(\beta K^*) && (10.2)\\ &= (\beta\theta_*)(\beta J^*) && (\text{above})\\ &= \beta I^* . \end{aligned}$$

This completes the induction and finishes Step 3.

<u>Step 4</u>. The result is true in general.

Suppose given the data $I\theta = \chi J$. From Step 3 we can construct a diagram of data

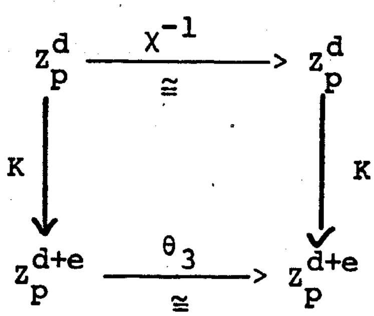


in which $K = 1^d \times i^e$, and so that

$$(\beta\theta_{1*})(\beta I^*) = (\beta K^*) ,$$

$$(\beta\theta_2)_*(\beta K^*) = (\beta J^*) .$$

From Step 2 we get the diagram



and the equation

$$(\beta\theta_{3*})(\beta K^*)(\beta\chi_*) = (\beta K^*) .$$

So Step 1 applies to the isomorphism

$$\theta_1\theta\theta_2\theta_3 \colon Z_p^{d+e} \longrightarrow Z_p^{d+e}$$

and gives (using (10.2))

$$(\beta\theta_{1*})(\beta\theta_*)(\beta\theta_{2*})(\beta\theta_{3*})(\beta K^*) = (\beta K^*) .$$

Multiplying on the right by $\beta\chi_*$ and substituting, we get

$$(\beta\theta_{1*})(\beta\theta_*)(\beta\theta_{2*})(\beta K^*) = (\beta K^*)(\beta\chi_*) ,$$

that is

$$(\beta\theta_{1*})(\beta\theta_*)(\beta J^*) = (\beta\theta_{1*})(\beta I^*)(\beta\chi_*) ,$$

so that

$$(\beta\theta_*)(\beta J^*) = (\beta I^*)(\beta\chi_*) .$$

This completes the proof of Lemma 10.5.

We proceed to define β on morphisms ϕ^*. Let $\phi: V \longrightarrow U$ be a monomorphism. Then we can find (in many ways) a diagram of the following form, in which I is a monomorphism of the special form considered above.

$$\begin{array}{ccc} Z_p^d & \xrightarrow[\cong]{\chi} & V \\ I \downarrow & & \downarrow \phi \\ Z_p^{d+e} & \xrightarrow[\cong]{\theta} & U \end{array}$$

We define

$$\beta\phi^* = (\beta\theta_*^{-1})(\beta I^*)(\beta\chi_*) .$$

The only point to check is that this is independent of the choice of the diagram; this follows immediately from (10.5).

<u>Lemma 10.6</u>. This construction of $\beta\phi^*$ secures the following properties.

(a) If ϕ is an isomorphism then

$$\beta\phi^* = \beta\phi_*^{-1} .$$

(b) If $W \xrightarrow{\psi} V$ and $V \xrightarrow{\phi} U$ are monomorphisms then

$$\beta(\psi\phi)^* = (\beta\phi^*)(\beta\psi^*) .$$

(c) If $\phi_1 : V_1 \longrightarrow U_1$ and $\phi_2 : V_2 \longrightarrow U_2$ are monomorphisms then

$$\beta(\phi_1 \times \phi_2)^* = (\beta\phi_1^*) \otimes (\beta\phi_2^*) .$$

(d) Suppose given a pull back diagram

$$\begin{array}{ccc} S & \xrightarrow{\chi} & V \\ \psi \downarrow & & \downarrow \phi \\ T & \xrightarrow{\theta} & U \end{array}$$

in which ϕ is a monomorphism (and therefore ψ is mono). Then

$$(\beta\theta_*)(\beta\phi^*) = (\beta[p])^{\lambda}(\beta\psi^*)(\beta\chi_*)$$

where

$$p^{\lambda} = \left(\frac{\text{index } \phi}{\text{index } \psi}\right) .$$

<u>Proof</u>. We begin with (a). Suppose ϕ is iso; then in the construction above we have

$$e = o , I = 1 , \qquad \phi = \chi^{-1}\theta \quad ;$$

then $\beta I^* = 1$ and the definition reduces to

$$\beta\phi^* = (\beta\theta_*^{-1})(\beta\chi_*)$$

$$= \beta\phi_*^{-1} \qquad (10.2).$$

We turn to (b). Suppose given monomorphisms

$$W \xrightarrow{\psi} V \xrightarrow{\phi} U .$$

We easily find a commutative diagram of the following form.

$$\begin{array}{ccc}
Z_p^d & \xrightarrow[\cong]{\xi} & W \\
{\scriptstyle 1^d \times i^e}\downarrow & & \downarrow{\scriptstyle \psi} \\
Z_p^{d+e} & \xrightarrow[\cong]{\eta} & V \\
{\scriptstyle 1^{d+e} \times i^f}\downarrow & & \downarrow{\scriptstyle \phi} \\
Z_p^{d+e+f} & \xrightarrow[\cong]{\zeta} & U
\end{array}$$

The definitions are

$$\begin{aligned}
&(\beta\phi^*)(\beta\psi)^* \\
&= (\beta\zeta_*^{-1})(1^{d+e} \otimes (\beta i^*)^f)(\beta\eta_*)(\beta\eta_*^{-1})(1^d \otimes (\beta i^*)^e)(\beta\xi_*) \\
&= (\beta\zeta_*^{-1})(1^d \otimes (\beta i^*)^{e+f})(\beta\xi)_* \\
&= \beta(\psi\phi)^* \ .
\end{aligned}$$

We turn to (c). Suppose given the following diagrams.

$$\begin{array}{ccc}
Z_p^{d_1} & \xrightarrow[\cong]{\chi_1} & V_1 \\
{\scriptstyle I_1}\downarrow & & \downarrow{\scriptstyle \phi_1} \\
Z_p^{d_1+e_1} & \xrightarrow[\cong]{\theta_1} & U_1
\end{array}
\qquad\qquad
\begin{array}{ccc}
Z_p^{d_2} & \xrightarrow[\cong]{\chi_2} & V_2 \\
{\scriptstyle I_2}\downarrow & & \downarrow{\scriptstyle \phi_2} \\
Z_p^{d_2+e_2} & \xrightarrow[\cong]{\theta_2} & U_2
\end{array}$$

These give the following diagram.

$$\begin{array}{ccc} Z_p^{d_1+d_2} & \xrightarrow[\cong]{\chi_1 \times \chi_2} & V_1 \times V_2 \\ {\scriptstyle I_1 \times I_2}\downarrow & & \downarrow{\scriptstyle \phi_1 \times \phi_2} \\ Z_p^{d_1+e_1+d_2+e_2} & \xrightarrow[\cong]{\theta_1 \times \theta_2} & U_1 \times U_2 \end{array}$$

The definitions give

$$(\beta\phi_1^*) \otimes (\beta\phi_2^*)$$

$$= [(\beta\theta_{1*}^{-1})(\beta I_1^*)(\beta\chi_{1*})] \otimes [(\beta\theta_{2*}^{-1})(\beta I_2^*)(\beta\chi_{2*})]$$

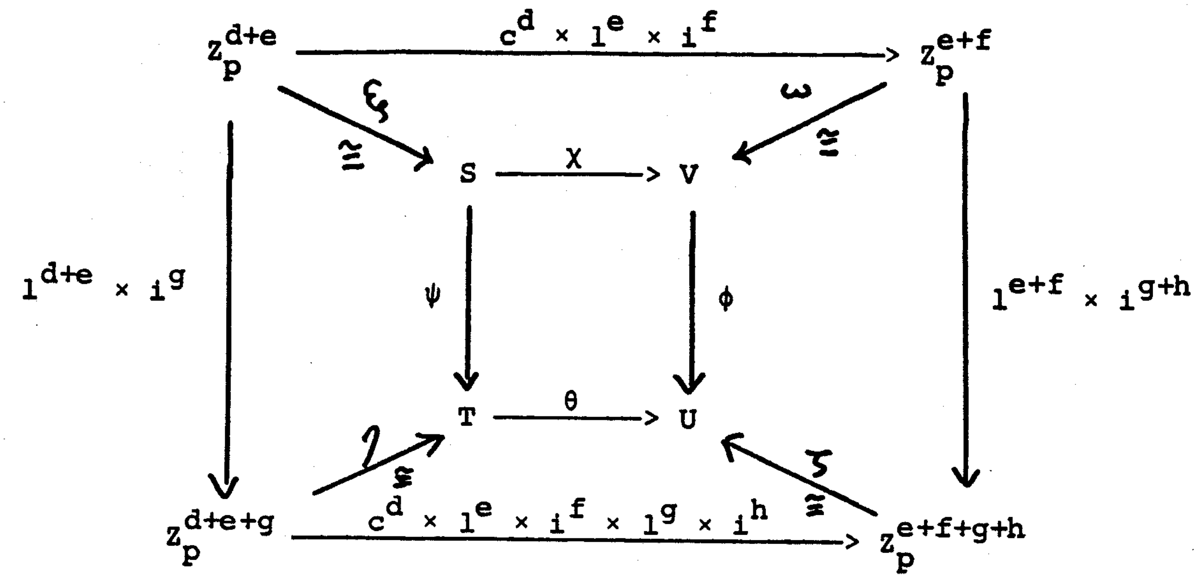
$$= (\beta\theta_{1*}^{-1} \otimes \beta\theta_{2*}^{-1})(\beta I_1^* \otimes \beta I_2^*)(\beta\chi_{1*} \otimes \beta\chi_{2*})$$

$$= (\beta(\theta_1 \times \theta_2)^{-1}{}_*)(\beta(I_1 \times I_2)^*)(\beta(\chi_1 \times \chi_2)_*)$$

((10.2) plus (10.4))

$$= \beta(\phi_1 \times \phi_2)^* .$$

We turn to (d). Suppose given a pullback diagram as in the enunciation. By choosing bases appropriately in S,T,U and V, we can obtain a commutative diagram of the following form, in which $c: Z_p \longrightarrow 1$ is the constant homomorphism.



Using the definitions, we get the following working.

$(\beta\theta)_*(\beta\phi)^* =$

$(\beta\eta_*^{-1})(\beta c_*^d \otimes 1^e \otimes \beta i_*^f \otimes 1^g \otimes \beta i_*^h)(\beta\zeta_*)(\beta\zeta_*^{-1})(1^{e+f} \otimes (\beta i^*)^{g+h})(\beta\omega_*)$

$\quad = (\beta\eta_*^{-1})(\beta c_*^d \otimes 1^e \otimes \beta i_*^f \otimes (\beta i^*)^g \otimes \beta[p]^h)(\beta\omega_*)$

$(\beta\psi^*)(\beta\chi_*) =$

$(\beta\eta_*^{-1})(1^{d+e} \otimes (\beta i^*)^g)(\beta\xi_*)(\beta\xi_*^{-1})(\beta c_*^d \otimes 1^e \otimes \beta i_*^f)(\beta\omega_*)$

$\quad = (\beta\eta_*^{-1})(\beta c_*^d \otimes 1^e \otimes \beta i_*^f \otimes (\beta i^*)^g)(\beta\omega^*)$.

Since h here is equal to λ in the enunciation, the result follows. This completes the proof of (10.6).

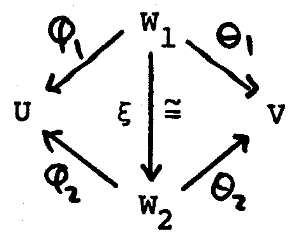
We proceed to define β on $[X]$ for each irreducible (U,V)-set X. By (9.2) we can write $[X] = \phi^*\theta_*$ for some diagram

$$U \xleftarrow[\text{mono}]{\phi} W \xrightarrow{\theta} V .$$

Of course we define

$$\beta[X] = (\beta\phi^*)(\beta\theta_*) .$$

The only point to check is that this is independent of the choice of the diagram. By (9.2) we have $\phi_1^*\theta_{1*} = \phi_2^*\theta_{2*}$ exactly when we have a diagram of the following form.



Then we have

$$
\begin{aligned}
(\beta\phi_1^*)(\beta\theta_{1*}) &= (\beta(\xi\phi_2)^*)(\beta\theta_{1*}) \\
&= (\beta\phi_2^*)(\beta\xi^*)(\beta\theta_{1*}) && (10.6b) \\
&= (\beta\phi_2^*)(\beta\xi_*^{-1})(\beta\theta_{1*}) && (10.6a) \\
&= (\beta\phi_2^*)(\beta\theta_{1*}) && (10.2).
\end{aligned}
$$

This shows that $\beta[X]$ is well-defined.

As we explained earlier, we now define

$$\beta[p^\lambda X] = (\beta[p])^\lambda(\beta[X]) .$$

We have to check that β , as defined on generators $[p^\lambda X]$, preserves composition and the monoidal operations. This follows easily from the results obtained above, using especially (10.6)(b), (c) and (d), which have to be compared with (9.1)(v) and (9.3).

Finally, we define β on all elements of $A^{gr}(U,V)$ by linearity over F_p . This completes the construction of β , and ensures that it has the properties stated in (10.1).

§11. Categorical book-keeping. In this section we will show how the categorical considerations introduced in §9, §10 enable us to give a conceptual statement of Theorem 1.1. We will carry the work just far enough to provide a foundation for the work in §12.

In homotopy-theory, the group $[T \wedge BG_1 , BG_2]$ is a representable functor of T ; the representing object is the function-spectrum of maps from BG_1 to BG_2 , and information about the functor is equivalent to information about the function-spectrum. Roughly speaking, we seek the algebraic analogue of a function-spectrum.

Suppose given a monoidal category C . We have in mind the following examples. (i) The Burnside category A of §9. (ii) The associated graded category A^{gr} of §9. (iii) The Ext category E of §10. We bear in mind that the hom-sets of C may be abelian groups, or graded abelian groups, or bigraded vector spaces over F_p ; for definiteness we will give the details for E whenever the grading makes a difference. In any case, our category C is at least preadditive.

We will explain the notion of a "function-object" in C . Let L and M be given objects in C ; we plan to consider functions from L to M . Suppose given further a finite number of objects W_i in C and morphisms

$$W_i \otimes L \xrightarrow{w_i} M .$$

In the bigraded case, the morphisms w_i may be of any bidegrees (s_i , t_i) . For each "test object" T in C we get a map

$$C^{s-s_i, t-t_i}(T,W_i) \xrightarrow{\omega_i} C^{s,t}(T \otimes L , M)$$

which carries

$$T \xrightarrow{f} W_i$$

to the composite

$$T \otimes L \xrightarrow{f \otimes 1} W_i \otimes L \xrightarrow{w_i} M .$$

With these maps as components we get a map

$$\bigoplus_i C^{s-s_i, t-t_i}(T , W_i) \xrightarrow{\omega} C^{s,t}(T \otimes L , M) .$$

(All these maps, of course, will be maps of abelian groups, or of vector spaces over F_p, according to the nature of the hom-sets in C .) If this map ω is an isomorphism for all objects T in C , we will say that the data $\{W_i , w_i\}$ are a "function-object" from L to M . In this case the data $\{W_i , w_i\}$ allow us to express the group $C(T \otimes L , M)$ in terms of representable functors of T .

Run on

Of course, if there were in C a categorical product of the objects W_i suitably regraded, then this object (with a suitable map) would be a function-object in the usual sense; but we do not assume that any such object exists in C .

If we have two distinct function-objects $\{W'_i , w'_i\}$, $\{W''_j , w''_j\}$ for the same L and M , then one can be thrown onto the other by an invertible matrix of maps $W'_i \longrightarrow W''_j$ (of suitable degrees).

Suppose given a suitable functor from one monoidal category to another; in our applications it will be the functor

$$\beta: A^{gr} \longrightarrow E$$

of §10. Suppose given a function-object from U to V in A^{gr} , say

$$\{W_i \, , \; w_i \in A^{gr}(W_i \times U \, , \; V)\} \, .$$

We will say that β "preserves this function-object" if

$$\{\beta W_i \, , \; \beta w_i \in E(\beta W_i \otimes \beta U \, , \; \beta V)\}$$

is a function-object from βU to βV in E . That is, in our applications, we have $\beta U = H^*(U)$, $\beta V = H^*(V)$ and we wish the appropriate induced map

$$\bigoplus_i \mathrm{Ext}_A^{s-s_i, t-t_i}(H^*(W_i), M) \longrightarrow \mathrm{Ext}_A^{s,t}(H^*(V), M \otimes H^*(U))$$

to be iso for every A-module M which is bounded below and finite-dimensional in each degree.

If β preserves one function-object from U to V , then it preserves all function-objects from U to V , since they are all equivalent and β carries an invertible matrix to an invertible matrix.

We will show that the category A^{gr} has function-objects, and we will show that the functor β of §10 preserves them.

Let U, V be any two objects of A^{gr} , that is, any two elementary abelian p-groups. Let X run over a set of representatives for the isomorphism classes of irreducible (U,V)-sets. For each X , let $W(X)$ be the automorphism group of X ; of course, we mean "automorphisms of X " to preserve the left G-action and the right H-action. We can consider X as a $(W(X) \times U, V)$-set; let

$$w(X) \in A^{gr}(W(X) \times U \, , \; V)$$

be the class of X .

Proposition 11.1. The data $\{W(X), w(X)\}$ constitute a function-object from U to V in A^{gr}.

Theorem 11.2. The functor $\beta: A^{gr} \longrightarrow E$ preserves function-objects.

These considerations explain those points about Theorem 1.1 which were left unexplained in §1. In particular, in Theorem 1.1, the homomorphism

$$\bigoplus_X \mathrm{Ext}_A^{s-s(X),\, t-s(X)}(H^*(W(X)), M) \xrightarrow{\omega}$$

$$\longrightarrow \mathrm{Ext}_A^{s,t}(H^*(V), M \otimes H^*(U))$$

is the one whose components are induced by the elements $\beta w(X)$, in the way described above. The results (11.1) and (11.2) between them show that this map ω is iso; when this is proved it will complete the proof of Theorem 1.1.

We devote the rest of this section to the proof of (11.1).

First we begin with the work which shows that function-objects exist in A. Let G and H be finite groups, and let X run over a set of representatives for the isomorphism classes of irreducible (G,H)-sets. Let F be a further finite group, and let Y be a typical $(F \times G, H)$-set. Let

$$Z_X = \mathrm{Inj}_{(G,H)}(X,Y)$$

be the set of injective (G,H)-maps from X to Y; this is an $(F, W(X))$-set. This construction assigns to each Y a collection $\{Z_X\}$ indexed by the representatives X.

Conversely, suppose given a collection $\{Z_X\}$ in which each Z_X is an $(F, W(X))$-set. Then we can form

$$Y = \coprod_X Z_X \times_{W(X)} X ,$$

and it is an $(F \times G,H)$-set.

<u>Lemma 11.3</u>. These constructions are natural for isomorphisms of Y or isomorphisms of each Z_X as the case may be; the two constructions are inverse up to natural isomorphism, and they preserve disjoint union.

The verification is elementary.

Let us now restrict attention to irreducible sets Y. Then clearly all the sets Z_X must be empty except one which is irreducible; that is, if we consider Y as a (G,H)-set, it will be isotypical for the type of just one X (as is obvious directly). Conversely, if Z_X is an irreducible $(F,W(X))$-set, then $Z_X \times_{W(X)} X$ is irreducible as an $(F\times G, H)$-set.

<u>Lemma 11.4</u>. These constructions give a (1-1) correspondence between isomorphism classes of irreducible $(F\times G, H)$-sets Y and isomorphism classes of irreducible $(F,W(X))$-sets Z_X (for all possible X).

This follows immediately from the discussion.

We now restrict attention to elementary abelian p-groups.

<u>Lemma 11.5</u>. Corresponding irreducible sets Y, Z_X have

$$s(Y) = s(Z_X) + s(X) .$$

<u>Proof</u>. $$p^{s(Y)} = |(Z_X \times_{W(X)} X)/H|$$

$$= |(Z_X/W(X)| \, |X/H|$$

$$= p^{s(Z_X)} p^{s(X)} .$$

<u>Proof of (11.1)</u>. The map

$$\bigoplus_X A^{gr}(F\,,\ W(X)) \xrightarrow{\ \omega\ } A^{gr}(F{\times}G\,,\ H)$$

yields a (1-1) correspondence from an $F_p[p]$-base of the left-hand side (given by the sets Z_X) to an $F_p[p]$-base of the right-hand side (given by the sets Y) .

§12. The case $U = Z_{p}$. In this section we will prove the following result.

Theorem 12.1. Theorem 1.1 is true in the special case $U = Z_p$.

The work will be arranged as follows. First we note an easy lemma from homological algebra. Secondly we explain how the general machinery of §11 specialises in the particular case $U = Z_p$. Thirdly we set up a certain diagram, Diagram 12.3, which is used in the proof. It takes two or three lemmas to discuss the commutativity of this diagram; but once that is done, Theorem 12.1 follows by easy diagram-chasing.

We begin with the lemma from homological algebra. Let L , M, P be left A-modules; let P^* be the dual of P , made into a left A-module in the usual way so that the evaluation map

$$P^* \otimes P \xrightarrow{ev} F_p$$

is A-linear. We have a natural transformation

$$\mathrm{Ext}_A^{**}(L, M \otimes P^*) \longrightarrow \mathrm{Ext}_A^{**}(L \otimes P, M)$$

which sends an element of the Ext category

$$L \xrightarrow{f} M \otimes P^*$$

to the composite

$$L \otimes P \xrightarrow{f \otimes 1} M \otimes P^* \otimes P \xrightarrow{1 \otimes ev} M .$$

Lemma 12.1. Suppose P is bounded above and finite-dimensional over F_p in each degree, and M is bounded below: then the natural transformation

$$\mathrm{Ext}_A^{**}(L, M \otimes P^*) \longrightarrow \mathrm{Ext}_A^{**}(L \otimes P, M)$$

is iso.

Sketch proof. The conditions ensure that the obvious map

$$M \otimes P^* \longrightarrow \mathrm{Hom}(P, M)$$

is iso. Thus the result for Ext^0 is an instance of the adjunction

$$\mathrm{Hom}_A(L, \mathrm{Hom}(P, M)) \xleftrightarrow{\cong} \mathrm{Hom}_A(L \otimes P, M) .$$

In this result for Hom_A we can replace L by the modules C_s of a free resolution for L ; then we pass to cohomology groups.

We now explain how the general machinery of §11 specialises to the present case. Suppose that $U = Z_p$ and V is of rank n . Then p^n of the indices X correspond to the homomorphisms

$$\theta_k : Z_p \longrightarrow V$$

$(k = 1, 2, \ldots, p^n)$. There is one more, which corresponds to the diagram

$$Z_p \xleftarrow{i} 1 \longrightarrow V ;$$

we assign it the number $k = 0$.

The group W_0 corresponding to $k = 0$ is $V \times Z_p$. The group W_k corresponding to $k > 0$ is V .

For $k = 0$ the element

$$X_0 \in A(V \times Z_p \times Z_p , V)$$

is $1_V \times (\mu i^*)$, where $\mu : Z_p \times Z_p \longrightarrow Z_p$ is the multiplication map and $i^* \in A(Z_p, 1)$ is as in §9. For $k > 0$ the element

$$X_k \in A(V \times Z_p, V)$$

is the homomorphism of groups $V \times Z_p \longrightarrow V$ which carries (v,z) to $v + \theta_k(z)$.

We turn to the diagram we need, and we begin with the following exact sequence.

$$0 \longrightarrow H^*(Z_p) \longrightarrow H^*(Z_p)_{loc} \xrightarrow{j} \frac{H^*(Z_p)_{loc}}{H^*(Z_p)} \longrightarrow 0$$

Tensoring with $H^*(V)$, we obtain the following exact sequence.

$$0 \longrightarrow H^*(V) \otimes H^*(Z_p) \longrightarrow H^*(V) \otimes H^*(Z_p)_{loc} \xrightarrow{1 \otimes j} H^*(V) \otimes \frac{H^*(Z_p)_{loc}}{H^*(Z_p)} \longrightarrow 0$$

This yields a long exact sequence of Ext groups, which provides the vertical sequence for the following diagram.

$$\begin{array}{ccc}
\uparrow & & \\
\mathrm{Ext}_A^s(H^*(V) \otimes H^*(Z_p)_{loc}, M) & \xleftarrow{\{res_k\}} & \bigoplus_{k=1}^{p^n} \mathrm{Ext}_A^s(H^*(W_k), M) \\
1 \otimes j \uparrow & & \downarrow \{X_k\} \\
\mathrm{Ext}_A^s\left(H^*(V) \otimes \frac{H^*(Z_p)_{loc}}{H^*(Z_p)}, M\right) & \underset{\cong}{\xleftarrow{D}} & \mathrm{Ext}_A^s(H^*(V), M \otimes H^*(Z_p)) \\
\Delta \uparrow & \nearrow X_0 & \\
\mathrm{Ext}^{s-1}(H^*(V) \otimes H^*(Z_p), M) & & \\
\uparrow & &
\end{array}$$

Diagram 12.3.

We will explain the remaining maps. The maps labelled X_k , X_o are induced by the elements X_k , X_o , as in §11. The map

$$res_k : H^*(V) \otimes H^*(Z_p)_{loc} \longrightarrow H^*(W_k)$$

is a residue of the sort described in §7. In fact, the group W_k is a quotient of $V \times Z_p$ via the map

$$V \times Z_p \xrightarrow{X_k} W_k$$

which carries (v,z) to $v + \theta_k z$; therefore $H^*(W_k)$ is a subalgebra of $H^*(V) \otimes H^*(Z_p)$ and we can consider formal Laurent series with coefficients in $H^*(W_k)$.

The map marked D (for "duality") is an instance of the map in (12.2). Here we must explain how we consider $H^*(Z_p)$ as the dual of $H^*(Z_p)_{loc}/H^*(Z_p)$. We define a map

$$\bar{\Delta} : Z_p \longrightarrow Z_p \times Z_p$$

by

$$\bar{\Delta}(z) = (-z , z) .$$

(The sign is necessary in order to get the details correct in what follows.) This map induces

$$\bar{\Delta}^* : H^*(Z_p) \otimes H^*(Z_p) \longrightarrow H^*(Z_p) ,$$

so that

$$e_1 \longmapsto -e , \quad e_2 \longmapsto e$$

$$x_1 \longmapsto -x , \quad x_2 \longmapsto x$$

(with an obvious notation). Localising, we get

$$\bar{\Delta}^*: H^*(Z_p)_{loc} \otimes H^*(Z_p)_{loc} \longrightarrow H^*(Z_p)_{loc} .$$

We now have the following A-map (of degree +1) , which is also a dual pairing.

$$H^*(Z_p)_{loc} \otimes H^*(Z_p)_{loc} \xrightarrow{\bar{\Delta}^*} H^*(Z_p)_{loc} \xrightarrow{res} F_p .$$

Since $H^*(Z_p)$ annihilates $H^*(Z_p)$, this map yields the following A-map (of degree +1) , which is again a dual pairing.

$$H^*(Z_p) \otimes \frac{H^*(Z_p)_{loc}}{H^*(Z_p)} \xrightarrow{\bar{e}} F_p .$$

We now apply the work of (12.2) with the map

$$P^* \otimes P \xrightarrow{ev} F_p$$

replaced by $\bar{e}$.

Lemma 12.4. The upper square of Diagram 12.3 is commutative.

Proof. It is sufficient to consider the k^{th} summand of the sum. Given k , we have a map

$$\theta_k: Z_p \longrightarrow V$$

and a map

$$\chi_k: V \times Z_p \longrightarrow V$$

carrying (v,z) to $v + \theta_k(z)$. We have the following diagram of groups.

$$
\begin{array}{ccc}
V \times Z_p & \xrightarrow{X_k} & V \\
{\scriptstyle X_k \times 1}\uparrow & & \\
V \times Z_p \times Z_p & & \uparrow{\scriptstyle 1} \\
{\scriptstyle 1 \times \overline{\Delta}}\uparrow & & \\
V \times Z_p & \xrightarrow{\pi_1} & V
\end{array}
$$

(Here of course we have $\pi_1(v,z) = v$.) Passing to cohomology and localising on the left, we get the following diagram.

$$
\begin{array}{ccc}
H^*(V) \otimes H^*(Z_p)_{loc} & \xleftarrow{X_k} & H^*(V) \\
{\scriptstyle X_k \otimes 1}\downarrow & & \\
H^*(V) \otimes H^*(Z_p) \otimes H^*(Z_p)_{loc} & & \downarrow{\scriptstyle 1} \\
{\scriptstyle 1 \otimes \overline{\Delta}^*}\downarrow & & \\
H^*(V) \otimes H^*(Z_p)_{loc} & \xleftarrow{\pi_1^*} & H^*(V)
\end{array}
$$

At the top we have a subalgebra $\operatorname{Im} X_k$, which is the subalgebra $H^*(W_k)$ embedded in $H^*(V) \otimes H^*(Z_p)$ via X_k . This corresponds, under the isomorphism on the left, to the subalgebra $H^*(V)$ at the bottom. This shows that res_k is the following composite.

$$\begin{array}{c}
H^*(V) \otimes H^*(Z_p)_{loc} \\
\Big\downarrow {\scriptstyle X_k \otimes 1} \\
H^*(V) \otimes H^*(Z_p) \otimes H^*(Z_p)_{loc} \\
\Big\downarrow {\scriptstyle 1 \otimes \overline{\Delta}^*} \\
H^*(V) \otimes H^*(Z_p)_{loc} \xrightarrow{1 \otimes res} H^*(V) \; .
\end{array}$$

This composite can be factored as in the following diagram.

$$\begin{array}{ccc}
H^*(V) \otimes H^*(Z_p)_{loc} & \xrightarrow{1 \otimes j} & H^*(V) \otimes \dfrac{H^*(Z_p)_{loc}}{H^*(Z_p)} \\
\Big\downarrow {\scriptstyle X_k \otimes 1} & & \Big\downarrow {\scriptstyle X_k \otimes 1} \\
H^*(V) \otimes H^*(Z_p) \otimes H^*(Z_p)_{loc} & \xrightarrow{1 \otimes j} & H^*(V) \otimes H^*(Z_p) \otimes \dfrac{H^*(Z_p)_{loc}}{H^*(Z_p)} \\
\Big\downarrow {\scriptstyle 1 \otimes \overline{\Delta}^*} & & \Big\downarrow {\scriptstyle 1 \otimes \overline{e}} \\
H^*(V) \otimes H^*(Z_p)_{loc} & \xrightarrow{1 \otimes res} & H^*(V)
\end{array}$$

On the other hand, composition with

$$(1 \otimes j)(X_k \otimes 1)(1 \otimes \overline{e})$$

gives the other three sides of the upper square in Diagram 12.3. This proves Lemma 12.4.

Before going on to the lower triangle of Diagram 12.3, we need an intermediate result. In the Ext category we can form the following composite.

$$
\begin{array}{c}
F_p \otimes \dfrac{H^*(Z_p)_{loc}}{H^*(Z_p)} \\
\Big\downarrow {\scriptstyle \beta i^* \otimes 1} \\
H^*(Z_p) \otimes \dfrac{H^*(Z_p)_{loc}}{H^*(Z_p)} \\
\Big\downarrow {\scriptstyle \mu^* \otimes 1} \\
H^*(Z_p) \otimes H^*(Z_p) \otimes \dfrac{H^*(Z_p)_{loc}}{H^*(Z_p)} \\
\Big\downarrow {\scriptstyle 1 \otimes \bar{e}} \\
H^*(Z_p) \otimes F_p
\end{array}
$$

Here $\beta i^* = [E]$, as in §10, and μ^* is the map of cohomology induced by $\mu\colon Z_p \times Z_p \longrightarrow Z_p$.

Lemma 12.5. This element of

$$\mathrm{Ext}_A^{1,1}\left(\frac{H^*(Z_p)_{loc}}{H^*(Z_p)},\ H^*(Z_p)\right)$$

is, up to a fixed sign, the class of the extension

$$0 \longrightarrow H^*(Z_p) \longrightarrow H^*(Z_p)_{loc} \longrightarrow \frac{H^*(Z_p)_{loc}}{H^*(Z_p)} \longrightarrow 0 .$$

Proof. We shall construct a diagram of extensions of the following form.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^*(Z_p) \otimes \dfrac{H^*(Z_p)_{loc}}{H^*(Z_p)} & \longrightarrow & E \otimes \dfrac{H^*(Z_p)_{loc}}{H^*(Z_p)} & \longrightarrow & F_p \otimes \dfrac{H^*(Z_p)_{loc}}{H^*(Z_p)} & \longrightarrow & 0 \\
 & & \Big\downarrow \theta & & \Big\downarrow \phi & & \Big\downarrow \chi & & \\
0 & \longrightarrow & H^*(Z_p) & \longrightarrow & H^*(Z_p)_{loc} & \longrightarrow & \dfrac{H^*(Z_p)_{loc}}{H^*(Z_p)} & \longrightarrow & 0
\end{array}
$$

Here E is the extension introduced in §10; thus the top line is the extension $[E] \otimes 1 = \beta i^* \otimes 1$. Of course the map θ has to be the composite of the maps $\beta\mu \otimes 1$ and $1 \otimes \bar{e}$ in the enunciation, and χ has to be ± 1.

We begin as in the proof of (12.4), using the following diagram of groups.

$$\begin{array}{ccc}
Z_p \times Z_p & \xrightarrow{\ \mu\ } & Z_p \\
\mu \times 1 \Big\uparrow & & \\
Z_p \times Z_p \times Z_p & & \Big\uparrow 1 \\
1 \times \bar{\Delta} \Big\uparrow & & \\
Z_p \times Z_p & \xrightarrow{\ \pi_1\ } & Z_p
\end{array}$$

Passing to cohomology and localising on the left, we get the following commutative diagram.

$$\begin{array}{ccc}
H^*(Z_p) \otimes H^*(Z_p)_{loc} & \xleftarrow{\ \mu^*\ } & H^*(Z_p) \\
\mu^* \otimes 1 \Big\downarrow & & \\
H^*(Z_p) \otimes H^*(Z_p) \otimes H^*(Z_p)_{loc} & & \Big\downarrow 1 \\
1 \otimes \bar{\Delta}^* \Big\downarrow & & \\
H^*(Z_p) \otimes H^*(Z_p)_{loc} & \xleftarrow{\ \pi_1^*\ } & H^*(Z_p)
\end{array}$$

As in the proof of (12.4), we have at the top a subalgebra $\mathrm{Im}\ \mu^*$; this corresponds, under the isomorphism on the left, to the subalgebra $H^*(Z_p)$ at the bottom. Let res_μ be the residue corresponding to the subalgebra $\mathrm{Im}\ \mu^*$. As in the proof of (12.4), res_μ is the following composite.

$$\begin{array}{c} H^*(Z_p) \otimes H^*(Z_p)_{loc} \\ \downarrow \mu^* \otimes 1 \\ H^*(Z_p) \otimes H^*(Z_p) \otimes H^*(Z_p)_{loc} \\ \downarrow 1 \otimes \bar{\Delta}^* \\ H^*(Z_p) \otimes H^*(Z_p)_{loc} \xrightarrow{1 \otimes res} H^*(Z_p) . \end{array}$$

Thus res_μ will certainly pass to the quotient and define the required map

$$H^*(Z_p) \otimes \frac{H^*(Z_p)_{loc}}{H^*(Z_p)} \xrightarrow{\theta} H^*(Z_p) .$$

Now res_μ certainly extends to

$$H^*(Z_p)_{loc} \otimes H^*(Z_p)_{loc} \xrightarrow{res_\mu} H^*(Z_p)_{loc} ,$$

and in particular to

$$E \otimes H^*(Z_p)_{loc} \xrightarrow{res_\mu} H^*(Z_p)_{loc} .$$

Unfortunately, it does not pass to the quotient to give a map

$$E \otimes \frac{H^*(Z_p)_{loc}}{H^*(Z_p)} \longrightarrow H^*(Z_p)_{loc} .$$

In fact, with the obvious notation we have

$$res_\mu(e_1 x_1^{-1} \otimes x_2^r) = \begin{cases} x^r & (r \geq 0) \\ 0 & (r < 0) \end{cases}$$

$$res_\mu(e_1 x_1^{-1} \otimes e_2 x_2^r) = \begin{cases} ex^r & (r \geq 0) \\ 0 & (r < 0) . \end{cases}$$

Therefore, we define

$$\phi = \text{res}_\mu - \text{res} \otimes 1 .$$

This agrees with res_μ on $H^*(Z_p) \otimes H^*(Z_p)_{loc}$, since $\text{res} \otimes 1$ is certainly zero there. It does pass to the quotient and define a map

$$E \otimes \frac{H^*(Z_p)_{loc}}{H^*(Z_p)} \xrightarrow{\phi} H^*(Z_p)_{loc} .$$

Thus we obtain the required diagram, with $\chi = -1$. This completes the proof of Lemma 12.5.

Lemma 12.6. The upper triangle of Diagram 12.3 commutes up to a (fixed) sign.

Proof. By the properties of β (§10) plus our account of X_o, the element βX_o is the following composite in the Ext category.

$$\begin{array}{c} H^*(V) \otimes F_p \\ \quad\downarrow 1 \otimes \beta i^* \\ H^*(V) \otimes H^*(Z_p) \\ \quad\downarrow 1 \otimes \mu^* \\ H^*(V) \otimes H^*(Z_p) \otimes H^*(Z_p) \end{array}$$

From this and (12.5), we see that (up to a sign) the maps $X_o D$ and Δ in (12.3) both carry $f \in \text{Ext}_A^{**}(H^*(V) \otimes H^*(Z_p), M)$ to the following composite in the Ext category.

$$
\begin{array}{c}
H^*(V) \otimes \dfrac{H^*(Z_p)_{loc}}{H^*(Z_p)} \\
\Big\downarrow {\scriptstyle 1 \otimes \beta i^* \otimes 1} \\
H^*(V) \otimes H^*(Z_p) \otimes \dfrac{H^*(Z_p)_{loc}}{H^*(Z_p)} \\
\Big\downarrow {\scriptstyle 1 \otimes \mu^* \otimes 1} \\
H^*(V) \otimes H^*(Z_p) \otimes H^*(Z_p) \otimes \dfrac{H^*(Z_p)_{loc}}{H^*(Z_p)} \\
\Big\downarrow {\scriptstyle f \otimes \bar{e}} \\
M \otimes F_p
\end{array}
$$

This proves Lemma 12.6.

Proof of Theorem 12.1. By (12.4) and (12.6), Diagram 12.3 is commutative, up to a fixed sign for the triangle. In this diagram, the map $\{res_k\}$ is iso by Corollary 7.8 plus (1.2). Therefore the map $1 \otimes j$ is epi, and the left-hand vertical sequence is short exact. Now the result follows by diagram-chasing.

§13. Proof of the Main Theorem. In this section we will complete the proof of Theorem 11.2 and so complete the proof of Theorem 1.1.

First we will return to the considerations of §11, and show how to make new function-objects from old. Suppose given a monoidal category C, and suppose given three objects F, G, H in C. Suppose that we have a function-object

$$\{W_j , W_j \otimes G \xrightarrow{w_j} H\}$$

from G to H, and that for each W_j we have a function-object

$$\{V_{ij} , V_{ij} \otimes F \xrightarrow{v_{ij}} W_j\}$$

from F to W_j. Then we can form the morphism

$$V_{ij} \otimes F \otimes G \xrightarrow{v_{ij} \otimes 1} W_j \otimes G \xrightarrow{w_j} H .$$

Lemma 13.1 $\{V_{ij} , (v_{ij} \otimes 1) w_j\}$ is a function-object from $F \otimes G$ to H.

In fact, the assumptions give us isomorphisms

$$\bigoplus_{i,j} C(E , V_{ij}) \longrightarrow \bigoplus_{j} C(E \otimes F , W_j)$$

$$\bigoplus_{j} C(E \otimes F , W_j) \longrightarrow C(E \otimes F \otimes G , H) ;$$

and their composite is the map which has to be proved iso, for its components are induced by the elements $(v_{ij} \otimes 1) w_j$.

We will call this construction of a function-object from $F \otimes G$ to H the "product construction".

Lemma 13.2 Suppose that a functor β preserves the function-object $\{W_j , w_j\}$ from G to H and also preserves the function-object $\{V_{ij}, v_{ij}\}$ from F to W_j for all j. Then it preserves the function-object $\{V_{ij}, (v_{ij} \otimes 1) w_j\}$ from $F \otimes G$ to H given by the product construction.

Proof. Let us write $\mathcal{E}$ for the target category of β. Then the assumptions give us isomorphisms

$$\bigoplus_{i,j} E(M, \beta V_{ij}) \longrightarrow \bigoplus_j E(M \otimes \beta F, \beta W_j)$$

$$\bigoplus_j E(M \otimes \beta F, \beta W_j) \longrightarrow E(M \otimes \beta F \otimes \beta G, \beta H)$$

and their composite is the map we need to prove iso.

Proof of Theorem 11.2. Let us consider a function-object in A^{gr} from U to V. If U is of rank 0 the result is trivial; the map we need to prove iso is essentially the identity map from $E(M, H^*(V))$ to itself. If U is of rank 1 then the result is true by Theorem 12.1. We may therefore proceed by induction over the rank of U. Suppose $U = U' \times U''$ where U' and U'' are of less rank. Then by (11.1) there is a function-object $\{W_j, w_j\}$ from U'' to V and there is also a function-object $\{V_{ij}, v_{ij}\}$ from U' to W_j for each j. By the inductive hypothesis β preserves these function-objects; so by (13.2) it preserves the function-object from $U' \times U''$ to V given by the product construction. Therefore it preserves any other function-object from $U' \times U''$ to V. This completes the induction and proves Theorem 11.2. This finishes the proof of all the results stated.

In the proof above, it can be shown that the class of function-objects given by (11.1) is closed under the product construction. The argument we have given makes it unnecessary to show this, so we omit it.

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