Smooth representations of $GL_n(F)$

- Bernstein-Zelevinsky classification

- L-factors & ε-factors

- Uniqueness of LLC

- Bernstein center

$F/\mathbb{Q}_p$ fin. ext'n , $G = GL_n(F)$, $\psi : F \longrightarrow U(1)$

$R(G)$ = cat. of smooth rep'n of G over $\mathbb{C}$

$$\begin{array}{ccccccccccc} \mathrm{Irr}^{unit,sc}(G) & \subset & \mathrm{Irr}^{ds}(G) & \subset & \mathrm{Irr}^{temp}(G) & \subset & \mathrm{Irr}^{unit}(G) & \subset & \mathrm{Irr}(G) & \subset & R(G) \\ \cap & & \cap & & \cap & & & & \cup & & \\ \mathrm{Irr}^{sc}(G) & \subset & \mathrm{Irr}^{ess.ds}(G) & \subset & \mathrm{Irr}^{ess.temp}(G) & & & \subset & \mathrm{Irr}^{gen}(G) & & \end{array}$$

§1. Bernstein-Zelevinsky classification

Notations

For $\overset{\text{parabolic}}{P} = \underset{\text{Levi}}{M}\overset{\text{nilp. radical}}{N} \subset G$, $\pi \in R(M)$

define $\iota_P^G \pi := \mathrm{Ind}_P^G(\pi \delta_P^{\frac{1}{2}})$

For $M = GL_{n_1} \times \cdots \times GL_{n_m}$, $\pi_i \in R(GL_{n_i})$

define $\pi_1 \times \cdots \times \pi_m := \iota_P^G(\pi_1 \boxtimes \cdots \boxtimes \pi_m) \in R(GL_{\Sigma n_i})$

For $\pi \in R(GL_n)$, denote $\pi(s) := \pi \otimes |\det|^s$ $s \in \mathbb{C}$ (unramified twist)

Def · An interval Δ is of the form

$$\Delta = \Delta(\pi, m) := (\pi, \pi(1), \ldots, \pi(m-1)) \in \mathrm{Irr}^{sc}(GL_n)^m$$

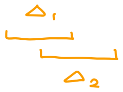
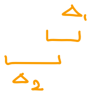
$$\rightsquigarrow \pi(\Delta) := \pi \times \pi(1) \times \cdots \times \pi(m-1) \in R(GL_{mn})$$

Denote $\mathrm{Int}_n^m \subset \mathrm{Irr}^{sc}(GL_n)^m$ to be the set of intervals.

$$\mathrm{Int} = \bigsqcup_{n,m} \mathrm{Int}_n^m$$

- Δ_1, Δ_2 are called linked if

  $\Delta_1 \not\subset \Delta_2$, $\Delta_2 \not\subset \Delta_1$, $\Delta_1 \cup \Delta_2$ is an interval

i.e.  or  linked

$\Delta_1 < \Delta_2$ $\quad$ $\Delta_2 < \Delta_1$

- Say $\Delta_1 < \Delta_2$ (Δ_1 precedes Δ_2) if

  Δ_1, Δ_2 are linked & $\min(\Delta_1) < \min(\Delta_2)$

Thm (Bernstein-Zelevinsky classification)

(1) For $\Delta \in \mathrm{Int}_n^m$

- $l(\pi(\Delta)) = 2^{m-1}$

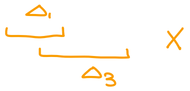
  length

- $\pi(\Delta)$ has a unique irreducible quotient $Q(\Delta) \in \mathrm{Irr}(GL_{mn})$, which is essentially square-integrable

  unique irreducible sub $\quad Z(\Delta) \in \mathrm{Irr}(GL_{mn})$

(2) Suppose $\Delta_1, \dots, \Delta_r \in \mathrm{Int}$ satisfies

$$i < j \Rightarrow \Delta_i \not< \Delta_j$$

i.e. 

Then · $Q(\Delta_1) \times \cdots \times Q(\Delta_r)$ has a unique irreducible quotient $Q(\Delta_1, \cdots \Delta_r)$

· $Z(\Delta_1) \times \cdots \times Z(\Delta_r)$ has a unique irreducible sub $\quad Z(\Delta_1, \cdots, \Delta_r)$

(3) Define $\quad \mathcal{O} = \mathbb{Z}^{\oplus \mathrm{Int}}$

Then
$$\begin{aligned} \mathcal{O} &\xrightarrow{\sim} \bigcup_n \mathrm{Irr}(GL_n(F)) \\ (\Delta_1, \cdots, \Delta_r) &\longmapsto Q(\Delta_1, \cdots \Delta_r) \end{aligned}$$
is a well-defined bijection

can arrage $\Delta_1, \cdots \Delta_r$ s.t. $i<j \Rightarrow \Delta_i \not< \Delta_j$

(4) $Q(\Delta_1) \times \cdots \times Q(\Delta_r)$ irreducible $\Leftrightarrow$ Δ_i, Δ_j are not linked for $i \neq j$

free polynomial ring $\quad$ $[\pi_1] \cdot [\pi_2] = [\pi_1 \times \pi_2]$

(5)
$$\begin{aligned} \mathbb{Z}[\Delta \mid \Delta \in \mathrm{Int}] &\longrightarrow \bigoplus_n K_0(\mathcal{R}(GL_n)) =: \mathcal{R}(F) \\ \Delta &\longmapsto Q(\Delta) \end{aligned}$$

defines a ring isomorphism

Bernstein-Zelevinsky duality

$G = G(F)$ reductive group $/F$

Define $\mathbb{D}_{BZ} : D^b(R(G))^{adm} \longrightarrow (D^b(R(G))^{adm})^{op}$

$$M \longmapsto R\mathrm{Hom}(M, C_c^\infty(G))$$

Thm For $M \in \mathrm{Irr}(G)$, $\exists\ k_M \in \mathbb{Z}$ ($k_M = \dim \mathrm{Spec}\, \mathfrak{Z}_s$)

s.t. $\mathbb{D}_{BZ}(M)[k_M] \in \mathrm{Irr}(G)$

Thm $\cdot$ $\mathbb{D}_{BZ}(Q(\Delta_1, \dots, \Delta_r))[\cdot] = (Z(\Delta_1, \dots, \Delta_r))^\vee = Z(\Delta_1^\vee, \dots, \Delta_r^\vee)$

where for $\Delta = (\pi, \pi(1), \dots, \pi(m-1)) \rightsquigarrow \Delta^\vee = (\pi^\vee(1-m), \dots, \pi^\vee)$

each regarded as a segment

Fact For $\Delta = (\pi, \dots, \pi(m-1))$, $Z(\Delta) = Q(\pi(m-1), \dots, \pi)$

e.g. $\Delta = (1\cdot|\;|^{\frac{1-n}{2}}, \dots, 1\cdot|\;|^{\frac{n-1}{2}})$

$$\begin{tikzcd} \rightsquigarrow Z(\Delta) = \mathbb{1} = Q(1\cdot|\;|^{\frac{n-1}{2}}, \dots, 1\cdot|\;|^{\frac{1-n}{2}}) \arrow[d, leftrightarrow, bend left, "{\mathbb{D}_{BZ}[\cdot]}"] \\ Q(\Delta) = \mathrm{St}_n \end{tikzcd}$$

not essentially tempered, not generic.

square-integrable, generic

e.g. For $\chi_i : F^\times \longrightarrow \mathbb{C}^\times$, suppose

$i < j \Rightarrow \chi_j \neq \chi_i \cdot |\cdot|$

$\Rightarrow Q(\chi_1, \dots, \chi_n)$ is unramified

Properties of $Q(\Delta_1, \dots, \Delta_r)$

$\pi = Q(\Delta_1, \dots, \Delta_r)$ is

- square-integrable $\Leftrightarrow$ $r=1$, $\Delta_1 = [\rho(\frac{m-1}{2}), \dots, \rho(\frac{1-m}{2})]$ for ρ unitary ($\Leftrightarrow$ unitary central char)

  (essentially square integrable $\Leftrightarrow$ $r=1$)

- tempered $\Leftrightarrow$ $Q(\Delta_i)$ are square-integrable

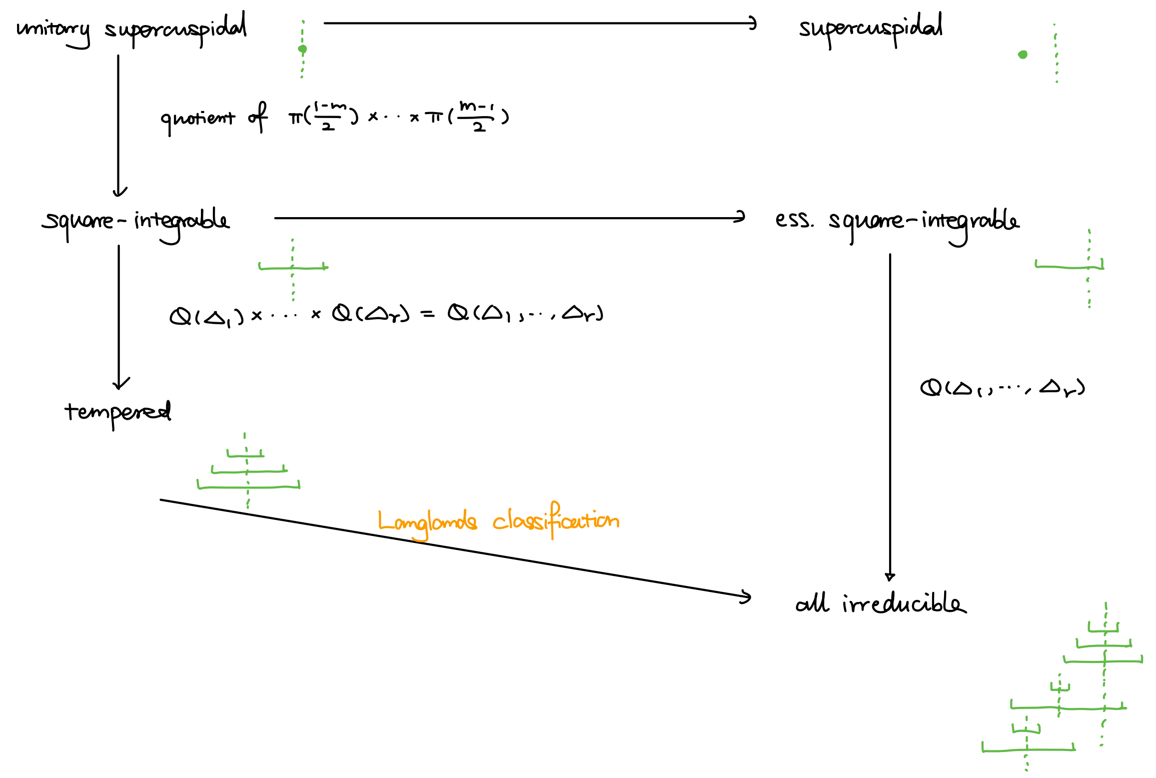
- unitary (see Tadić, Classification of unitary representations in irreducible representations of general linear groups)

- generic $\Leftrightarrow$ Δ_i, Δ_j are not linked

  i.e. $\mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \theta) \neq 0$, where $\theta : N \longrightarrow \mathbb{C}^\times$
  $$n \longmapsto \psi(\textstyle\sum n_{i,i+1})$$

  Cor essentially tempered $\Rightarrow$ generic

Summary



§2. L-factors, ε-factors

Goal Define $L(\pi\times\pi', s)$, $\varepsilon(\pi\times\pi', \psi, s)$ for $\pi\in \mathrm{Irr}(GL_n(F))$, $\pi'\in\mathrm{Irr}(GL_{n'}(F))$

They are symmetric w.r.t π, π' $\Rightarrow$ WLOG $n \geq n'$

L, ε-factors for generic rep'n

Recall that $\pi\in\mathrm{Irr}(GL_n(F))$ is called generic if

$$\exists\ \pi \simeq W(\pi,\psi)\subset C^\infty((\underset{\begin{pmatrix}1&*&*\\&1&*\\&&1\end{pmatrix}}{\underset{\shortparallel}{N_n(F)}},\psi)\backslash GL_n(F))$$

write $\pi\in\mathrm{Irr}^{gen}(GL_n(F))$

Def (Local Zeta integrals)

Case $n=n'$

Input $W\in W(\pi,\psi)$, $W'\in W(\pi',\bar\psi)$, $\Phi\in C_c^\infty(F^n)$

$$\rightsquigarrow Z(W,W',\Phi,s) := \int_{N_n(F)\backslash GL_n(F)} W(g)W'(g)\cdot\Phi(\underset{(0,\cdots,0,1)\in F^n}{\underset{\shortparallel}{v_n}}\cdot g)\cdot|\det g|^s dg$$

Case $n>n'$

Input $W\in \mathcal{W}(\pi,\psi)$, $W'\in\mathcal{W}(\pi',\bar\psi)$, $j\in\{0,1,\cdots,n-n'-1\}$

$$Z(W,W',j,s) = \int_{N_{n'}(F)\backslash GL_{n'}(F)}\int_{M_{j\times n'}(F)} W(\begin{pmatrix} g & & \\ x & I_j & \\ & & I_{n-n'-j}\end{pmatrix})W'(g)\cdot|\det g|^{s-\frac{n-n'}{2}}dx\,dg$$

Case $n=n'+1$

$$Z(W,W',s) = \int_{N_{n'}(F)\backslash GL_{n'}(F)} W(\begin{pmatrix} g & \\ & 1\end{pmatrix})W'(g)|\det g|^{s-\frac{1}{2}}dg$$

Prop/Def (L-factors) (1) $\underset{(j)}{Z(W,W',\Phi,s)}$ converges for $\mathrm{Re}(s)>>0$, defines an elem in $\mathbb{C}(q^{\pm s})$

(2) $\{\underset{(j)}{Z(W,W',\Phi,s)} \mid W,W',\Phi\}$ = fractional ideal of $\mathbb{C}[q^{\pm s}]$ $= L(\pi\times\pi',s)\cdot\mathbb{C}[q^{\pm s}]\subset\mathbb{C}(q^{\pm s})$

for some $L(\pi\times\pi',s) = \dfrac{1}{P(q^{-s})}$, $P\in\mathbb{C}[X]$, $P(0)=1$

Idea of proof $(n=n')$

Asymptotic behavior of W $\Rightarrow$ (1)

Kirillov model $\Rightarrow$ (2)

Denote $A =$ maximal torus, fix W, W', Φ, suppose $W \in \mathcal{W}(\pi, \psi)^K$

(1) For α positive simple root, $a \in A$

<u>Claim</u> When $|\alpha(a)| >> 0$, $W(a) = 0$

$$\text{In fact, } \underset{\substack{\| \\ W(a)}}{W(a \cdot \overset{n_\alpha \in U_\alpha \cap K}{n_\alpha})} = W(a n_\alpha a^{-1} \cdot a) = \underset{\neq 1 \text{ when } |\alpha(a)| >> 0}{\psi(\alpha(a) \cdot n_\alpha)} \cdot W(a) \quad \Rightarrow W(a) = 0$$

root space of α

Assume π supercuspidal

<u>Claim</u> When $|\alpha(a)| << 1$, $W(a) = 0$

$$\text{In fact, } r^G_{P_{\Delta \setminus \{\alpha\}}} \pi = 0 \Rightarrow W = \sum_i (u_i \cdot W_i - W_i) \quad \text{for } u_i \in U_{\Delta \setminus \{\alpha\}}, \ W_i \in \mathcal{W}(\pi, \psi)$$

Jacquet module assoc. to maximal parabolic attached to α

When $|\beta(a)| >> 0$ for some $\beta \in \Delta$, (a) $\Rightarrow$ ✓. Can assume $|\beta(a)|$ not large.

$$(u_i \cdot W_i - W_i)(a) = (\underbrace{\psi(a u_i a^{-1})}_{=1 \text{ when } |\alpha(a)| << 1} - 1) W_i(a) = 0$$

<u>Cor</u> When π is supercuspidal, $\mathcal{W}(\pi, \psi) \subset C_c^\infty((N(F), \psi) \backslash GL_n(F))$

hence $Z(W, W', \Phi, s) \in \mathbb{C}[q^{\pm s}] \quad \Rightarrow L(\pi \times \pi', s) = 1$

<u>Rmk</u> In general, if π is only irr. generic

$r^G_{P_{\Delta \setminus \{\alpha\}}} \pi$ is of finite length $\Rightarrow$ asymptotic behavior when $|\alpha(a)| \to 0$ "only have finitely many modes"

(2) $Z(gW, gW', g\Phi, s) = |\det g|^{-s} Z(W, W', \Phi, s) \quad \Rightarrow$ LHS is a fractional ideal of $\mathbb{C}[q^{\pm s}]$

Looking for test functions W, W', Φ s.t. $Z(W, W', \Phi, s) = 1$.

$P_n =$ mirabolic subgroup $\left(\begin{array}{cc} \overbrace{*}^{n-1} & \overbrace{*}^{1} \\ \hline & 1 \end{array}\right) \subset GL_n(F)$

<u>Thm</u> (Kirillov model)

For $\pi \in \mathrm{Irr}^{\mathrm{gen}}(GL_n(F))$, then $c\text{-}\mathrm{ind}_{N_n}^{P_n} \psi \subset \mathrm{Res}_{P_n}^{GL_n(F)} \pi \subset \mathrm{Ind}_{N_n}^{P_n} \psi$ i.e. $C_c^\infty((N_n, \psi) \backslash P_n) \subset \mathcal{W}(\pi, \psi)|_{P_n} \subset C^\infty((N_n, \psi) \backslash P_n)$

If π is supercuspidal, then $c\text{-}\mathrm{ind}_{N_n}^{P_n} \psi = \mathrm{Res}_{P_n}^{GL_n(F)} \pi$ i.e. $C_c^\infty((N_n, \psi) \backslash P_n) = \mathcal{W}(\pi, \psi)|_{P_n}$

$$Z(W, W', \Phi, s) = \int_{N_n\backslash GL_n} W(g)W'(g)\Phi(v_n\cdot g)|\det g|^s\, dg$$

$G_n = P_n \cdot F^\times \cdot K$

center

$g = pak$

$$= \int_K \int_{F^\times} \int_{N_n\backslash P_n} \underbrace{W(pak)W'(pak)}_{\overset{\shortparallel}{W(pk)W'(pk)\cdot\omega_\pi(a)\omega_{\pi'}(a)}} \Phi(v_n\cdot ak)\, |\det p|^s \cdot |a|^{ns} \cdot \underbrace{|\det p|^{-1}}_{\text{modulus character of } P_n} \cdot d_r p\, d^\times a\, dk$$

For any $f, f' \in C_c^\infty((N_n,\psi)\backslash P_n)$, choose $W|_{P_n} = f$, $W'|_{P_n} = f'$

Suppose $W, W' \in W(\pi,\psi)^{K'}$, choose $\Phi = \mathbb{1}_{v_n\cdot K'}$

Then $\Phi(v_n\cdot ak) \neq 0 \Rightarrow ak \in P_n\cdot K' \Rightarrow ak \in (P_n\cap K)\cdot K'$, $a \in \mathcal{O}_K^\times$

$$= \text{const}\cdot \int_{N_n\backslash P_n} f(p)f'(p)|\det p|^{s-1}\, d_r p$$

Can choose f, f'

$$= 1$$

Consider

$$\begin{aligned} W(\pi,\psi) &\longrightarrow W(\pi^\vee, \bar\psi) \\ W &\longmapsto \widetilde W : g \longmapsto W(w_n {}^t g^{-1}) \\ C_c^\infty(F^n) &\longrightarrow C_c^\infty(F^n) \\ \Phi &\longmapsto \hat\Phi : x \longmapsto \int_{F^n} \Phi(y)\psi(x\cdot {}^t y)\, dy \end{aligned}$$

Prop/Def (ε-factors)

$\exists$ $\varepsilon(\pi\times\pi', \psi, s) \in \mathbb{C}[q^{\pm s}]^\times$ s.t.

Case $n = n'$

$$\frac{Z(\widetilde W, \widetilde W', \hat\Phi, 1-s)}{L(\pi^\vee\times\pi'^\vee, 1-s)} = (\omega_{\pi'}(-1))^n \cdot \varepsilon(\pi\times\pi', \psi, s)\cdot \frac{Z(W, W', \Phi, s)}{L(\pi\times\pi', s)}$$

$$(Z(\widetilde W, \widetilde W', \hat\Phi, 1-s) = \omega_{\pi'}(-1)^n \cdot \gamma(\pi\times\pi', \psi, s)\cdot Z(W, W', \Phi, s) \,)$$

Case $n > n'$

$$\frac{Z(\begin{pmatrix} I_{n'} & \\ & w_{n-n'}\end{pmatrix}\cdot \widetilde W, \widetilde W', n-n'-1-j, 1-s)}{L(\pi^\vee\times\pi'^\vee, 1-s)} = (\omega_{\pi'}(-1)^{n-1})\cdot \varepsilon(\pi\times\pi', \psi, s)\cdot \frac{Z(W, W', j, s)}{L(\pi\times\pi', s)}$$

$$Z(\begin{pmatrix} I_{n'} & \\ & w_{n-n'}\end{pmatrix}\cdot \widetilde W, \widetilde W', n-n'-1-j, 1-s) = \omega_{\pi'}(-1)^{n-1}\cdot\gamma(\pi\times\pi', \psi, s)\cdot Z(W, W', j, s)$$

Idea of proof ($n = n'$)

$$0 \longrightarrow c\text{-ind}_{P_n}^{GL_n}\mathbb{1} \longrightarrow C_c^\infty(F^n) \longrightarrow \mathbb{1} \longrightarrow 0$$

Except finitely many s, both sides define trilinear form $B : W(\pi,\psi)\otimes W(\pi',\bar\psi)\otimes \overset{\shortparallel}{C_c^\infty(F^n)} \longrightarrow \mathbb{C}$

Use Frobenius reciprocity

Reduce to Kirillov model.

s.t. $B(gW, gW', g\Phi) = |\det g|^{-s} B(W, W', \Phi)$

which is unique up to scalar.

Rmk For $\pi \in \mathrm{Irr}(GL_n(F))$

$$L(\pi, s) := L(\pi \times \mathbb{1}, s)$$

$$\varepsilon(\pi, \psi, s) := \varepsilon(\pi \times \mathbb{1}, \psi, s)$$

L, ε-factors for arbitrary irrep

Def/Prop

(a) (Any irr. in terms of (ess.) discrete series)

$$L(Q(\Delta_1, \cdots, \Delta_r) \times \pi', s) = \prod_{i=1}^{r} L(Q(\Delta_i) \times \pi', s)$$

$$\varepsilon(Q(\Delta_1, \cdots \Delta_r) \times \pi', \psi, s) = \prod_{i=1}^{r} \varepsilon(Q(\Delta_i) \times \pi', \psi, s)$$

(b) ((ess.) discrete series in terms of supercuspidal)

$$\Delta = [\sigma, \cdots, \sigma(r-1)], \quad \Delta' = [\sigma', \cdots, \sigma'(r'-1)], \quad r' \geq r$$

$$L(Q(\Delta) \times Q(\Delta'), s) = \prod_{i=1}^{r} L(\sigma \times \sigma', s + r + r' - 1 - i)$$

$\left(\text{think of } \; L(r-1) \otimes L(r'-1) = \bigoplus_{j=1}^{r} L(r+r'-2j) \;\right)$

$$\gamma(Q(\Delta) \times Q(\Delta'), \psi, s) = \prod_{\substack{0 \leq i \leq r-1 \\ 0 \leq j \leq r'-1}} \underbrace{\gamma(\sigma \times \sigma', \psi, s+i+j)}_{\text{"}\gamma(\sigma(i) \times \sigma'(j), \psi, s)}$$

Prop · If $\pi \in \mathrm{Irr}^{sc}(GL_n(F))$, $\pi' \in \mathrm{Irr}(GL_{n'}(F))$, $n \geq n'$

$$L(\pi \times \pi', s) = \begin{cases} \displaystyle\prod_{\substack{\chi : F^\times / \mathcal{O}_F^\times \to \mathbb{C}^\times \\ \text{s.t. } \chi \cdot (\pi')^\vee = \pi}} L(\chi, s) & \text{if } n = n' \\ 1 & n > n' \end{cases}$$

· $\pi \in \mathrm{Irr}(GL_n(F))$

$$\varepsilon(\pi, \psi, s) = \varepsilon(\pi, \psi, 0) \cdot q^{-(s f(\pi) + n \cdot n(\psi))}$$

where $n(\psi) = \min\{ n \mid \psi|_{\mathcal{O}_F \cdot \varpi^{-n}} = \mathbb{1} \}$

$= \bmod \varpi^f = \begin{pmatrix} * & * \\ & 1 \end{pmatrix}$

$$f(\pi) = \min\{ f \mid \pi^{K_1(\varpi^f)} \neq 0 \} = \text{conductor of } \pi$$

$$\varepsilon(\pi \times \pi^\vee, \psi, \tfrac{1}{2}) = \omega_\pi(-1)^{n-1}$$

§3. Uniqueness of LLC

Thm1 Suppose $\pi, \pi' \in \mathrm{Irr}^{sc}(GL_n(F))$ satisfies

$$\varepsilon(\pi\times\tau,\psi,s) = \varepsilon(\pi'\times\tau,\psi,s) \quad \text{for all } \tau\in\mathrm{Irr}^{sc}(GL_{n'}(F)),\ n'<n$$

Then $\pi \simeq \pi'$

Cor There exists atmost one collection of bijections

$$\mathrm{Irr}^{sc}(GL_n(F)) \xrightarrow[\text{rec}]{\sim} \text{Irr. n-dim'l } WD_F\text{-rep'n} \quad , n\geq 1 \qquad (*)$$

s.t. $\varepsilon(\pi\times\tau,\psi,s) = \varepsilon(\mathrm{rec}(\pi)\otimes\mathrm{rec}(\tau),\psi,s)$

Rmk One extends $(*)$ to

$$\mathrm{Irr}(GL_n(F)) \xrightarrow{\text{rec}} \text{n-dim'l } WD_F\text{- rep'n}$$

$$\underset{[\pi,\cdots,\pi(m-1)]}{\underset{\shortparallel}{Q(\Delta)}} \longmapsto \mathrm{rec}(\pi)\otimes \underset{\mathrm{Spam}(e_1,\cdots,e_m)}{\underset{\shortparallel}{\underline{sp(m)}}}$$

$Ne_i = e_{i+1}$

$w\cdot e_i = |w|^{i-1}e_i$

$$Q(\Delta_1,\cdots,\Delta_r) \longmapsto \bigoplus_{i=1}^{r} \mathrm{rec}(Q(\Delta_i))$$

Thm1 can be proved by the following thm2

Thm2 Suppose $\pi,\pi'\in\mathrm{Irr}^{gen}(GL_n(F))$ satisfies

$$\gamma(\pi\times\tau,\psi,s) = \gamma(\pi'\times\tau,\psi,s) \quad \text{for all } \tau\in\mathrm{Irr}^{gen}(GL_{n-1}(F)),$$

then $\pi\simeq\pi'$

Proof of thm1

Recall $L(\pi\times\tau,s) = 1$ for $\pi\in\mathrm{Irr}^{sc}(GL_n(F))$, $\tau\in\mathrm{Irr}(GL_{n'}(F))$, $n'<n$

$\Rightarrow \gamma(\pi\times\tau,\psi,s) = \gamma(\pi'\times\tau,\psi,s)$ for all $\tau\in\mathrm{Irr}^{sc}(GL_{n'}(F))$, $n'<n$

$\Rightarrow \gamma(\pi\times\tau,\psi,s) = \gamma(\pi'\times\tau,\psi,s)$ for all $\tau\in\mathrm{Irr}^{gen}(GL_{n-1}(F))$ $\square$

$\tau = Q(\Delta_1,\cdots,\Delta_r)$

$\Rightarrow \gamma(\pi\times\tau,\psi,s) = \prod_i \gamma(\pi\times\sigma_i,\psi,s+k_i)$, $\sigma_i\in\mathrm{Irr}^{sc}$

Proof of thm 2

Note that for $W \in \mathcal{W}(\pi,\psi)$, $W' \in \mathcal{W}(\pi',\psi)$, $V \in \mathcal{W}(\tau,\bar{\psi})$

If $Z(W,V,s) = Z(W',V,s) \Rightarrow Z(\widetilde{W},\widetilde{V},1-s) = Z(\widetilde{W}',\widetilde{V},1-s)$

$$Z(W,V,s) = \int_{N_{n-1}\backslash GL_{n-1}} W\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right)\cdot V(g)\,|\det g|^s\,dg$$

$\widetilde{W}(g) = W(w_n \cdot {}^t g^{-1})$ $\qquad$ $\widetilde{V}(g) = W(w_{n-1} {}^t g^{-1})$

Take $W|_{GL_{n-1}} = W'|_{GL_{n-1}}$ (in fact $W|_{GL_{n-1}}$ exausts $C_c^\infty((N_{n-1},\psi)\backslash GL_{n-1})$ by "Kirillov model")

Then $Z(W,V,s) = Z(W',V,s)$ $\quad$ for all $\tau \in \mathrm{Irr}^{\mathrm{gen}}(GL_{n-1}(F))$, $V \in \mathcal{W}(\tau,\bar{\psi})$

$\Rightarrow Z(\widetilde{W},\widetilde{V},1-s) = Z(\widetilde{W}',\widetilde{V},1-s)$ for all $\tau \in \mathrm{Irr}^{\mathrm{gen}}(GL_{n-1}(F))$, $V \in \mathcal{W}(\tau,\bar{\psi})$

$\Rightarrow \widetilde{W}|_{GL_{n-1}} = \widetilde{W}'|_{GL_{n-1}}$

Define $S_\psi(\pi,\pi') \subset \mathcal{W}(\pi,\psi) \oplus \mathcal{W}(\pi',\psi)$

$$S_\psi(\pi,\pi') = \{(W,W') \mid W|_{GL_{n-1}} = W'|_{GL_{n-1}}\}$$

Above argument gives

$$\begin{array}{ccc} S_\psi(\pi,\pi') & \xrightarrow{\sim} & S_{\bar{\psi}}(\pi^\vee,\pi'^\vee) \\ (W,W') & \longmapsto & (\widetilde{W},\widetilde{W}') \end{array} \qquad (*)$$

Both sides are P_n-stable $\Rightarrow$ both sides are $(P_n)^t$-stable

$\Rightarrow$ both sides are $GL_n(F)$-stable

$\Rightarrow \pi \simeq \pi'$ $\qquad \square$

§4. Bernstein center

G = any connected reductive gp / F

Block decomposition of $R(G)$

Def · A cuspidal pair of G is

(L, σ)

- L is a Levi of G
- $\sigma \in \mathrm{Irr}^{sc}(L)$

- $X_{nr}(G) := \{ |\chi|^s : G \longrightarrow \mathbb{C}^\times \mid \chi \in X^*(G) \}$ called unramified characters

  For $\pi \in R(G)$, $\{\pi \otimes \chi\}_{\chi \in X_{nr}(G)}$ are called unramified twists of π.

- (Inertial equivalence)

  $(L, \sigma) \sim (M, \tau) \Leftrightarrow (\mathrm{Ad}_g L, \mathrm{Ad}_g \sigma) = (M, \tau \cdot \chi)$ for some $g \in G$, $\chi \in X_{nr}(M)$

  $B(G) := \{\text{cuspidal pairs}\}/\sim$

- For $s \in B(G)$

  $R_s(G)$ = thick subcat of $R(G)$ gen. by factors of $\overset{\text{normalized induction}}{\downarrow}$ $\iota_P^G \sigma$ for all $(L, \sigma) \in s$, $P \supset L$ parabolic

Thm 1 (Bernstein decomposition)

$$R(G) = \prod_{s \in B(G)} R_s(G)$$ as abelian categories

In particular, $Z(R(G)) \xrightarrow{\sim} \prod_{s \in B(G)} \underbrace{Z(R_s(G))}_{\overset{\shortparallel}{\mathfrak{Z}_s}}$, this is the Bernstein center

$(Z(\mathcal{C}) = \mathrm{End}(\mathrm{id}_{\mathcal{C}}))$

Description of blocks

For $s = (L, \sigma)$, $W_L := N_G(L)/L$, $W^s = \mathrm{Stab}_{W_L}([\sigma]) = \{ w \in W_L \mid \mathrm{Ad}_w(\sigma) = \sigma \cdot \chi \}$

$\mathrm{Irr}^{[\sigma]}(L) := \{ \sigma \cdot \chi \mid \chi \in X_{nr}(L) \}$ (unramified twists of σ)

$\circlearrowleft$
W^s

Denote $L^\circ = \bigcap_{\chi \in X_{nr}(L)} \operatorname{Ker}(\chi)$

$$\mathfrak{S}(\sigma) := \{ \chi \in X_{nr}(L) \mid \sigma \cdot \chi \simeq \sigma \} \subset \{ \chi \in X_{nr}(L) \mid \chi|_{\underset{\uparrow \text{ center of } L}{Z_L}} = \mathrm{triv} \}$$

Then $\# \mathfrak{S}(\sigma) < [L : Z_L \cdot L^\circ] < \infty$

$$\Rightarrow \underset{\underset{W^s}{\circlearrowleft}}{\mathrm{Irr}^{[\sigma]}(L)} = X_{nr}(L) / \mathfrak{S}(\sigma) = \left(X^*(L) \otimes \left(\mathbb{C} / \tfrac{2\pi i}{\ln q} \mathbb{Z} \right) \right) / \mathfrak{S}(\sigma) \approx (\mathbb{C}^\times)^r \qquad r = \operatorname{rk} X^*(L)$$

<u>Thm 2</u>

$$\begin{aligned} \mathcal{Z}_s &\xrightarrow{\sim} \mathcal{O}(\mathrm{Irr}^{[\sigma]}(L))^{W^s} \quad \text{is an isomorphism} \\ \varphi &\longmapsto \Big(\sigma \cdot \chi \longmapsto \underbrace{\varphi|_{L_P^G(\sigma \cdot \chi)}}_{} \Big) \end{aligned}$$

<u>Fact</u> This is a scalar for generic χ

<u>Rmk</u> For $s = [(L, \sigma)] \in \mathcal{B}(G)$, we have $\sigma|_{L^\circ} = (\sigma^\circ)^{\oplus m}$ for $\sigma^\circ \in \mathrm{Irr}(L^\circ)$

$\rightsquigarrow$ $P_s := \iota_P^G(\text{c-ind}_{L^\circ}^L \sigma^\circ)$ is a compact projective generator of $R_s(G)$

$$\mathcal{A}_s := \operatorname{End}(P_s)$$

$$\rightsquigarrow \begin{aligned} R_s(G) &\xrightarrow{\sim} \text{Mod-}\mathcal{A}_s \\ \pi &\longmapsto \operatorname{Hom}_G(P_s, \pi) \end{aligned}$$

Then $\mathcal{A}_s$ is free of rk $m^2 \cdot (\# W^s)^2$ over $Z(\mathcal{A}_s) = \mathcal{Z}_s \approx \mathcal{O}((\mathbb{C}^\times)^{\operatorname{rk} X^*(L)})$

e.g. For $G = GL_n$, $s = [\pi] \in \mathrm{Irr}^{sc}(G)$

$$\mathcal{A}_s = \mathcal{Z}_s \approx \mathbb{C}[t, t^{-1}]$$

<u>Compatibility with K-types</u>

Say an open compact subgroup $K \subset G$ is <u>strongly decomposable</u> if

$$(\underset{\underset{\text{minimal Levi}}{\uparrow}}{K \cap L_0}) \cdot \prod_{\alpha \in \Phi} (\underset{\underset{\text{root wrt } A}{\nwarrow}}{K \cap U_\alpha}) \twoheadrightarrow K$$

e.g. Iwahori ✓, hyperspecial ×

These K form a neighborhood basis of $1 \in G$

$\mathcal{H} = (C_c^\infty(G), *)$. $e_K = \mathbb{1}_K \in \mathcal{H}$, $\mathcal{H}_K = e_K \mathcal{H} e_K$

$R_K(G) := \{ V \in \operatorname{Rep}(G) \mid \mathcal{H} \cdot V^K = V \} \subset R(G)$ full subcat

Thm 3 Assume K strongly decomposable, then

$\exists$ finite subset $S(K) \subset B(G)$ s.t.

$$\mathcal{H}_K\text{-mod} \simeq R_K(G) \simeq \prod_{s \in S(K)} R_s(G)$$

Cor · $Z(\mathcal{H}_K) \simeq \prod_{s \in S(K)} \mathfrak{Z}_s$

· $\varprojlim_K Z(\mathcal{H}_K) \simeq Z(R(G)) \simeq \prod_{s \in B(G)} \mathfrak{Z}_s$