0.1 Local Langlands for $\mathrm{GL}(n)$

References are [B-K93, Hen93, Kud94, Rog00, H-T01, Wed08, Sch13].

Notation (0.1.0.1).

- Let $p \in \texttt{Prime}$.
- Let $\ell \in \texttt{Prime} \setminus \{p\}$ and an isomorphism $\iota_\ell : \overline{\mathbb{Q}_\ell} \cong \mathbb{C}$.
- Let $(K, \mathcal{O}_K, \varpi_K, \kappa) \in p\text{-}\texttt{NField}$.
- Fix a square root $|-|_K^{1/2} : K^\times \to \overline{\mathbb{Q}_\ell}^\times$ s.t. $\iota \circ |-|_K^{1/2}$ is valued in $\mathbb{R}_{>0}$.
- Let $\mathbf{1} \neq \psi \in K^\vee$.
- Let $(K_r, \mathcal{O}_r, \varpi_K, \kappa_r)/K$ be an unramified extension of degree r.

⌟

1 Local Langlands for p-adic $\mathrm{GL}(n)$

Remark (0.1.1.1). The local Langlands correspondence for essentially-L^2 and generalized Speh representations of $\mathrm{GL}(n;K)$ are realized in the cohomology of the moduli space of 1-dimensional p-divisible groups of height n.

To construct Galois representations, Harris and Taylor consider the cohomology of the Shimura varieties associated to unitary groups of signature $(1, n-1)$ and split at p, uses it to construct ℓ-adic Galois representations associated to certain cuspidal automorphic representations of $\mathrm{GL}(n)$, and proves a local-global compatibility result at places of bad reduction. This is achieved by counting points on Igusa varieties.

This method and its generalizations allow one to compute the trace of arbitrary Hecke correspondences at p on a Shimura variety. But Scholze restrict attention to Hecke operators at p coming from the maximal compact subgroup and reduce all counting problems to counting problems for the maximal compact level structure, for which one can just appeal to the classical work of [**?**]. ⌟

Thm. (0.1.1.2) [Test Function Characterization of Local Langlands, Scholze[Sch13]].

(a) For each $n \in \mathbb{Z}_+$, there is a unique map

$$\sigma_n : \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K)) \to \mathfrak{wd}^n_{\psi\text{-ss}}(W_K)$$

s.t. for any $\tau \in W_K$ and any "cut-off" function $h \in C_c^\infty(\mathrm{GL}(n;K))$,

$$\mathrm{tr}(f_{\tau,h}|\pi) = \mathrm{tr}(\tau|\sigma_n(\pi))\,\mathrm{tr}(h|\pi)(0.1.2.4),$$

Write $\mathrm{rec}'(\pi) = \sigma_n(\pi)(\frac{1-n}{2})$.

(b) If $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$ is a constituent of $\pi_1 \times \ldots \times \pi_r$, then $\mathrm{rec}'(\pi) = \mathrm{rec}'(\pi_1) \oplus \ldots \oplus \mathrm{rec}'(\pi_r)$.

(c) rec' induces a bijection between $\mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n;K))$ and $\mathrm{Irr}^n_{\varphi\text{-ss}}(W_K)$.

(d) rec' is compatible with twists, central characters, duals, and L- and ϵ-factors of pairs, hence $\mathrm{rec}' = \mathrm{rec}$ as in**??**.

⌟

Proof: (a) and (b) follow from(0.1.1.3) and(0.1.1.5).

For (c), use computation of I_K-invariant nearby cycles for simple Shimura varieties. This computation leads to a direct proof of the bijective correspondence for supercuspidal representations, without using the numerical local Langlands correspondence.

Finally, for the proof of (d): By Brauer induction and linearity, it suffices to assume that π is induced from characters. It suffices to show that: For any $\pi_1 \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n_1;K)), \pi_2 \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n_2;K))$, there exists $F \in \texttt{NField}$ with $w \in \Sigma_F^{\mathrm{fin}}$ s.t. $K \cong F_w$, and two potentially Abelian $\pi_i \in \mathrm{Irr}^{\mathrm{auto}}(\mathrm{GL}(n_i)/F), \pi_2 \in \mathrm{Irr}^{\mathrm{auto}}(\mathrm{GL}(n_2)/F)$ s.t. $(\Pi_i)_w$ is an unramified twist of π_i. Cf. proof of[H-T01]VII.2.10.

Then the compatibility follows from Henniart's method of twisting with highly ramified characters, cf. Corollary 2.4 of [**?**]. □

Prop. (0.1.1.3) [Dévissage for Constructing Galois Representations]. For $n \in \mathbb{Z}_+$, suppose (a) and (b) of(0.1.1.2) hold for all $n' < n$ and the following hold:

1. If $\pi = \pi_1 \times \ldots \times \pi_r \in \mathrm{Rep}^{\mathrm{adm}}(\mathrm{GL}(n;K))$ where $\pi_i \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n_i;K))$, then
$$\mathrm{tr}(f_{\tau,h}|\pi) = \mathrm{tr}\left(\tau\Big|\bigoplus_{1\le i\le r}\sigma(\pi_i)(\frac{n-n_i}{2})\right)\mathrm{tr}(h|\pi).$$
2. For $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$ that is either essentially square-integrable or a generalized Speh representation, then there exists a virtual finite dimensional representation $\sigma(\pi)$ of W_K with $\mathbb{Q}_+$ coefficients of dimension n s.t.
$$\mathrm{tr}(f_{\tau,h}|\pi) = \mathrm{tr}(\tau|\sigma(\pi))\,\mathrm{tr}(h|\pi).$$
3. If $\pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n;K))$, then $\sigma(\pi)$ is a genuine representation of W_K.

Then (a) and (b) of(0.1.1.2) hold true for n, by defining $\sigma(\pi)$ as follows: If π has supercuspidal support $\{\pi_1,\ldots,\pi_r\}$(with multiplicity), then we define
$$\sigma_n(\pi) = \bigoplus \sigma(\pi_i)(\frac{1-n_i}{2}).$$
⌟

Proof: Firstly, the uniqueness of σ_n follows from the fact the elements mapping to a positive power of σ_K determines a Weil-Deligne representation up to semisimplification.

What is Bernstein center Let f_τ be the function on the Bernstein center that acts through the scalar $\mathrm{tr}(\tau|\sigma(\pi)(\frac{n-1}{2}))$ on any $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\pi)$.

Then to prove (a) and (b) of(0.1.1.2), it suffices to show that
$$\mathrm{tr}(f_{\tau,h}|\pi) = \mathrm{tr}(f_\tau * h|\pi)$$
for any $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$. If π is induced or supercuspidal, then this is true by the hypothesis listed. By(0.1.1.4), it suffices to check for any generalized Speh representation $\pi = \mathrm{St}(\pi_0;t)$. Choose $h_0 \in C_c^\infty(\mathrm{GL}(n;K))$ as in item1 of(0.1.1.4), then
$$\mathrm{tr}(f_{\tau,h_0}|\pi') = \mathrm{tr}(f_\tau * h_0|\pi')$$
holds for any tempered $\pi' \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$: This is true □

Lemma (0.1.1.4) [Testing on Tempered Representations and Generalized Speh Representations].

- Let $d \in \mathbb{Z}_+, t \in \mathbb{Z}_{\geq 2}, n = dt, \pi_0 \in \mathrm{Irr}^{\mathrm{cusp,uni}}(\mathrm{GL}(d;K))$ and $\pi = \mathrm{St}(\pi_0, t)$. Then there exists $h \in C_c^\infty(\mathrm{GL}(n;\mathcal{O}_K))$ s.t. $\mathrm{tr}(h|\pi) = 0$ for any $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;F))$ that is tempered but not of the form $\pi = \pi_0(\mathrm{i}y_1) \times \ldots \times \pi_0(\mathrm{i}y_t)$ where $y_i \in \mathbb{R}$. And for these π, $\mathrm{tr}(h|\pi) \neq 0$.
- If $h \in C_c^\infty(\mathrm{GL}(n;K))$ s.t. for all $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(d;K))$ that is tempered but non-square-integrable or $\pi = \mathrm{St}(\pi_0, t)$ for some $d \in \mathbb{Z}_+, t \in \mathbb{Z}_{\geq 2}, n = dt, \pi_0 \in \mathrm{Irr}^{\mathrm{cusp,uni}}(\mathrm{GL}(d;K))$; then $\mathrm{tr}(h|\pi) = 0$ for any $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$.

⌟

Proof: Kazhdan's density theorem and □

Prop. (0.1.1.5) [Constructing Galois Representations]. The hypotheses of(0.1.1.3) are true for any $n \in \mathbb{Z}_+$. In particular, (a) and (b) of(0.1.1.2) hold true for any $n \in \mathbb{Z}_+$. ⌟

Proof: 1 is proved in Theorem 6.4. of [Sch13], by relating the deformation spaces of one-dimensional p-divisible groups to the deformation spaces of their infinitesimal parts.

2. Firstly by(0.1.3.3), K can be realized as $K = F_w$ with notation as in(0.1.3.1) and(0.1.3.1). Then we can apply(0.1.5.2) to $\pi^\vee$ to get $\pi_f \in \mathrm{Irr}^{\mathrm{adm}}(\mathbf{G}(\mathbf{A}_f))$ s.t.

- $H_\xi^*(\pi_f) \neq 0$.
- $\pi_{p,0}$ is unramified.
- $\pi_w = \pi \otimes \chi$ for some unramified quasi-character χ of $K^\times$.

Then we apply(0.1.5.1) to get

$$\mathrm{tr}(f_{\tau,h}^\vee|\pi^\vee \otimes \chi) = \frac{1}{a(\pi_f)} \mathrm{tr}\left(\tau|[H_\xi[\pi_f]] \otimes \chi_{\pi_{p,0}}\right) \mathrm{tr}\left(h^\vee|\pi^\vee \otimes \chi\right).$$

Then we can take $\sigma(\pi) = \frac{1}{a(\pi_f)}[H_\xi[\pi_f]]^\vee \otimes \chi_{\pi_{p,0}}^{-1} \otimes \chi^{-1}$, which has dimension n by(0.1.3.7).

3: It suffices to show that $\sigma(\pi) \in K_0(\mathrm{Rep}^{\mathrm{fd}}(W_K))$. For this, Cf.[Sch13]P705. □

<u>**Vanishing Cycles**</u>

Thm. (0.1.1.6) [Formal Vanishing Cycles on the Deformation Tower]. Consider the formal vanishing cycles on the tower $\{\mathrm{Spf}\, R_{\beta,m}\}_{m\in\mathbb{Z}_+}$:

$$\Psi_\beta^i = \varinjlim_{m\in\mathbb{Z}_+} H^0(\overline{\mathrm{Spf}\, R_{\beta,m}}, \Psi^i_{\mathrm{Spf}\, R_{\beta,m}} \overline{\mathbb{Q}_\ell}).$$

Then

$$H^0(\overline{\mathrm{Spf}\, R_{\beta,m}}, R^i\psi_{\mathrm{Spf}\, R_{\beta,m}} \overline{\mathbb{Q}_\ell}) \in \mathrm{Rep}^{\mathrm{fd}}\left(W_{K_r} \times \mathrm{GL}(n;\mathcal{O}_K/(\varpi^m))\right),$$

and it vanishes unless $i \in [0, n-1]$.

In particular $\Psi_\beta^i, [\Psi_\beta] \in \mathrm{Rep}^{(\mathrm{cont,adm})}(W_{K_r} \times \mathrm{GL}(n;\mathcal{O}_K))$. ⌟

Proof: Cf.[Sch13]P673. □

Def. (0.1.1.7) [Vanishing Cycles on the Lubin-Tate Tower]. Define the vanishing cycles as

$$\Psi_n = \varinjlim_{m\in\mathbb{Z}_+} H^0(\overline{\mathrm{Spf}\, R_{n,m}}, \Psi^i_{\mathrm{Spf}\, R_m} \overline{\mathbb{Q}_\ell}),$$

with an action of $A_{K,n}$ induced by**??**. Then $\Psi_n \in \mathrm{Rep}^{(\mathrm{adm,alg,cont})}(\mathrm{GL}(n;\mathcal{O}_K) \times \mathcal{O}^*_{D_{K,1/n}} \times I_K)$, and it vanishes unless $i \in [0, n-1]$. ⌟

Proof: The smoothness of $\mathcal{O}^*_{D_{K,1/n}}$-action follows from comparison theorem Corollary 4.5 of [Vanishing cycles for formal schemes. 2, Berkovich], Cf. proof of [H-T01], Lemma II.2.8. The rest follows from(0.1.1.6). □

2 Cyclic Base Change

Prop. (0.1.2.1). K_r/K be the unramified extension of degree r, then for any $x \in \mathrm{GL}(n; K_r)$, we can define $\mathrm{Nm}(x) = x.x^{\sigma}.\dots.x^{\sigma^{l-1}}$.

Then N defines an injection from the σ-conjugacy classes of $\mathrm{GL}(n; K_r)$ to the conjugacy classes of $\mathrm{GL}(n; K)$. In fact, if $\gamma = \mathrm{Nm}\, x$, then $G_{x,\delta}$ is an inner form of G_γ. ⌟

Proof: Cf.[Arthur-Clozel]P3. □

Prop. (0.1.2.2) [Cyclic Base Change]. If $\gamma = \mathrm{Nm}(x)$, we can define orbital integrals and twisted orbital integrals:

$$\mathrm{TOrb}_\delta(\varphi) = \int_{G_{\delta,\sigma}\backslash\,\mathrm{GL}(n;K_r)} \varphi(g^{-1}\delta g^{\sigma})\frac{\mathrm{d}g_r}{\mathrm{d}t},$$

$$\mathrm{Orb}_\gamma(f) = \int_{G_\gamma\backslash\,\mathrm{GL}(n;K)} \varphi(g^{-1}\delta g)\frac{\mathrm{d}g}{\mathrm{d}t}.$$

Then for any $\varphi \in C_c^\infty(\mathrm{GL}(n; K_r))$, there exists $f \in C_c^\infty(\mathrm{GL}(n; K))$ s.t. for any regular $\gamma \in \mathrm{GL}(n; K)$,

$$\mathrm{Orb}_\gamma(f) = \begin{cases} \mathrm{TOrb}_{\gamma,\sigma}(\varphi) & ,\gamma = \mathrm{Nm}\,\delta, \delta \in \mathrm{GL}(n; K_r) \\ 0 & ,\text{otherwise} \end{cases}.$$

⌟

Proof: Cf.[Arthur-Clozel]P20. □

Test Functions

Def. (0.1.2.3) [Test Functions]. For any $\tau \in \sigma_k^r I_K$ and $h \in C_c^\infty(\mathrm{GL}(n; \mathcal{O}_K); \mathbb{Q})$, define

$$\psi_{\tau,h}(\beta) = \begin{cases} \mathrm{tr}(\tau \times h^\vee | [\Psi_\beta]) & ,\beta \in \mathrm{GL}(n; \mathcal{O}_{K_r})\,\mathrm{diag}(\varpi_K, 1, \dots, 1)\,\mathrm{GL}(n; \mathcal{O}_{K_r}) \\ 0 & ,\text{otherwise} \end{cases}.$$

Then $\varphi_{\tau,h} \in C_c^\infty(\mathrm{GL}(n; K_r); \mathbb{Q})$, and is independent of ℓ. ⌟

Proof: Cf.[Sch13]P674. □

Def. (0.1.2.4) [Base Change Test Function, $f_{\tau,h}$]. Define $f_{\tau,h} \in C_c^\infty(\mathrm{GL}(n; K); \mathbb{Q})$ s.t. it has matching twisted orbital integral with $\varphi_{\tau,h} \in C_c^\infty(\mathrm{GL}(n; K_r); \mathbb{Q})$(0.1.2.2) w.r.t. the Haar measures that give the hyperspecial subgroups volume 1. ⌟

Def. (0.1.2.5). For any $\tau \in W_{K_r}$ and $h \in C_c^\infty(\mathrm{GL}(n; \mathcal{O}_K); \mathbb{Q}), h' \in C_c^\infty(\mathcal{O}^*_{D'}; \mathbb{Q})$, define the function

$$\varphi_{\tau,h,h'}(\delta_0) = \mathrm{tr}\left(\tau \times h^\vee \times h' \mid [\Psi_{\underline{\overline{H}}}]\right) \mathbf{??}.$$

if $\underline{\overline{H}}$ has Dieudonné paramter δ_0**??**, and 0 if there is not such $\underline{\overline{H}}$. Then by(0.1.2.3), $\varphi_{\tau,h,h'} \in C_c^\infty(\mathbf{G}_D(W(\kappa_r)[\frac{1}{p}]))$ and is independent of ℓ. ⌟

Prop. (0.1.2.6)[Base Change Test Functions]. $\varphi_{\tau,h,h'} \in C_c^\infty(\mathbf{G}_D(W(\kappa_r)[\frac{1}{p}]); \mathbb{Q})$ corresponds to $f^\vee_{\tau,h} \times h' \in C_c^\infty(\mathbf{G}_D(\mathbb{Q}_p); \mathbb{Q})$. ⌟

Proof: Cf.[Sch13]P688. □

3 Simple Shimura Varieties

Notation (0.1.3.1).

- Let $F_0 \in \mathtt{NField}$ be totally real with $2|[F_0 : \mathbb{Q}]$.
- Let $\tau \in \Sigma_{F_0}^{\infty}$ and $x_0 \in \Sigma_{F_0}^{\mathrm{fin}}$.
- Let $\mathcal{K} \subset \mathbb{C}$ be an imaginary quadratic field s.t. the rational prime below x_0 splits in $\mathcal{K}$.
- Let $F = F_0\mathcal{K}$ and $S_F(x_0) = \{x, x^{\mathsf{c}}\}$.
- Let $w \in \Sigma_F^{\mathrm{fin}}$ with $u = w \cap \mathcal{K}$ and $p = w \cap \mathbb{Q}$ s.t. p is split in $\mathcal{K}$ and $w \notin \{x, x^{\mathsf{c}}\}$(in particular, $\mathbb{D}$ is split at w).
- Let $K = F_w$ with residue field κ. Let $n \in \mathbb{Z}_+$.

⌟

Prop. (0.1.3.2) [Division Rings and Simple Unitary Groups]. There exists $\mathbb{D} \in \mathrm{Az}_F$ of dimension n^2 with an involution $*$ of second kind, and a homomorphism $h_0 : \mathbb{C} \to \mathbb{D}_\tau$ s.t.

- $x \to h_0(\mathrm{i})^{-1}x^* h_0(\mathrm{i})$ is a positive involution on $\mathbb{D}_{\mathbb{R}}$.
- $\mathbb{D}$ splits at all places $v \in \Sigma_F \setminus \{x, x^{\mathsf{c}}\}$.
- Define $\mathbf{G}_0 \subset \mathrm{Res}_{F/F_0}(\mathbb{D}^*) \in \mathcal{A}\mathrm{lg}\mathcal{G}\mathrm{rp}\,/F_0$ representing the functor

$$\mathbf{G}_0 : R \mapsto \{g \in (\mathbb{D} \otimes_{F_0} R)^* | gg^* = 1\},$$

  then $\mathbf{G}_0$ is quasi-split at all non-split places of F_0, unitary of signature $(1, n-1)$ at τ and unitary of signature $(0, n)$ at all other infinite places of F_0.

Define

$$\mathbf{G} \in \mathcal{R}\mathrm{ed}\mathcal{G}\mathrm{rp}\,/\mathbb{Q} : R \mapsto \{\, g \in (\mathbb{D} \otimes R)^* \mid gg^* \in R^* \,\},$$

together with the amplitude map $\nu : \mathbf{G} \to \mathbb{G}_m$ with kernel $\mathbf{G}_1$.

⌟

Proof: Take $\mathbb{D} = B^{\mathrm{op}}$ as in [H-T01]Lemma 1.7.1. □

Prop. (0.1.3.3) [Globalization]. Any $K \in p\text{-}\mathtt{NField}$ can be realized as $K = F_w$ for some F and w with the setting as in(0.1.3.1) and(0.1.3.1). ⌟

Proof:

□

Prop. (0.1.3.4) [Shimura Varieties]. Situation as in(0.1.3.2), we can regard h_0 as a map $\mathbb{S} \to \mathbf{G}_{\mathbb{R}}$. Then the datum $(\mathbf{G}, h^{-1})$ defines a tower of Shimura varieties

$$\varprojlim_{\mathcal{K} \subset \mathbf{G}(\mathbf{A}_f)} \mathcal{S}\mathrm{h}_{\mathcal{K}}(\mathbf{G}, h^{-1})$$

with reflex field $E = \tau(F)$. ⌟

Proof: By [**?**] and[Sch13]P691. □

Automorphic Vector Bundles

Thm. (0.1.3.5) [Matsushima's formula].

$$\operatorname{inj}\lim_{U} H^i(X_{U,\tau}(\mathbb{C}), \mathcal{L}^{top}_{\iota_\ell(\xi)}) \cong \bigoplus_{\pi\in \mathrm{Irr}^{\mathrm{auto}}(\mathbf{G}/\mathbb{Q})} \pi_f \otimes H^i(\mathrm{Lie}\,\mathbf{G}(\mathbb{R}), U_\tau, \pi_\infty \otimes \iota_\ell(\xi))^{\#\ker^1(\mathbb{Q},\mathbf{G})}.$$

⌟

Proof:

□

Def. (0.1.3.6) [Cohomology of Automorphic Vector Bundles]. For any $\xi \in \mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(\mathbf{G})$, we can get a lisse sheaf $\mathcal{L}_{\xi,\mathcal{K}} \subset \mathrm{Loc}^{\text{ét}}_{\overline{\mathbb{Q}_\ell}}(\mathcal{S}\mathrm{h}_{\mathcal{K}}(\mathbf{G}))$ for any open compact $\mathcal{K} \subset \mathbf{G}(\mathbf{A}_f)$ that is small enough. The action of $\mathbf{G}(\mathbf{A}_f)$ on $\mathcal{S}\mathrm{h}_{\mathcal{K}}(\mathbf{G})$ extends to $\mathcal{L}_{\xi,\mathcal{K}}$, and we can consider the cohomologies

$$H^*_\xi = \varinjlim_{\mathcal{K}} \mathrm{H}^*_{\text{ét}}(\mathcal{S}\mathrm{h}_{\mathcal{K}}(\mathbf{G})_{\overline{F}}, \mathcal{L}_{\xi,\mathcal{K}}) \in \mathrm{Rep}^{(\mathrm{cont,adm})}(\mathrm{Gal}_F \times \mathbf{G}(\mathbf{A}_f)).$$

Then:

- There is a decomposition $H^i_\xi = \bigoplus_{\pi_f} \pi_f \otimes H^i_\xi[\pi_f]$, where π_f runs over $\mathrm{Irr}^{\mathrm{adm}}_{\overline{\mathbb{Q}_\ell}}(\mathrm{G}(\mathbf{A}_f))$ and $H^i_\xi[\pi_f] \in \mathrm{Rep}^{\mathrm{fd}}_{\overline{\mathbb{Q}_\ell}}(\mathrm{Gal}_F)$.
- $$\dim H^i_\xi[\pi_f] = \#\ker^1(\mathbb{Q},\mathbf{G}) \sum_{\pi_\infty} m_\tau(\iota_\ell(\pi)\otimes\pi_\infty) \dim H^i(\mathrm{Lie}\,\mathbf{G}(\mathbb{R}), U_\tau, \pi_\infty \otimes \iota_\ell(\xi)),$$
  where π_∞ runs through $\mathrm{Irr}(\mathbf{G}(\mathbb{R}))$ and $m(\pi)$ is the multiplicity of π in $\mathcal{A}(\mathbf{G}/\mathbb{Q})$.
- $H^i_\xi[\pi_f] \in \mathrm{Rep}^{\mathrm{fd}}_{\overline{\mathbb{Q}_\ell}}(\mathrm{Gal}_F)$ is pure of weight $i + w(\xi)$ and deRham.
-

⌟

Proof: Cf.[H-T01]P104. □

Prop. (0.1.3.7) [Kottwitz]. For any $\pi_f \in \mathrm{Irr}^{\mathrm{adm}}(\mathbf{G}(\mathbf{A}_f))$ and $\xi \in \mathrm{Irr}^{\mathrm{adm}}_{\overline{\mathbb{Q}_\ell}}(\mathbf{G})$, consider

$$H^*_\xi[\pi_f] \in K_0(\mathrm{Rep}(\mathrm{Gal}_F))(0.1.3.6).$$

Then either π_f only appears in odd dimensions or only appears in even dimensions. Thus $\pm H^*_\xi[\pi_f]$ is an genuine representation.

And there exists $a(\pi_f) \in \mathbb{N}$ s.t. $\dim\left(\pm H^*_\xi[\pi_f]\right) = a(\pi_f)n$ for any ξ. ⌟

Proof: Cf.[**?**]. □

4 Langlands-Kottwitz Method

Def. (0.1.4.1). Define

$$f = h^\vee \times h' \times \mathbf{1}_{\mathbb{Z}_p^*} \times f^p \in C_c^\infty(\mathbf{G}(\mathbf{A}_f))$$

where

$$h \in C_c^\infty(\mathrm{GL}(n;\mathcal{O}_K);\mathbb{Q}), \quad h' \in C_c^\infty(\mathcal{O}_{D'}^*;\mathbb{Q}), \quad f^p = \frac{1}{\mathrm{Vol}(K^p)} \mathbf{1}_{K^p g^p K^p}$$

where $g^p \in \mathbf{G}(\mathbf{A}_f^p)$ and $K^p \subset \mathbf{G}(\mathbf{A}_f^p)$ is a sufficiently small compact open subgroup.

Fix $m \in \mathbb{Z}_+$ s.t. $h^\vee \times h' \times \mathbf{1}_{\mathbb{Z}_p^*}$ is bi-$\mathcal{K}_p^m$-invariant. ⌟

For $\tau \in W_K^+$ with $\deg(\tau) = r$, we want to evaluate $\mathrm{tr}(\tau \times f|[H_\xi])$ via Lefschetz trace formula.

Cor. (0.1.4.2) [Test Function Trace Formula].

$$n\,\mathrm{tr}(\tau \times h^\vee \times h' \times \mathbf{1}_{\mathbb{Z}_p^*} \times f^p|[H_\xi]) = \mathrm{tr}(\mathbf{1} \times f^\vee_{\tau,h} \times h' \times \mathbf{1}_{p^{-r'}\mathbb{Z}_p^*} \times f^p|[H_\xi]).$$

where $r' = r.[\kappa : \mathbb{F}_p]$. ⌟

Proof: Cf.[Sch13]P698. □

5 Global methods

Prop. (0.1.5.1). Notation as in Simple Shimura Varieties, assume that $\pi_f \in \mathrm{Irr}^{\mathrm{adm}}(\mathbf{G}(\mathbf{A}_f))$ s.t. $H^*_\xi(\pi_f) \neq 0$(0.1.3.6), and in the decomposition $\pi_p = \pi_w \otimes \pi_p^w \otimes \pi_{p,0}$ corresponding to

$$\mathbf{G}(\mathbb{Q}_p) = \mathrm{GL}(n; F_w) \times (D')^\times \times \mathbb{Q}_p^\times\textbf{??},$$

assume $\pi_{p,0}$ is unramified and let $\chi_{\pi_{p,0}} = \left(\pi_{p,0} \circ \mathrm{Art}^{-1}_{\mathbb{Q}_p}\right)|_{W_K}$ be a quasi-character of W_K. Then for any $\tau \in W_K^+$ and $h \in C_c^\infty(\mathrm{GL}(n; \mathcal{O}_K))$, we have

$$\mathrm{tr}(f^\vee_{\tau,h}|\pi_w) = \frac{1}{a(\pi_f)}\,\mathrm{tr}\left(\tau|\,[H_\xi[\pi_f]] \otimes \chi_{\pi_{p,0}}\right)\mathrm{tr}(h^\vee|\pi_w).$$

⌟

Proof: Take $h' \in C_c^\infty(\mathcal{O}^*_{D'}; \mathbb{Q})$ s.t. $\mathrm{tr}(h'|\pi_p^w) = 1$, and take $m \in \mathbb{Z}_+$ s.t. both $h^\vee \times h' \times \mathbf{1}_{\mathbb{Z}_p^*}$ and $f^\vee_{\tau,h} \times h' \times \mathbf{1}^*_{p^{-r'}\mathbb{Z}_p}$ are both bi-$\mathcal{K}_p^m$-invariant**??**, and $\pi_p^{\mathcal{K}_p^m} \neq 0$. And we take $\mathcal{K}^p \subset \mathbf{G}(\mathbf{A}_f^p)$ s.t. $(\pi_f^p)^{\mathcal{K}^p} \neq 0$, and let $\mathcal{K} = \mathcal{K}^p\mathcal{K}_p^m$.

Because $\dim H^*(\mathcal{S}\mathrm{h}_{\mathcal{K}}, \mathcal{L}_{\xi,\mathcal{K}}) < \infty$, there are only f.m. $\pi_f \in \mathrm{Irr}^{\mathrm{adm}}(\mathbf{G}(\mathbf{A}_f))$ that is invariant under $\mathcal{K}$ and $H^*_\xi[\pi_f] \neq 0$. So we can choose $f^p \in C_c^\infty(\mathbf{G}(\mathbf{A}_f^p))$ bi-invariant under $\mathcal{K}^p$ s.t. $\mathrm{tr}(f^p|\pi_f^p) = 1$, and satisfying the following: whenever $\pi'_f \in \mathrm{Irr}^{\mathrm{adm}}(\mathbf{G}(\mathbf{A}_f))$ is $\mathcal{K}$-spherical and $\mathrm{tr}(f^p|\pi'^p_f) \neq 0$, we have $\pi'^p_f \cong \pi^p_f$, thus also $\pi_f \cong \pi'_f$ by (0.1.5.6).

Applying(0.1.4.2), we get

$$n\,\mathrm{tr}(\tau|[H^*_\xi[\pi_f]])\,\mathrm{tr}(h^\vee|\pi_w) = na(\pi_f)\,\mathrm{tr}(f^\vee_{\tau,h}|\pi_w)\pi_{p,0}(p^{-r'}),$$

and the assertion follows. □

Thm. (0.1.5.2) [Globalization, Harris-Taylor]. If $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n)/K)$ that is either essentially square-integrable or generalized Speh, then there exists $\pi_f \in \mathrm{Irr}^{\mathrm{adm}}(\mathbf{G}(\mathbf{A}_f))$ and $\xi \in \mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(\mathbf{G})$ s.t. (notation as in(0.1.5.1)):

- $H^*_\xi(\pi_f) \neq 0$(0.1.3.6).
- $\pi_{p,0}$ is unramified.
- π_w is an unramified twist of π.

⌟

Proof: Cf.[H-T01]6.2.5 and 6.2.11. □

Cor. (0.1.5.3) [A variant Globalization]. If $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n)/K)$ is essentially square-integrable, then there exists $\Pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F)$ s.t.

- $\Pi^\vee \cong \Pi^{\mathsf{c}}$.
- Π is regular C-algebraic(0.1.6.8).
- Π_x is supercuspidal.
- Π_w is an unramified twist of π.

⌟

Proof: Cf.[H-T01]Cor. 6.2.6. □

Thm. (0.1.5.4) [Clozel's Base Change]. Suppose $\pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathbf{G}/\mathbb{Q})$ is cohomological for $\xi' \in \mathrm{Irr}_{\mathbb{C}}(\mathbf{G})$, then there is a unique $\mathrm{BC}(\pi) = (\psi, \Pi) \in \mathrm{Irr}^{\mathrm{auto}}(\mathrm{Res}_{\mathcal{K}/\mathbb{Q}} \mathbb{G}_m \times \mathrm{Res}_{F/\mathbb{Q}} D^\times)$ s.t.

- $\psi = \psi_\pi|^{\mathsf{c}}_{\mathbf{I}_{\mathcal{K}}}$.
- If $p \in$ `Prime` is split in $\mathcal{K}$, then $\mathrm{BC}(\pi)_p = \mathrm{BC}(\pi_p)$.
- For a.e. $p \in$ `Prime` that is inert in $\mathcal{K}$, we have $\mathrm{BC}(\pi_p) = \mathrm{BC}(\pi)_p$.
- Π is cohomological for $\xi'_{\mathcal{K}}$.
- $\psi|^{\mathsf{c}}_{\mathcal{K}^\times_\infty} = \xi'|^{-1}_{\mathcal{K}^\times_\infty}$.
- $\psi_\Pi|_{\mathbf{I}_E} = \psi^{\mathsf{c}}/\psi$.

⌟

Proof: Cf.[H-T01]6.2.1. □

Cor. (0.1.5.5). It follows from strong multiplicity one?? and the theorem that: if $\pi, \pi' \in \mathrm{Irr}^{\mathrm{auto}}(G/\mathbb{Q})$ are cohomological for $\xi' \in \mathrm{Irr}_{\mathbb{C}}(\mathbf{G})$ and $\pi_p \cong \pi'_p$ for a.e. $p \in$ `Prime`, then $\pi_p \cong \pi'_p$ for any $p \in$ `Prime` that is split in $\mathcal{K}$. ⌟

Cor. (0.1.5.6) [Harris-Taylor]. If $\pi, \pi' \in \mathrm{Irr}^{\mathrm{adm}}(\mathbf{G}(\mathbf{A}_f))$ and $\xi \in \mathrm{Irr}_{\overline{\mathbb{Q}_\ell}}(\mathbf{G})$ s.t.

- $\pi^p \cong (\pi')^p$.
- $[H^*_\xi[\pi_f]] \neq 0$.
- $[H^*_\xi[\pi'_f]] \neq 0$.

Then $\pi \cong \pi'$. ⌟

Proof: It follows from(0.1.3.6) that both π, π' are automorphic and cohomological for $\iota_\ell(\xi)$. Then the assertion follows from(0.1.5.5), noticing that p is split in $\mathcal{K}$(0.1.3.1). □

Thm. (0.1.5.7) [Clozel's Descent]. Suppose $\Pi \in \mathrm{Irr}^{\mathrm{auto}}(D^\times/F)$ and ψ is a Hecke character of $\mathcal{K}$ s.t.

- $\Pi \cong \Pi^*$.
- $\psi_\Pi|_{\mathbf{I}_{\mathcal{K}}} = \psi^{\mathsf{c}}/\psi$.
- Π is cohomological for $\xi'_{\mathcal{K}}$.
- $\xi'|^{-1}_{\mathcal{K}^\times_\infty} = \psi|^{\mathsf{c}}_{\mathcal{K}^\times_\infty}$.

Then there is a $\pi \in \mathrm{Irr}^{\mathrm{auto}}(\mathbf{G}/\mathbb{Q})$ s.t.

- $\mathrm{BC}(\pi) = (\psi, \Pi)$.
- π is cohomological for ξ'.
- $\dim H_{\iota_\ell^{-1}(\xi')}[\iota_\ell^{-1}\pi_f] \neq 0$.

⌟

Proof: Cf.[H-T01]6.2.9. □

Prop. (0.1.5.8) [Descending from $\mathbb{D}^\times$ to $\mathbf{G}$]. Let $\Pi \in \mathrm{Irr}^{\mathrm{auto}}(\mathbb{D}^\times)$ s.t. $\Pi \cong \Pi^*$ and Π is regular C-algebraic, then there exists a Hecke character ψ of $\mathcal{K}$ and $\xi' \in \mathrm{Irr}(\mathbf{G}/\mathbb{C})$ s.t.

- $\psi_\Pi|_{\mathbf{I}_\mathcal{K}} = \psi^{\mathsf{c}}/\psi$.
- Π is cohomological for $\xi'_\mathcal{K}$.
- $\xi'|^{-1}_{\mathcal{K}^\times_\infty} = \psi|^{\mathsf{c}}_{\mathcal{K}^\times_\infty}$.
- $\psi(\mathcal{O}^\times_{\mathcal{K},x_0}) = 1$.

⌟

Proof: Cf.[H-T01]Lemma6.2.10. □

Cor. (0.1.5.9) [Jacquet-Langlands plus Base Change, Harris-Taylor]. Let $\Pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F)$ s.t.

- $\Pi^\vee \cong \Pi^{\mathsf{c}}$.
- Π is regular C-algebraic(0.1.6.8).
- Π_x is square-integrable.

Then there exists

- Some $\pi_f \in \mathrm{Irr}^{\mathrm{adm}}(\mathbf{G}(\mathbf{A}_f))$,
- Some $\xi \in \mathrm{Rep}^{\mathrm{fd}}(\mathbf{G})$ s.t. $H^*_\xi[\pi_f] \neq 0$(0.1.3.6).
- Some algebraic Hecke character ψ of $\mathcal{K}$**??** satisfying the following: For any $w \in \Sigma_F^{\mathrm{fin}}$ with $u = w \cap \mathcal{K}$ and $p' = w \cap \mathbb{Q}$ s.t. p' is split in $\mathcal{K}$ and $w \notin \{x, x^{\mathsf{c}}\}$, we have

$$\pi_w \cong \Pi_w, \quad \pi_{p',0} \cong \psi_u (0.1.5.1).$$

Moreover, for any $p' \in$ `Prime` that is split in $\mathcal{K}$, we can arrange (π_f, ξ, ψ) s.t. $\pi_{p',0}$ is unramified.

⌟

Proof: Use global Jacquet-Langlands correspondence**??** to obtain $\rho \in \mathrm{Irr}^{\mathrm{auto}}(\mathbb{D}^\times)$ s.t. $\text{J-L}(\rho) \cong \Pi$(0.1.3.2), then $\rho \cong \rho^*$ and ρ_∞ is regular C-algebraic. Then(0.1.5.7)(0.1.5.8) and the properties of the base-change map(0.1.5.4) finish the proof. □

Cor. (0.1.5.10)[Deligne-Brylinski(86)/Carayol(86)/Harris-Taylor[H-T01]/Scholze[Sch13]]. Let $\Pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F)$ s.t.

- $\Pi^\vee \cong \Pi^{\mathsf{c}}$.
- Π is regular C-algebraic(0.1.6.8).
- Π_x is square-integrable.

Then there exists $a \in \mathbb{Z}_+$ and $R(\Pi) \in \mathrm{Rep}^{an}_{\mathbb{Q}_\ell}(\mathrm{Gal}_K)$ s.t. for any $v \in \Sigma_F^{\mathrm{fin}} \setminus S(\ell)$, we have

$$R(\Pi)|_{W_{F_v}} = a\sigma_{n,v}(\Pi_v)(0.1.1.2).$$

Moreover, for each $v \in \Sigma_F^{\mathrm{fin}}$, the representation Π_v is tempered. ⌟

Proof: This is Theorem C of Harris-Taylor.

Take π_f, ξ, ψ as in(0.1.5.9), $a = a(\pi_f)$(0.1.5.1). Take χ_ψ be a character of Gal_F corresponding to ψ under the global class field theory. And let

$$R(\Pi) = [H^*_\xi(\pi_f)]^\vee \otimes \chi_\psi^{-1}.$$

Then the assertion is trivial if $v \in \Sigma_F^{\text{fin}} \setminus (S_F(\ell) \cup \{x, x^{\mathsf{c}}\})$ is split over F_0 and $\pi_{p',0}$ is unramified, where $p' = v \cap \mathbb{Q}$. Then by Chebotarev density theorem, this determines $\frac{1}{a}R(\Pi)$ uniquely. But for any such $p' \in \mathtt{Prime}$, we can arrange that $\pi_{p',0}$ is unramified, so this condition can be dropped.

Now for an arbitrary v, we can take a real quadratic field R that is linearly disjoint with F s.t. the rational prime under x splits in R, and the completion of R and $\mathcal{K}$ at the rational prime under v are isomorphic. Let $\widetilde{F}_0 = F_0 R$ and $\widetilde{F} = FR$, and choose $\widetilde{v}$ and $\widetilde{x}$ above v and x in $\widetilde{F}$ s.t. $\widetilde{v} \notin \{\widetilde{x}, \widetilde{x}^{\mathsf{c}}\}$.

Then we can take $\widetilde{\Pi} \in \mathrm{Irr}^{\text{auto}}(\mathrm{GL}(n)/\widetilde{F})$ to be the cyclic base change of Π, which continues to have the properties listed. Hence we get an $R(\widetilde{\Pi})$ that satisfies the assertion for any $\widetilde{v} \in \Sigma_{\widetilde{F}}^{\text{fin}} \setminus (S_{\widetilde{F}}(\ell) \cup \{\widetilde{x}, \widetilde{x}^{\mathsf{c}}\})$ split over $\widetilde{F}_0$. By Chebotarev density theorem, $\frac{1}{a}R(\widetilde{\Pi}) = \frac{1}{a}R(\Pi)|_{\mathrm{Gal}_{\widetilde{F}}}$. Then applying the above split case to $\widetilde{v}$ finishes the proof.

Using p-adic Hodge theory, it would be no problem to show that one can choose $a = 1$, cf. proof of Proposition VII.1.8 in [H-T01]. Also, this holds for any CM field F, cf. proof of Theorem VII.1.9 of [H-T01]. □

Cor. (0.1.5.11). Let $\Pi \in \mathrm{Irr}^{\text{cusp}}(\mathrm{GL}(n)/F)$ s.t.

- $\Pi^\vee \cong \Pi^{\mathsf{c}}$.
- $\Pi(\frac{n-1}{2})$ is regular C-algebraic(0.1.6.7).
- Π_x is square-integrable for some $x \in \Sigma_F^{\text{fin}}$ that is split over F_0.

Then there exists $a \in \mathbb{Z}_+$ and $R(\Pi) \in \mathrm{Rep}_{\mathbb{Q}_\ell}^{an}(\mathrm{Gal}_K)$ s.t. for any $v \in \Sigma_F^{\text{fin}} \setminus S(\ell)$, we have

$$R(\Pi)|_{W_{F_v}} = a\,\mathrm{rec}'_v(\Pi_v)$$

⌟

Proof: It suffices to find a Hecke character χ of F s.t. $\chi^{-1} = \chi^{\mathsf{c}}$ and $\chi_\infty(\frac{n-1}{2})$ is C-algebraic, which can be done as in the proof of Corollary 7.2.8 of [H-T01]. □

6 Geometric Galois Representations

Def. (0.1.6.1)[Geometric Representations, Fontaine-Mazur[?]]. $(\rho, V) \in \mathrm{Rep}_{\overline{\mathbb{Q}}_p}^{\text{fd}}(\mathrm{Gal}_F)$ is called a **geometric representation** if it satisfies

- For a.e. $v \in \Sigma_F^{\text{fin}}$, ρ_v is unramified.
- For any $v \in S_F(p)$, the representation ρ_v is deRham**??**.

⌟

Def. (0.1.6.2) [Algebraic Weil Representations]. For $\tau \in \Sigma_F^\infty$, an admissible Weil representation $r : W_{F_\tau} \to {}^{\mathrm{L}}G(\mathbb{C})$ is called

- an ***L*-algebraic Weil representation** if $\lambda_\sigma \in X^*(T)$.
- a ***C*-algebraic Weil representation** if $\lambda_\sigma - \delta \in X^*(T)$.

⌟

Prop. (0.1.6.3)[Algebraic Weil Representations and Galois Representations]. ⌟

<u>**Automorphy of Galois Representations**</u>

Def. (0.1.6.4)[Admissible Representations attached to Galois Representations]. For a geometric p-adic Galois representation(0.1.6.1) $\rho : \mathrm{Gal}_F \to \mathrm{GL}(n; \overline{\mathbb{Q}_p})$ and an isomorphism $\iota_p : \overline{\mathbb{Q}_p} \cong \mathbb{C}$, we can associate a $\Pi(\rho) = \otimes'_v \Pi_v(\rho) \in \mathrm{Irr}^{\text{adm}}(\mathrm{GL}(n)/F)$ s.t.

- For any $v \in \Sigma_F^{\text{fin}} \setminus S_F(p)$, $\Pi_v(\rho)$ corresponds to the Weil-Deligne representation attached to the restriction of ρ to the Weil group W_v.
- For any $v \in S_F(p)$, we can also associate a Weil-Deligne representation $\mathrm{WD}_v(\rho)$ via Fontaine's D_{pst} functor.
- For any $v \in \Sigma_F^\infty$, we can also define Hodge-Tate numbers and thus an irreducible $(\mathfrak{g}_v, \mathcal{K}_v)$-module Π_v.

⌟

Conj. (0.1.6.5) [Automorphy of Galois Representations, Clozel[?]/Fontaine-Mazur[?]]. For any irreducible geometric p-adic Galois representation(0.1.6.1) $\rho : \mathrm{Gal}_F \to \mathrm{GL}(n; \overline{\mathbb{Q}_\ell})$,

- $w(\rho) = \frac{2}{\dim \rho} \sum_{a \in \text{H-T}(\rho)} a \in \mathbb{Z}$,
- $m_i(\rho) = m_{w(\rho)-i}(\rho)$ for all i.
- $\Pi(\rho) \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F)$(0.1.6.4).
- $\Pi_v(\rho)$ is C-algebraic(0.1.6.8).

⌟

Conj. (0.1.6.6) [Langlands-Fontaine-Mazur]. Any geometric representation of Gal_F(0.1.6.1) is automorphic(0.1.6.5). ⌟

Proof:

□

Def. (0.1.6.7) [Algebraic Automorphic Representations]. For $v \in \Sigma_F^\infty$, an irreducible representation π_v of $G(F_v)$ is called an L**-algebraic representation**(resp. a C**-algebraic representation**) if the Weil representation associated to it by Langlands correspondence is so(0.1.6.2).

And $\Pi \in \mathrm{Irr}^{\mathrm{auto}}(G/F)$ is called an L**-algebraic representation**(resp. a C**-algebraic representation**) if Π_v is so for any $v \in \Sigma_F^\infty$. ⌟

Def. (0.1.6.8)[Regular Algebraic Automorphic Representations]. For $v \in \Sigma_F^\infty$, $\Pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F)$ is called a **regular C-algebraic automorphic representation** at v if λ_{Π_v} is just the highest weight of an irreducible algebraic representation of $\mathrm{GL}(n; F_v)$.

A regular C-algebraic representation is C-algebraic. ⌟

Proof: This follows from the definition of the Harish-Chandra isomorphism, Cf.[Kna02]5.43. and [B-G14]Section2.3. □

Example (0.1.6.9) [Algebraic Representations for GL(n)]. $\Pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F)$ L-algebraic if $\Pi(\frac{n-1}{2})$ is C-algebraic(0.1.6.7).

The notion of L-algebraic and C-algebraic coincides for n odd and differs by a twist for n even. ⌟

Proof:

□

Galois Representations attached to Automorphic Forms

Conj. (0.1.6.10)[Automorphic Galois Representations(weak version), Buzzard-Gee[B-G14]]. If $\pi \in \mathrm{Irr}^{\mathrm{auto}}(G/F)$ is L-algebraic(0.1.6.7), then for any $p \in \texttt{Prime}$ and $\iota_p : \overline{\mathbb{Q}_\ell} \cong \mathbb{C}$, there exists a continuous geometric Galois representation

$$\rho_\pi = \rho_{\pi,\iota_p} : \mathrm{Gal}_F \to {}^{\mathrm{L}}G(\overline{\mathbb{Q}_p})$$

s.t.

- The composition of ρ_π with the natural projection ${}^{\mathrm{L}}G(\overline{\mathbb{Q}_p}) \to \mathrm{Gal}_F$ is the identity map.
- For a.e. $v \in \Sigma_F^{\mathrm{fin}}$, $\rho_\pi|_{W_{F_v}}$ is unramified and $\iota \circ \rho_\pi(\sigma_v)$ is $\widehat{G}(\mathbb{C})$-conjugate to $c(\pi_v)$.
- For $v \in S_F(p)$, the Hodge-Tate character of $\rho_\pi|_{\mathrm{Gal}_{F_v}}$ can be read off π(Cf.[B-G14]P156).
- If $v \in \Sigma_F^{\mathbb{R}}$, then $\iota \circ \rho_\pi(\mathsf{c}_v)$ is $\widehat{G}(\mathbb{C})$-conjugate to the element $\alpha_v = \lambda_{\sigma_v}(i)\lambda_{\tau_v}(i)r_{\pi_v}(j)$

⌟

Proof:

□

Conj.Cor. (0.1.6.11)[Automorphic Galois Representations for $\mathrm{GL}(n)$, Clozel[?]]. For $p \in \texttt{Prime}$ and any $\Pi \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n)/F)$ that is isobaric**??** and C-algebraic(0.1.6.7), then

- π_f is defined over a number field $E \in \texttt{NField}$(i.e. $\pi_f^\sigma \cong \pi_f$ for any $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$).
- there exists an irreducible geometric p-adic Galois representation

$$\rho_{\ell,\Pi} : \mathrm{Gal}_F \to \mathrm{GL}(n; \overline{\mathbb{Q}_p})$$

  s.t.

$$\Pi(\rho_{p,\Pi}) \cong \Pi(\frac{1-n}{2})(0.1.6.4).$$

- Moroever, such $\{\rho_{p,\Pi}\}$ form a strongly compatible system $\mathcal{R}$ of representations of Gal_F.

Such representations $\rho_{p,\Pi}$ and $\overline{\rho}_{p,\Pi}$(its reduction modulo p) are called **automorphic Galois representations**. ⌟

Proof:

□

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