SEMINAR ON PROOF OF THE LOCAL LANGLANDS CORRESPONDENCE FOR GL(n) OVER p-ADIC FIELDS

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In this seminar, we want to understand Scholze's proof of the local Langlands conjecture for GL(n) over p-adic fields, cf. [Sch13], which simplifies substantially some arguments in the proof given by Harris-Taylor, cf. [H-T01]. The proof uses some global arguments, which are based on the study of some unitary Shimura varieties. For the history behind Harris-Taylor's proof, see [Har15]Chapter 9.

Thm. (0.0.1)[LLC for GL(n), Hasse(30)/Tunnell(78)/Kutzko(80)/Harris-Taylor[H-T01]/Henniart(84, 86, 88, 93, 00)[Hen00]]. Let p be a prime, K be a finite extension of $\mathbb{Q}_p$ and $\psi : K \to \mathbb{C}^\times$ be a non-trivial additive character. Then there exists a unique collection of bijections $\{\mathrm{rec}_n\}_{n\in\mathbb{Z}_+}$ between sets:

$$\mathrm{rec}_n : \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K)) \xrightarrow{\cong} \mathrm{Rep}^n_{\varphi\text{-ss}}(\mathrm{WD}_K)$$

satisfying the following properties:

1. For a quasi-character χ of $K^\times$, $\mathrm{rec}_1(\chi) = \chi \circ \mathrm{Art}_K^{-1}$.
2. For a quasi-character χ of $K^\times$ and $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$,
$$\mathrm{rec}_n(\pi(\chi)) = \mathrm{rec}_n(\pi) \otimes \mathrm{rec}_1(\chi).$$
3. For any $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$ with central character ω,
$$\det(\mathrm{rec}_n(\pi)) = \mathrm{rec}_1(\omega).$$
4. For any $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$, $\mathrm{rec}_n(\pi^\vee) = \mathrm{rec}_n(\pi)^*$.
5. For any two $\pi_1 \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n_1;K)), \pi_2 \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n_2;K))$
$$L(\pi_1 \times \pi_2; s) = L(\mathrm{rec}_{n_1}(\pi_1) \otimes \mathrm{rec}_{n_2}(\pi_2); s), \quad \epsilon(\pi_1 \times \pi_2; s) = \epsilon(\mathrm{rec}_{n_1}(\pi_1) \otimes \mathrm{rec}_{n_2}(\pi_2); s).$$

Scholze defined some test functions, which appear naturally in the point-counting formula for bad reductions of Shimura varieties, generalizing the formula in [Kot92]. These test functions are constructed by moduli spaces of p-divisible groups and matching orbital integral, and can be generalized to more general PEL setting.

The rough idea is that one might hope to associate to any $\tau \in W_K$ a function $f_\tau \in C_c^\infty(\mathrm{GL}(n;K))$ such that for any $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$, we have

$$\mathrm{tr}(f_\tau|\pi) = \mathrm{tr}\left(\tau\middle|\, \mathrm{rec}_n(\pi)(\frac{n-1}{2})\right)$$

But this is too much to hope for, as then f_τ would have non-zero trace on each component of the Bernstein center. So instead we add a "cut-off" function $h \in C_c^\infty(\mathrm{GL}(n;K))$ add associate a function $f_{\tau,h} \in C_c^\infty(\mathrm{GL}(n;K))$ s.t.

$$\mathrm{tr}(f_{\tau,h}|\pi) = \mathrm{tr}(\tau|\sigma_n(\pi))\,\mathrm{tr}(h|\pi).$$

Then conversely, Scholze can use these test functions to characterize the local Langlands correspondence:

Thm. (0.0.2) [Theorem 1.2. of [Sch13]].

(a) For each $n \in \mathbb{Z}_+$, there is a unique map
$$\sigma_n : \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K)) \to \mathrm{Rep}^n_{\psi\text{-ss}}(\mathrm{WD}_K)$$
s.t. for any $\tau \in W_K$ and any "cut-off" function $h \in C_c^\infty(\mathrm{GL}(n;K))$,
$$\mathrm{tr}(f_{\tau,h}|\pi) = \mathrm{tr}(\tau|\sigma_n(\pi))\,\mathrm{tr}(h|\pi),$$
where $f_{\tau,h} \in C_c^\infty(\mathrm{GL}(n;K))$ has matching twisted orbital integral with $\varphi_{\tau,h} \in C_c^\infty(\mathrm{GL}(n;K_{\deg\tau}))$ via Clozel's base change(to be defined). Write $\mathrm{rec}'(\pi) = \sigma_n(\pi)(\frac{1-n}{2})$.

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(b) If $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\pi)$ is a constituent of $\pi_1 \times \ldots \times \pi_r$, then $\mathrm{rec}'(\pi) = \mathrm{rec}'(\pi_1) \oplus \ldots \oplus \mathrm{rec}'(\pi_r)$.

(c) rec' induces a bijection between $\mathrm{Irr}^{\mathrm{sup.cusp}}(\mathrm{GL}(n;K))$ and $\mathrm{Irr}^{n}_{\varphi\text{-ss}}(W_K)$.

(d) rec' is compatible with twists, central characters, duals, and L- and ϵ-factors of pairs, hence $\mathrm{rec}' = \mathrm{rec}$ as in(0.0.1).

The proof of (a) and (b) of this main theorem uses induction on n:

Lemma (0.0.3) [Lemma 3.2. of [Sch13]]. For $n \in \mathbb{Z}_+$, suppose (a) and (b) of(0.0.2) hold for all $n' < n$ and the following hold:

(i) If $\pi = \pi_1 \times \ldots \times \pi_r \in \mathrm{Rep}^{\mathrm{adm}}(\mathrm{GL}(n;K))$ where $\pi_i \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n_i;K))$, then

$$\mathrm{tr}(f_{\tau,h}|\pi) = \mathrm{tr}\left(\tau\Big| \bigoplus_{1\le i\le r} \sigma(\pi_i)(\frac{n-n_i}{2})\right)\mathrm{tr}(h|\pi).$$

(ii) For $\pi \in \mathrm{Irr}^{\mathrm{adm}}(\mathrm{GL}(n;K))$ that is either essentially square-integrable or a generalized Speh representation, then there exists a virtual representation $\sigma(\pi)$ of W_K with $\mathbb{Q}_+$ coefficients of dimension n s.t.

$$\mathrm{tr}(f_{\tau,h}|\pi) = \mathrm{tr}(\tau|\sigma(\pi))\,\mathrm{tr}(h|\pi).$$

(iii) If $\pi \in \mathrm{Irr}^{\mathrm{sup.cusp}}(\mathrm{GL}(n;K))$, then $\sigma(\pi)$ is a genuine representation of W_K.

Then (a) and (b) of(0.0.2) hold true for n, by defining $\sigma(\pi)$ as follows: If π has supercuspidal support $\{\pi_1,\ldots,\pi_r\}$(with multiplicity), then we define

$$\sigma(\pi) = \bigoplus \sigma(\pi_i)(\frac{1-n_i}{2}).$$

For the induction process, (i) is proved in Theorem 6.4. of [Sch13], by relating the deformation spaces of one-dimensional p-divisible groups to the deformation spaces of their infinitesimal parts.

(ii) is proved in Corollary 10.3 using global methods, especially the Langlands-Kottwitz point-counting method.

(iii) is proved by passing to the Lubin-Tate tower and then using the Jacquet-Langlands correspondence and the theory of newforms for $\mathrm{GL}(n)$:

Thm. (0.0.4) [Supercuspidal Representations are Realized on Lubin-Tate Space, Theorem 1.4 of Scholze[Sch13]]. Let $[R\psi_n]$ be the alternating sum of the global section of the vanishing cycles for the Lubin-Tate tower, then it's endowed with an admissible action of the subgroup $A_{K,n} \subset \mathrm{GL}(n;K) \times D^{\times}_{K,1/n} \times W_K$ consisting of elements (γ,δ,σ) s.t.

$$|\det\gamma|^{-1}.|\mathrm{Nmrd}(\delta)|.|\mathrm{Art}_K^{-1}\sigma| = 1$$

.

Let $\rho \in \mathrm{Irr}^{\mathrm{adm}}(D^{*}_{K,1/n})$ s.t. $\pi = \mathrm{JL}(\rho) \in \mathrm{Irr}^{\mathrm{cusp}}(\mathrm{GL}(n;K))$. Then as virtual representations of $\mathrm{GL}(n;\mathcal{O}_K)\times W_K$,

$$[R\psi](\rho) = (-1)^{n-1}\pi^{\vee}|_{\mathrm{GL}(n;\mathcal{O}_K)} \otimes \sigma(\pi).$$

The proof of (c) of (0.0.2) uses computation of I_K-invariant nearby cycles for simple Shimura varieties. This computation leads to a direct proof of the bijective correspondence for supercuspidal representations, without using the numerical local Langlands correspondence in [Hen93], in contrast to both [H-T01] and [Hen00].

Finally, the proof of (d) of (0.0.2) follows from Harris' arguments constructing cuspidal automorphic representations, cf. [Har93] and Henniart's method of twisting with highly ramified characters, cf. [Hen00].

Throughout the proof, we will refer back to other papers, for example [A-C89, Har93, Hen00, H-T01, Sch13a]. We want to mention Fargues-Scholze's geometric construction of the L-parameters, cf. [F-S21], but we mostly likely won't have time.

0. Introduction. Give overview of the local Langlands conjecture. Then outline the proof given in [Sch13]Section 1 and 3.

1. Weil-Deligne representations and L-, ϵ-factors. Recall the definition and basic properties of complex Weil-Deligne representations of W_K. Then prove Theorem 4.2.1 and Corollary 4.2.2 of [Tat79], see [Del72]section 8 or [Lan], which gives the bijection between complex Weil-Deligne representations and continuous ℓ-adic representations of W_F. Define the L- and ϵ-factors for a Weil-Deligne representation of W_K, cf. [Wed08]3.2. In particular, proof the theorem of Deligne on the existence of ϵ-factors, cf. [H-T01]section 4.

2. Bernstein-Zelevinsky classification for $\mathrm{GL}(n)$ and L-, ϵ-factors. Recall the Bernstein-Zelevinsky classification for $\mathrm{GL}(n)$ following [Wed08]2.2 and [B-Z76, B-Z77]. Define the Bernstein center as in [Ber84] and [Sch11]section 2. Define L- and ϵ-factors as in [Wed08]2.5 or [JPS83, J-S89]. Then give a sketch of the proof of the uniqueness, cf.[Hen93], in the formulation with generic representations; and deduce the supercuspidal version.

3. Automorphic forms on $D^\times$ and the simple unitary groups. Define the simple unitary group G associated to a central division algebra D over a CM field, as in [H-T01]1.7. State the global Jacquet-Langlands correspondence. Then study Clozel's base change in detail as in [H-T01]Section 6.2: state Theorems 6.1.1, 6.2.1 and 6.2.9 loc.cit and sketch the proofs.

4. Harris-Taylor's simple Shimura varieties. Introduce Harris-Taylor's Shimura varieties and their integral models, as in [H-T01]3.1, 3.4, and [Sch13]section 8. Apply Theorem 3.4 and Theorem 5.3 of [Sch13a] to prove Lemma 5.5 and Corollary 5.6 of [Sch13a].

5. Deformation spaces of p-divisible groups and the test functions. Introduce the deformation spaces of p-divisible groups, and also formal nearby(vanishing) cycles of [Ber96]. Define the test function $\varphi_{\tau,h}$ in section 2 of [Sch13]. Prove the base change identities on $D^\times$ in Section 4 of loc.cit.

6. Descent properties of the test function. Prove the descent properties of the test functions $\varphi_{\tau,h}$ by relating the deformation spaces of one-dimensional p-divisible groups to the deformation spaces of their infinitesimal parts, cf. section 5 and 6 of [Sch13]. Use these to prove Lemma 3.2(1) of loc.cit.

7. Langlands-Kottwitz Method. Relate the local test functions $\varphi_{\tau,h}$ to global test functions for Harris-Taylor's Shimura varieties by proving Lemma 7.5 of [Sch13]. Introduce the Langlands-Kottwitz point-counting method, following [Sch13]section 9 and [Kot92]. Prove Theorem 9.3 and Corollary 9.4 of [Sch13].

8. ℓ-adic Galois representations attached to automorphic forms. Recall Clozel's base change again as in [H-T01]Section 6.2, and use it to construct the virtual Weil representation $\sigma(\pi)$ with $\mathbb{Q}_+$ coefficients, cf. Corollary 10.3 of [Sch13]. If time permits, construct ℓ-adic Galois representations attached to some regular algebraic conjugate self-dual cuspidal automorphic representations, cf. Theorem 10.6 of loc.cit, assuming the statements (a) and (b) of Theorem 1.2 of loc. cit.

9. Bijective correspondence for irreducible supercuspidal representations. Pass to the Lubin-Tate tower, then use the Jacquet-Langlands correspondence and the theory of newforms for $\mathrm{GL}(n)$ to prove that $\sigma(\pi)$ is a genuine representation for π supercuspidal, cf. Corollary 11.5 of [Sch13]. Then prove the bijective correspondence for supercuspidal representations of $\mathrm{GL}(n)$, cf. Theorems 12.1 and 12.3 of loc.cit.

10. Compatibility of the correspondence. Follows Harris' arguments in [Sch13]Section 13 or [Har93] to construct cuspidal automorphic representations associated to some Weil representations induced from a character, cf. Theorems 13.6 of [Sch13]. Then use Henniart's method of twisting with highly ramified characters, cf. Corollary 2.4 of [Hen00], to prove the compatibility of L- and ϵ-factors and other compatibilities, cf. Theorem 14.1 of [Sch13].

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