Matrix World: The Picture of All Matrices

I am happy to tell the history of Matrix World—the creation of Kenji Hiranabe in Japan. In April 2020 his friend Satomi Joba asked if I would send him a birthday message as a surprise. He was happy (and very surprised). Kenji combines mathematics with art and with computing: three talents in one. I was the one to be surprised when he sent Matrix World in its first form—without a name, without many of the entries and ideas that you see now, but with the central idea of displaying the wonderful variety of matrices.

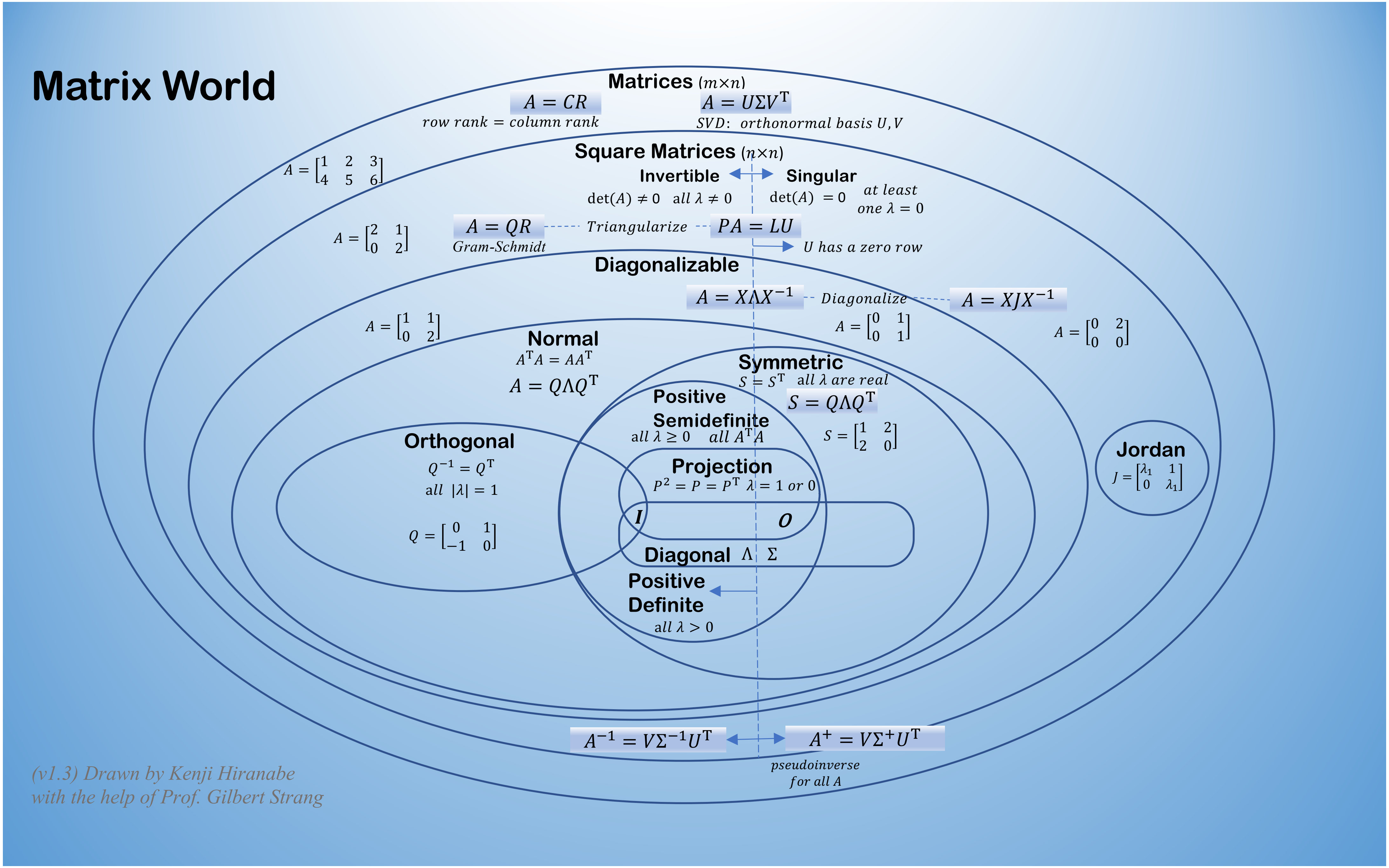
Since that first form, Matrix World has steadily grown. It includes every property that would fit and every factorization that would display that property. Interesting that the SVD is in the outer circle and the identity matrix is at the center—it has all the good properties: the matrix I is

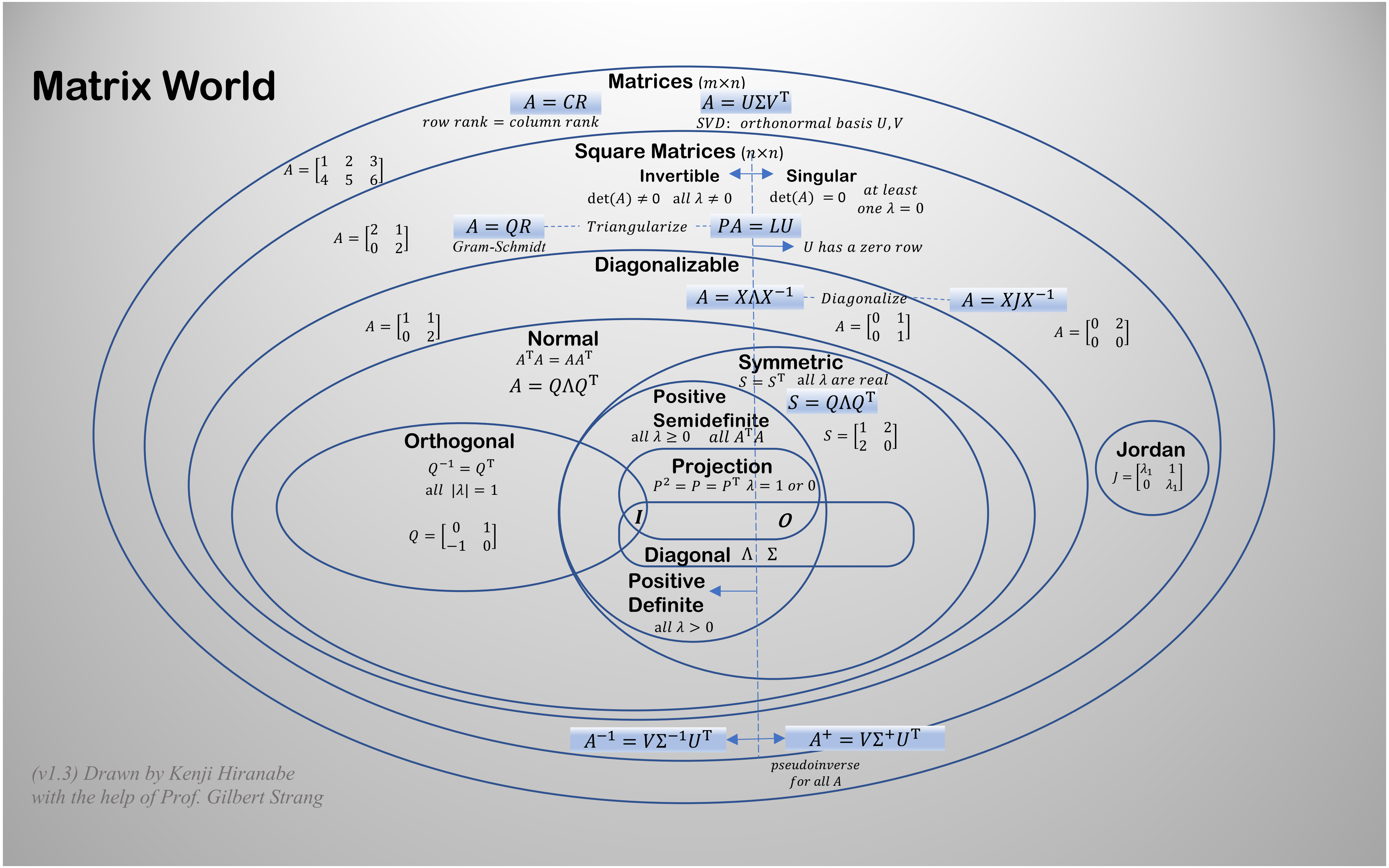
diagonal, positive definite symmetric, orthogonal, projection, normal, invertible, and square.

Lek-Heng Lim has pointed out the usefulness of matrices M that are **symmetric and orthogonal**—kings and also queens. Their eigenvalues are 1 and -1. They have the form $M = I - 2P$ ($P =$ symmetric projection matrix). There is a neat match between all those matrices M and all subspaces of $\mathbf{R}^n$.

You may see something interesting (or something missing) in Matrix World. We hope you will! Thank you to Kenji.

Gilbert Strang


Matrix World
Matrices $(m \times n)$
$A = CR$
row rank = column rank
$A = U\Sigma V^{\mathrm{T}}$
SVD: orthonormal basis U, V
$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$
Square Matrices $(n \times n)$
Invertible
Singular
$\det(A) \neq 0$ all $\lambda \neq 0$
$\det(A) = 0$ at least one $\lambda = 0$
$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$
$A = QR$
Gram-Schmidt
Triangularize
$PA = LU$
U has a zero row
Diagonalizable
$A = X\Lambda X^{-1}$
Diagonalize
$A = XJX^{-1}$
$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$
$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$
$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$
Normal
$A^{\mathrm{T}}A = AA^{\mathrm{T}}$
$A = Q\Lambda Q^{\mathrm{T}}$
Symmetric
$S = S^{\mathrm{T}}$ all λ are real
$S = Q\Lambda Q^{\mathrm{T}}$
$S = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$
Positive Semidefinite
all $\lambda \geq 0$ all $A^{\mathrm{T}}A$
Orthogonal
$Q^{-1} = Q^{\mathrm{T}}$
all $|\lambda| = 1$
$Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
Projection
$P^2 = P = P^{\mathrm{T}}$ $\lambda = 1$ or 0
I
O
Diagonal Λ Σ
Positive Definite
all $\lambda > 0$
Jordan
$J = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$
$A^{-1} = V\Sigma^{-1}U^{\mathrm{T}}$
$A^{+} = V\Sigma^{+}U^{\mathrm{T}}$
pseudoinverse for all A
(v1.3) Drawn by Kenji Hiranabe
with the help of Prof. Gilbert Strang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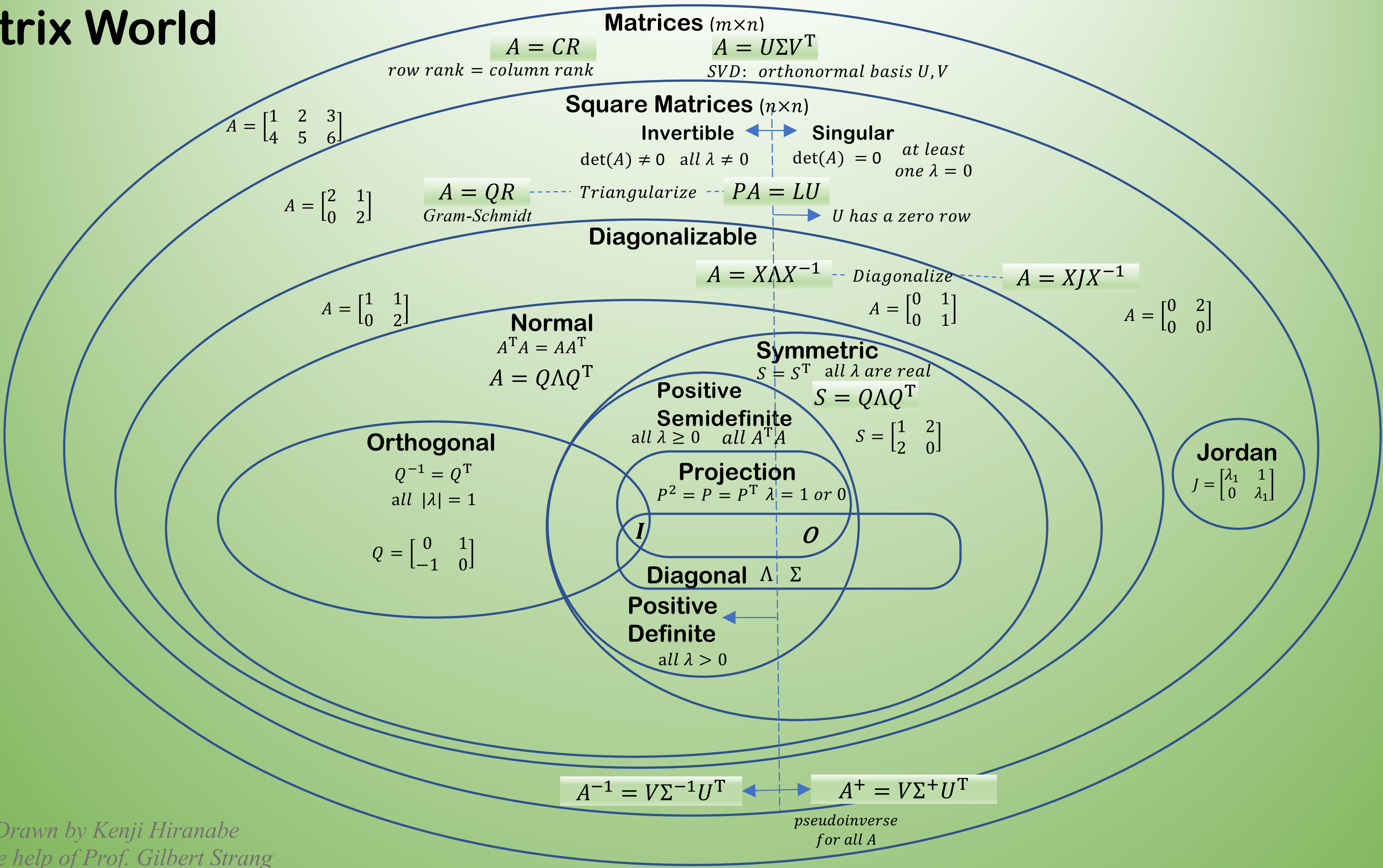

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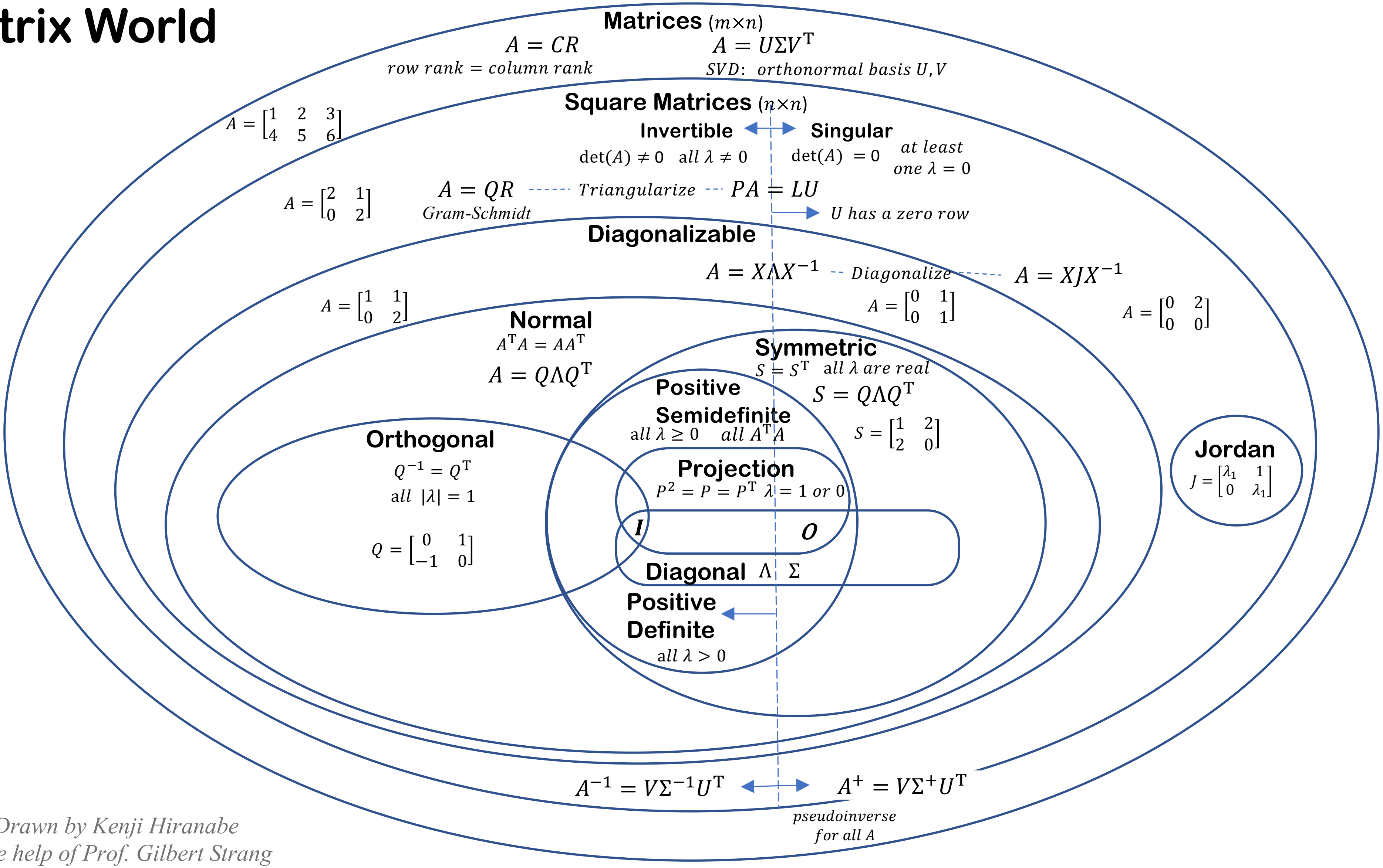

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