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Nonlinear Dispersive Equations and the Beautiful Mathematics that comes with them.

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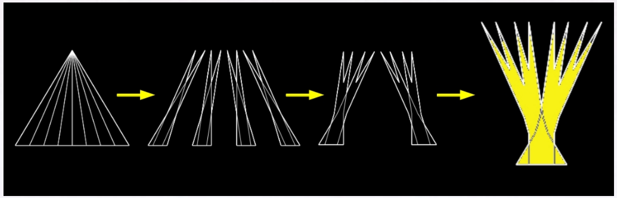
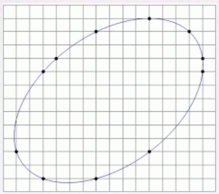
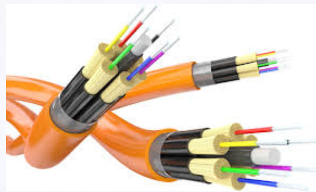
What do these pictures have in common?



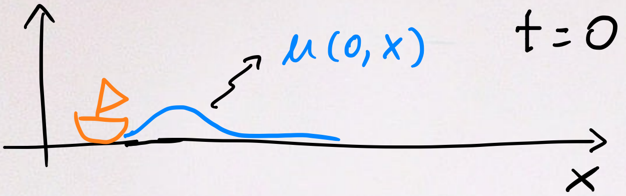


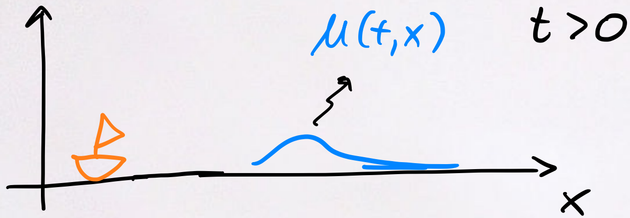


Goals of these lectures

* To introduce the equations representing certain fundamental wave phenomena
* To relate terms in the equations to physical quantities
* To give examples of mathematical tools used to study these PDE and their solutions
* To show how tools developed for a certain problem become key for a completely different setting.

A simple example: Soliton





$$\boxed{u(t,x) = -\frac{1}{2}\,c\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}\,(x-ct-a)\right]}$$

$a, c = \text{const}$

A little bit of history

- Scott Russell (1834)

He was a naval engineer who first described a soliton, the special solution to KdV introduced above.

Korteweg–de Vries

$$\partial_t u + \partial_{xxx} u - 6u\,\partial_x u = 0$$

- Lord Rayleigh, Boussinesq (1871)
- Korteweg & de Vries (1895)

Conservation Laws

The solutions to the KdV equation have infinitely many conserved integrals (Conservation Laws):

Momentum: $\int_{\mathbb{R}} u(t,x)\,dx = C_0$

Mass: $\int_{\mathbb{R}} |u(t,x)|^2\,dx = C_1$

Energy: $\underbrace{\int_{\mathbb{R}} \frac{1}{2}(\partial_x u)^2\,dx}_{\text{Kinetic energy } K(t)} - \underbrace{\int_{\mathbb{R}} u^3\,dx}_{\text{Potential energy } P(t)} = C_2$

$\vdots$

Remarks

$$(\mathrm{KdV}) \quad \underbrace{\partial_t u + \partial_{xxx} u}_{\text{linear part}} \underbrace{- 6 u \partial_x u}_{\text{nonlinear part}} = 0$$

* the soliton is the perfect balance of the kinetic (linear) and the potential (nonlinear) energies

* the ∞ many conservation laws gives also a very rich algebraic structure to the problem that has been studied "abstractly" very actively.

The Initial Value Problem

$$(\mathrm{KdV}) \quad \begin{cases} \partial_t u + \partial_{xxx} u - 6\, u \partial_x u = 0 & x \in \mathbb{R} \\ u\big|_{t=0} = u_0 \end{cases}$$

initial datum or profile

Questions: Given an initial datum $u_0(x)$, does the IVP have a unique solution? For how long? Is it stable under perturbations of $u_0(x)$?

$\overset{?}{=}$ Well-Posedness of the IVP.

the good and the bad

To study well-posedness the linear part of the equation

$$\partial_t v + \partial_{xxx} v \qquad \text{(Airy operator)}$$

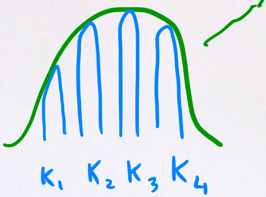
is very good since it encodes dispersion.

The nonlinear part $6u\partial_x u$ is bad since it encodes the interaction of u with $\partial_x u$, and as a consequence hard to control effects (resonance) could happen.

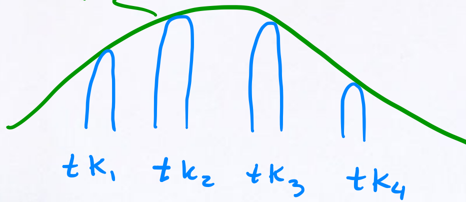
What is dispersion?

Dispersion = Broadening of wave packet

$t = 0$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $t \gg 1$



$\longrightarrow$



$$\boxed{|u(t,x)| \xrightarrow[t \to \infty]{} 0}$$

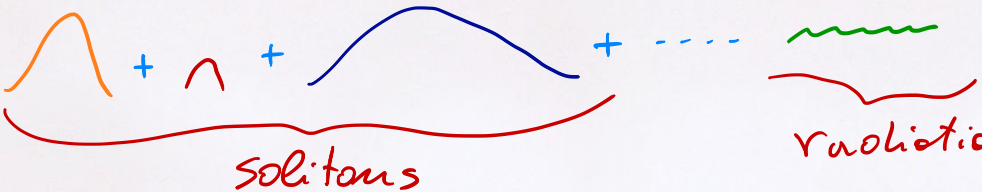
Remarks

* Wave components at higher frequencies move faster.
* Since solutions to the linear KdV equation die out in time, solitons must come from nonlinear interactions!
* The nonlinear term $u\partial_x u$, or equivalently the potential energy $P(t) = -\int_{\mathbb{R}} u^3(t,x)\,dx$, is what restores the wave signal.

A Major Conjecture

Any solution $u(t,x)$ of KdV should be the sum of solitons and radiation:

$u(t,x) =$



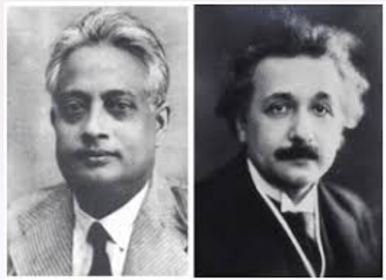
This is called : $\boxed{\text{The soliton resolution conjecture.}}$

The Schrödinger Equation (video)

This is arguably the most important dispersive PDE. It appears for example naturally in the study of the BEC.

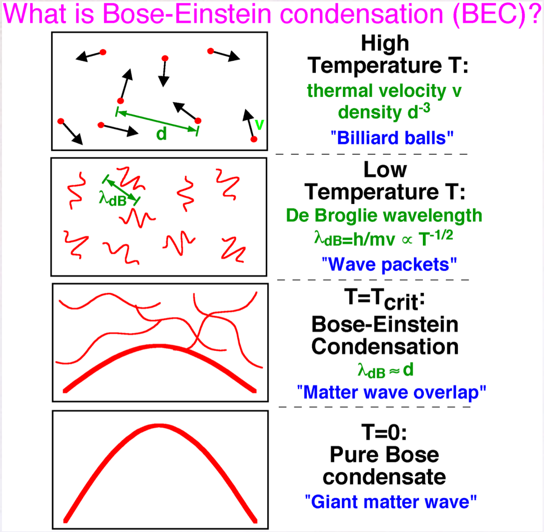
Bose-Einstein Condensate

this is the limit state of diluted gas of Bosons particles as the temperature approaches the absolute zero.



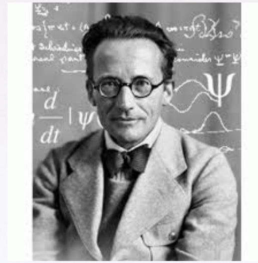
S. N. Bose (1894-1974) A. Eistein (1879-1955)

the limit process



"Combinations" of solutions to the Schrödinger equation

$$\boxed{i\partial_t u + \Delta u = \pm |u|^2 u}$$



can be used to describe certain

"giant matter waves".

Conservation Laws

Consider the Nonlinear Schrödinger (NLS) equation

$$\boxed{i\partial_t u + \Delta u = \pm |u|^2 u} \qquad M^d = \mathbb{R}^d, \mathbb{T}^d$$

$$u : \mathbb{R} \times M^d \longrightarrow \mathbb{C}$$

$$\text{Mass} = \int_{M^d} |u|^2(x,t)\, dx = C_0$$

$$\text{Energy} = \underbrace{\int_{M^d} \frac{1}{2} |\nabla u|^2(t,x)\, dx}_{\text{Kinetic}} \pm \overbrace{\int \frac{1}{4} |u(t,x)|^4\, dx}^{\text{Potential}} = C_1$$

$$\text{Momentum} = \text{----} = C_2$$

Note: If $d = 1$ then are ∞ many conservation laws.

Well-Posedness

Consider the initial value problem (IVP):

$$(\text{NLS}) \quad \begin{cases} \partial_t u + \Delta u = \pm |u|^2 u \\ u|_{t=0} = u_0(x) \end{cases}$$

$M(u_0)$ = mass of u_0

$E(u_0)$ = energy of u_0

Assume $M(u_0)$, $E(u_0) < \infty$. Is the IVP well-posed?

Well-posedness = $\begin{cases} \text{a) Solution exists and it is unique} \\ \text{b) the solution is stable under perturbation of the initial datum } u_0(x). \end{cases}$

Difficulties

* Given an initial datum $u_0(x)$ there is no "explicit" formula for the solution $u(t,x)$.

* The difficult part to handle is the nonlinearity

$$\boxed{|u|^2 u}$$

this is a 3 wave interaction and out of control "growth" may happen.

* Just assuming enough regularity to make sense of M and E is often too little.

Finding Solutions by Iterative Process

the goal is to define "a good" sequence which "limit" will give a solution:

$u_0(x)$ = initial profile

$u_1(x)$ = solution to $\begin{cases} i\partial_t v + \Delta v = 0 \\ v\big|_{t=0} = u_0 \end{cases}$ linear

$u_2(x)$ = solution to $\begin{cases} i\partial_t w + \Delta w = \pm |u_1|^2 u_1 \\ w\big|_{t=0} = 0 \end{cases}$ forcing term

$\vdots$

$u_n(x)$ = solution to $\begin{cases} i\partial_t \theta + \Delta \theta = \pm |u_{n-1}|^2 u_{n-1} \\ \theta\big|_{t=0} = 0 \end{cases}$

$$\boxed{u_n \longrightarrow ?}$$

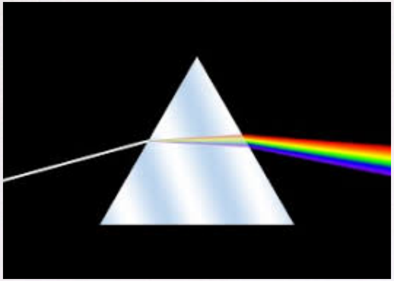
the power of linear Solutions

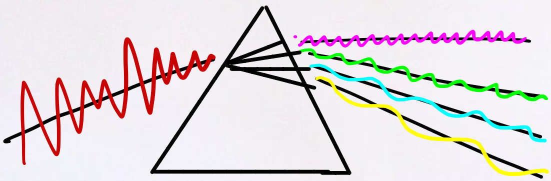
From the previous scheme it is clear that we need to understand the following

Question: If $u_0(x)$ has finite energy and mass and $v(t,x)$ is the solution of the linear Schrödinger IVP, in which space is $|v|^2 v$?

Remark: If $u_0(x) = \sin x$ or $\cos x$ then one can use explicit calculations and derive formulae for $|v|^2 v$. But what do we do in general ?

The Fourier Transform





The Fourier Transform is a mathematical tool that allows us to write complex signals as a sum of sin and cos.

Mathematically

Consider a periodic non signal $f(x)$, then

$$f(x) \text{ "=" } c_0 + \sum_{n\in\mathbb{Z}} b_n \sin(nx) + \sum_{n\in\mathbb{Z}} d_n \cos(nx) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$$

Fourier Series

$$a_n = \hat{f}(n) = \int_0^{2\pi} f(x)\, e^{-inx}\, dx$$

Fourier Coefficient

If $f(x)$ is not periodic, then

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x)\, e^{-ix\xi}\, dx$$

is the Fourier Transform and

$$f(x) = c \int_{\mathbb{R}} \hat{f}(\xi)\, e^{ix\xi}\, d\xi$$

(reconstruction formula)

Some well-known properties

$$\widehat{\frac{d}{dx}f}(\xi) = i\xi\,\hat{f}(\xi) \qquad\qquad \widehat{\frac{d}{dx}f}(n) = i n\, a_n$$

$$\left(\int_{\mathbb{R}} |f(x)|^2\,dx\right)^{\frac{1}{2}} =: \|f\|_{L^2(\mathbb{R})} \qquad\qquad \left(\int_{\mathbb{T}} |f(x)|^2\,dx\right)^{\frac{1}{2}} =: \|f\|_{L^2(\mathbb{T})}$$

$$\| \qquad\qquad\qquad\qquad \|$$

Plancherel

$$c\left(\int_{\mathbb{R}} |\hat{f}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} =: \|\hat{f}\|_{L^2(\mathbb{R})} \qquad\qquad c\left(\sum_n |a_n|^2\right)^{\frac{1}{2}} =: \|a_n\|_{\ell^2}$$

$$\|a_n\| = \left(\sum_n |a_n|^2\right)^{\frac{1}{2}}$$

Fourier Transform in Action

Consider the linear IVP

$$\begin{cases} i\partial_t v + \Delta v = 0 \\ v|_{t=0} = u_0(x) \end{cases}$$

assume for now $\boxed{x \in \mathbb{R}^d.}$

To solve take FT $\Updownarrow$ and fix the frequency ξ

$$\boxed{\begin{cases} i\,\dot{\hat{v}}(\xi) - |\xi|^2 \hat{v}(\xi) = 0 \\ \hat{v}(0,\xi) = \hat{u}_0(\xi) \end{cases}} \iff \boxed{\hat{v}(t,\xi) = \hat{u}_0(\xi)\, e^{it|\xi|^2}}$$

$$\hat{v}(t,\xi) = \hat{u}_0(\xi)\, e^{it|\xi|^2}$$

Note: For larger ξ $\hat{v}(t,\xi)$ has a faster velocity $t|\xi|$, hence the spreading of the wave pocket = dispersion.

$$S(t)u_0(x) := v(t,x) = \int_{\mathbb{R}^d} \underbrace{u_0(\xi)\, e^{i(x\cdot\xi + t|\xi|^2)}}_{\text{oscillatory integral}} d\xi \qquad (1)$$

One also has the formula:

$$S(t)u_0(x) = \frac{c}{|t|^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{2t}} u_0(y)\, dy \qquad (2)$$

dispersion

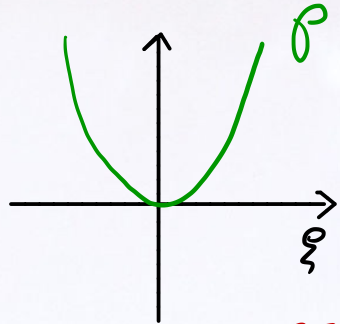
Good use of both formulae

From formula (2) we have

$$\boxed{|S(t)u_0(x)| \le \frac{c}{|t|^{\frac{d}{2}}} \int_{\mathbb{R}^d} |u(y)|\,dy = \frac{c}{|t|^{\frac{d}{2}}} \|u\|_{L^1(\mathbb{R}^d)}}$$

dispersive estimate

$$\mathcal{P} = \{(\xi, |\xi|^2) \,/\, \xi \in \mathbb{R}^d\}$$



From formula (1) we have that

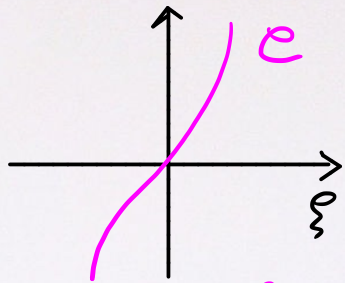
$$\boxed{S(t)u_0(x) = R^* u_0(x)}$$

R = restriction of FT on $\mathcal{P}$

The Airy Equation

Using a similar procedure we can also show that the solution to the Airy IVP

$$\begin{cases} i\partial_t v + \partial_{xxx} v = 0 \\ v\big|_{t=0} = u_0 \end{cases}$$



$\mathcal{C} = \{(\xi, \xi^3) \,/\, \xi \in \mathbb{R}\}$

$$\boxed{W(t)u_0(x) := v(t,x) = \int_{\mathbb{R}} \hat{u}_0(\xi)\, e^{i(t\xi^3 + x\xi)}\, d\xi}$$

$= \tilde{R}^* u_0$ where $\tilde{R}$ = Restriction on cubic $\mathcal{C}$.

Time for some definitions

Assume $p>0$, then

$$L^p(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{C} \;/\; \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{\frac{1}{p}} =: \|f\|_{L^p} < \infty \right\}$$

L^p space

Assume $k \in \mathbb{N}$, then

$$H^k(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{C} \;/\; \|D^\alpha f\|_{L^2} < \infty \text{ for } \underset{\alpha = (\alpha_1, \dots, \alpha_d)}{|\alpha| \le k} \right\}$$

Sobolev space

$$\|f\|_{H^k} := \sum_{|\alpha| \le k} \|D^\alpha f\|_{L^2} \approx \sum_{|\alpha| \le |k|} \| |\xi|^\alpha \hat{f} \|_{L^2} \approx \| (1+|\xi|)^k \hat{f} \|_{L^2}$$

So we can generalize the definition of $H^k(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ for any $s \in \mathbb{R}$!

The power of harmonic analysis

There are several beautiful results in harmonic analysis dealing with restrictions of Fourier transforms on hypersurfaces:

"Theorem": Let S be a "curved" surface in $\mathbb{R}^d$. Then the restriction operator R_S is well defined and there are good L^p estimates for it.

(Stein, Tomas, Wolff, Bourgain, Strichartz, Kenig ...)

Strichartz Estimates

For simplicity here I will state only those in $\mathbb{R}^2$:

Theorem: Assume $u_0 \in L^2(\mathbb{R}^2)$. Assume that (p,q) are s.t.

$$\frac{2}{p} = 2\left(\frac{1}{2} - \frac{1}{q}\right). \text{ then}$$

$$\| S(t) u_0 \|_{L^p_t L^q_x} \leq C \, \| u_0 \|_{L^2}$$

Remark: If $(p,q) = (4,4)$ we are looking at $|S(t) u_0|^4$

4 waves interaction

$$\int_{\mathbb{R} \times \mathbb{R}^2} |S(t) u_0(x)|^4 \, dt \, dx \leq C \, \| u_0 \|^4_{L^2} = C \, (\text{Mass})^2.$$

Rescaling

The IVP

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^2 u \\ u|_{t=0} = u_0 \end{cases}$$

can be "rescaled". In fact if we define

$$\boxed{u_\lambda(t,x) = \frac{1}{\lambda} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)} \quad (\lambda \to \infty)$$

then u_λ solves the IVP with datum $\boxed{u_{0,\lambda} = \frac{1}{\lambda} u_0\left(\frac{x}{\lambda}\right)}$

$$\boxed{\|u_{0,\lambda}\|_{\dot{H}^s} \approx \lambda^{-s} \|u_0\|_{\dot{H}^s}} \overset{\text{if } s=0}{\Longrightarrow} \boxed{\|u_{0,\lambda}\|_{L^2} \approx \|u_0\|_{L^2}}$$

$\Longrightarrow$ For this problem the "mass" is scalar invariant.

($s=0$ critical exponent)

A well-posedness theorem

$$(*) \quad \begin{cases} i\partial_t u + \Delta u = \pm |u|^2 u \\ u|_{t=0} = u_0 \end{cases}$$

Theorem [local well-posedness]

Assume $s>0$. Then $\forall\, u_0 \in H^s(\mathbb{R}^d)$

$\exists\, \delta = \delta(\|u_0\|_{H^s}^{-1})$ and $\exists!$ solution u to $(*)$ s.t.

$u \in \mathcal{C}([0,\delta], H^s(\mathbb{R}^d)) \cap X^s$ and it is "stable".

$\boxed{\text{What happens after time } \delta\,?}$

If $s=0$ same conclusion but $\delta = \delta(u_0)$. it depends also on the profile

From local to global

The question of longtime behaviour of solutions is a difficult one. One very useful ingredient is: conservation laws.

$$M = \text{Mass} = \int_{\mathbb{R}^2} |u|^2(t,x)\,dx = \|u(t)\|_{L^2}^2 = C_0$$

$$E = \text{Energy} = \frac{1}{2}\int_{\mathbb{R}^2} |Du(t,x)|^2 dx \pm \frac{1}{4}\int_{\mathbb{R}^2} |u(t,x)|^4 dx$$

$+$ = defocusing case

$-$ = focusing case.

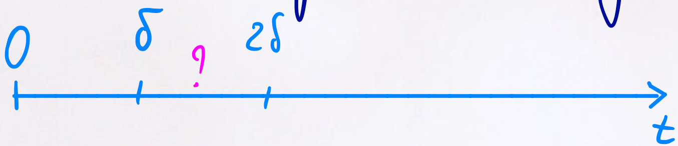
Defocusing Case

Assume $M(u_0)+E(u_0)<\infty$.

Then if $u(t,x)$ is solution to the defocusing NLS with datum u_0, we have: $M(u(t))+E(u(t)) = M(u_0)+E(u_0)<\infty$, and so

$$\|u(t)\|^2_{H^1(\mathbb{R}^2)} \le M+E. \qquad (**)$$

Recall from local well-posedness for $s=1$



$\delta \simeq \|u_0\|_{H^1}^{-\alpha}$, so the energy to move to 2δ would be the growth of $\|u(\delta)\|_{H^1}$ which by $(**)$ is prevented!

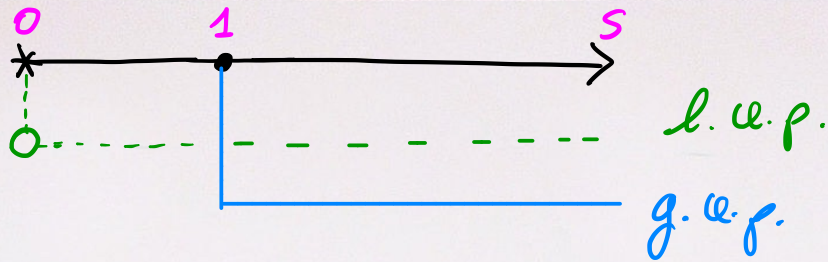
Hence by iteration we can extend the local well-posedness to a global one. In fact this can be done for all $s \geq 1$.

Theorem [global well-posedness]

Fix $s \geq 1$ and assume that NLS is defocusing. Then $\forall\, u_0 \in H^s(\mathbb{R}^2)$ $\exists!$ solution $u \in \mathcal{C}(\mathbb{R}, H^s) \cap X^s$ that is "stable". Moreover if $s > 1$

$$\boxed{\|u(t)\|_{H^s(\mathbb{R}^2)} \leq c_1 \exp(c_2 |t|) \quad \forall\, t \in \mathbb{R}}$$

Summary:



Question: We have a conservation law (mass) for $s=0$, why do not iterate with that?

Answer: Because when $s=0$ we have $\delta=\delta(u_0)$, it depends on the profile of u_0, not only its mass!

Global well-posedness with only finite mass is much harder!!

A beautiful theorem

Theorem (Dodson '14) Consider the defocusing cubic NLS in $\mathbb{R}^2$. Then $\forall u_0$ with $M(u_0) < \infty$, $\exists!$ solution $u(t,x)$ in $\mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^2)) \cap X^0$ that is "stable". Moreover $\exists\, u^+, u^- \in L^2$ such that

$$\boxed{\| u(t) - S(t)u^{\pm} \|_{L^2} \xrightarrow[t \to \pm\infty]{} 0} .$$

This last property is called **scattering**.

(See work of Killip-Tao-Visan-Zhang).

The focusing case

In this case the situation is much more complex. An important role is played by the ground state $Q(x)$. this is the unique positive solution of

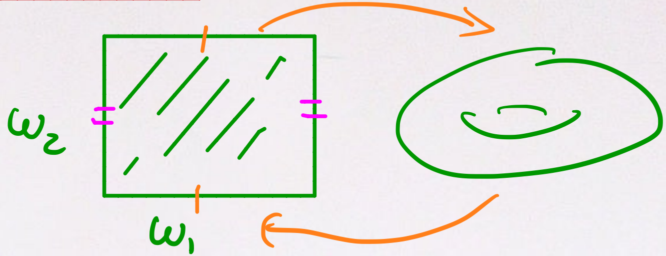
$$\boxed{\Delta Q + Q^3 = Q}$$

Theorem (Dodson '14) Assume that $M(u_0) < M(Q)$. Then $\exists!$ solution $u(t,x)$ to the focusing cubic NLS in $\mathbb{R}^2$ s.t. $u \in C(\mathbb{R}, L^2) \cap X^0$, it is "stable" and scatters.

(See also Killip-Visan-Zhang for radial case)

The periodic case

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^2 u \\ u|_{t=0} = u_0 \end{cases} \qquad \boxed{x \in \mathbb{T}^2}$$



Fact: This problem is much more complicated than the one in $\mathbb{R}^2$!

In fact the presence of the boundary increases nonlinear effects.

the linear solution

$$\begin{cases} i\partial_t v + \Delta v = 0 \\ v|_{t=0} = v_0 \end{cases} \quad \underset{\text{Fix } k \in \mathbb{Z}^2}{\overset{\text{FT}}{\Longrightarrow}} \quad \begin{cases} i\dot{\hat{v}} - |k|_*^2 \hat{v} = 0 \\ \hat{v}|_{t=0} = \hat{u}_0(k) \end{cases}$$

$$|k|_* = \omega_1 k_1^2 + \omega_2 k_2^2$$

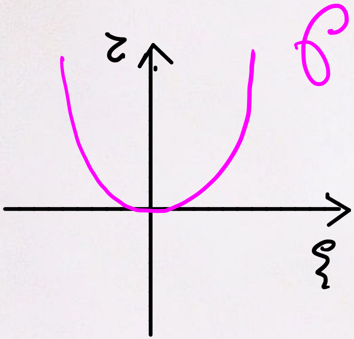
$$\hat{v}(t,k) = \hat{u}_0(k)\, e^{it|k|_*^2} \Longrightarrow \boxed{v(t,x) = \sum_{k\in\mathbb{Z}^2} \hat{u}_0(k)\, e^{i(t|k|_*^2 + x\cdot k)}}$$

this is an oscillatory series. These objects are studied in analytic number theory

Comparison

The $\mathbb{R}^2$ case

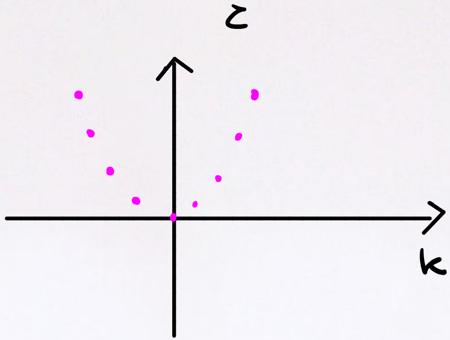
$$S(t)u_0 := \int \hat{u}_0(\xi)\, e^{i(t|\xi|^2 + x\xi)}\, d\xi$$



Strichartz Estimates via Fourier Restriction theorems.

The $\mathbb{T}^2$ case

$$S(t)u_0 := \sum_{k \in \mathbb{Z}^2} \hat{u}_0(k)\, e^{i(t|k|_*^2 + x\cdot k)}$$



Do Strichartz Estimates follow from analytic number theory?

Rational and irrational tori

Definition: A torus $\mathbb{T}^2$ of periods (ω_1, ω_2) is called

rational $\iff$ $\omega_1/\omega_2 \in \mathbb{Q}$

irrational $\iff$ $\omega_1/\omega_2 \in \mathbb{R} \setminus \mathbb{Q}$

Remark: If $\mathbb{T}^2$ is rational then $S(t)u_0$ is also periodic in time.

Theorem (Bourgain '95) Assume $\mathbb{T}^2$ is a $\boxed{\text{rational}}$ torus then

$\forall s > 0$
$$\|S(t)u_0\|_{L^4(\mathbb{T}\times\mathbb{T}^2)} \leq C \|u_0\|_{H^s}$$

the mass is not enough!

"Proof"

J. Bourgain

Step 1: $\|S(t)u_0\|_{L^4}^2 = \|S(t)u_0\cdot S(t)u_0\|_{L^2(\mathbb{T}\times\mathbb{T}^2)}$

$$\simeq \|\widehat{S(t)u_0\cdot S(t)u_0}\|_{\ell^2(\mathbb{Z}\times\mathbb{Z}^2)}$$

Step 2: Write $(\widehat{S(t)u_0\cdot S(t)u_0})(\tau,k)$ explicitly

Step 3: For simplicity assume $(\omega_1,\omega_2)\in\mathbb{N}\times\mathbb{N}$, one has to estimate $|\mathcal{E}|$ where

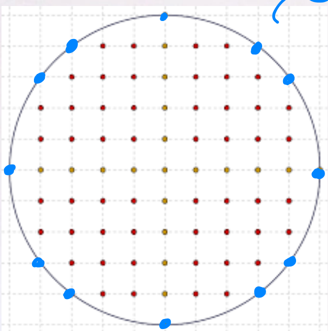
$$\boxed{\left\{(x,y)\in\mathbb{Z}^2 \,/\, \omega_1 x^2+\omega_2 y^2 = R^2\right\} =: \mathcal{E}}$$

$$\boxed{\mathcal{E} = \{(x,y)\in\mathbb{Z}^2 \,/\, w_1 x^2 + w_2 y^2 = R^2\}}$$

$w_i \in \mathbb{N}$ $\quad i=1,2$

Suppose $w_i = 1$, $i=1,2$, then we count lattice points on circles. Using a lemma by Gauss

$\mathcal{E}$



$$\boxed{|\mathcal{E}| \le \exp\left(\frac{\log R}{\log\log R}\right) \ll R^s}$$

for any $s>0$.

this is where the loss of derivative comes from.

Some Remarks

* If π^2 is irrational:
  - $S(t)u_0$ is no longer periodic in time
  - there are no good estimates of how many lattice points are on ellipses.

* In Bourgain's proof

$$\boxed{\text{Analytic Number Theory} \Longrightarrow \text{Harmonic Analysis}}$$

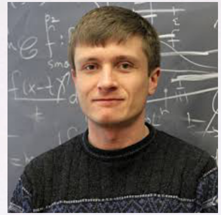
the irrational case

Strichartz estimates for general tori were proved by **Bourgain-Demeter** in 2014 !

Surprisingly ANT was not part of the proof. The Strichartz estimates were proved as a corollary of the

$\boxed{\ell^2\text{-decoupling theorem}}$

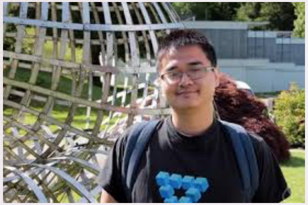
J. Bourgain



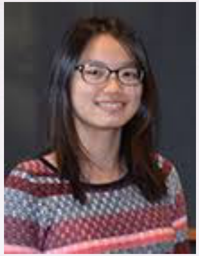
C. Demeter

this theorem had been a major open conjecture in HA for decades. This theorem is also related to the Fourier Restriction Theorem mentioned above.

Following Bourgain–Demeter work, improved Strichartz estimates were proved by myself with:



Chenjie Fan



Hong Wang

Bobby Wilson

Finally we can state:

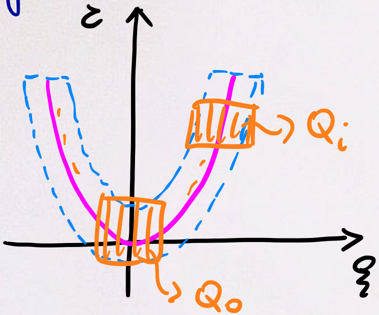
any torus!

Theorem Assume $u_0 \in H^s(\mathbb{T}^2)$, $s > 0$. Then the cubic NLS initial value problem is well posed in $[0, \delta]$

$\delta = \delta(\|u_0\|_{H^s}^{-1})$.

The ℓ^2 decoupling theorem

The main goal is to "reconstruct" in the right spaces of functions the size of a signal from the sizes of its parts.



What happens when we interact Q_i with Q_j? How do we make sure we do not overcount the interactions?

- Classical harmonic analysis
- Combinatorics
- Incidence geometry
- Polynomial method

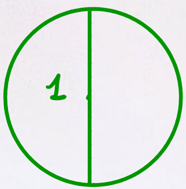
} Larry Guth

L. Guth

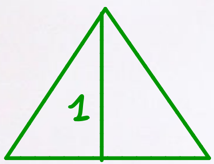
The Kakeya problem inspires

The work of Bourgain-Demeter is strictly connected to work of Bourgain-Guth on the Kakeya problem.

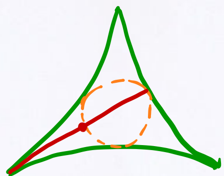
Definition: A Kakeya needle set is a set in the plane such that a unit line segment can be rotated continuously through 180° within it returning to its original position but with reversed direction.



Area $= \pi/4$



Area $= \frac{1}{\sqrt{3}}$

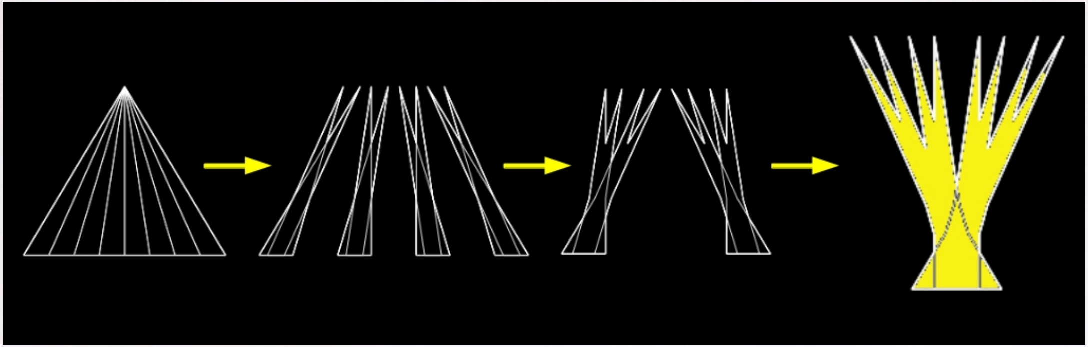


Area $= \frac{\pi}{8}$

?

Area $\to 0$

Besicovitch '28 demonstrated that on the plane there exist Kakeya sets of arbitrarely small area:



By subdividing the triangle by 2^n parts as above, and letting $n \to \infty$, one obtains a tree of arbitrarely small area. This is the Perron Tree.

Besicovitch actually proved even more: there are Kakeya sets of measure zero.

Kakeya Conjecture: Every Kakeya set in $\mathbb{R}^d$ has Minkowski dimension d.

* If $d=2$ the conjecture is proved (Davis '71)
* If $d \geq 3$ the conjecture is really hard!

(see Bourgain, Guth, Katz, Laba, Tao, Wolff ...)

From harmonic analysis to number theory

Recently Bourgain - Demeter - Guth implemented techniques from the proof of the "ℓ^2 decoupling theorem" to prove the "Vinogradov Mean Value Theorem":

Let $s, n, N \in \mathbb{N}$, $s \geq 1$, $n, N \geq 2$. Let $J_{s,n}(N)$ be the number of integral solutions to the system:

$$\boxed{x_1^i + \cdots + x_s^i = x_{s+1}^i + \cdots + x_{2s}^i} \qquad 1 \leq i \leq n$$

and $1 \leq x_1^i, \ldots, x_{2s}^i \leq N$. Then

$$\boxed{J_{s,n}(N) \underset{\varepsilon}{\lesssim} N^{s+\varepsilon} + N^{2s - \frac{n(n+1)}{2} + \varepsilon}} \qquad \forall\, \varepsilon > 0.$$

Back to NLS

$$(\mathrm{IVP})\begin{cases} i\partial_t u + \Delta u = \pm |u|^2 u \\ u|_{t=0} = u_0 \qquad x \in \mathbb{T}^2 \end{cases}$$

$$M(u) = \int_{\mathbb{T}^2} |u|^2(t,x)\,dx$$

$$E(u) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u|^2 dx \pm \frac{1}{4}\int |u|^4 dx$$

From the Strichartz estimate

$$\boxed{\|S(t)u_0\|_{L^4([0,1]\times\mathbb{T}^2)} \lesssim \|u_0\|_{H^s}}$$ and refinements:

Theorem: Assume $s>0$. Then $\forall\, u_0 \in H^s(\mathbb{T}^2)$ $\exists!$ solution $u(t,x) \in \mathcal{C}([0,\delta], H^s(\mathbb{T}^2)) \cap X^s$, "stable" and $\delta = \delta(\|u_0\|_{H^s}^{-1})$.

If the IVP is defocusing we can iterate and prove:

Theorem: Assume $s \geq 1$, then $\forall u_0 \in H^s(\mathbb{T}^2)$ $\exists !$ global stable solution $u(t,x) \in C(\mathbb{R}, H^s(\mathbb{T}^2)) \cap X^s$.

Moreover if $s > 1$

$$\boxed{\|u(t)\|_{H^s} \leq C_0 \exp(C_1 |t|) \quad \text{as } |t| \to \infty}$$

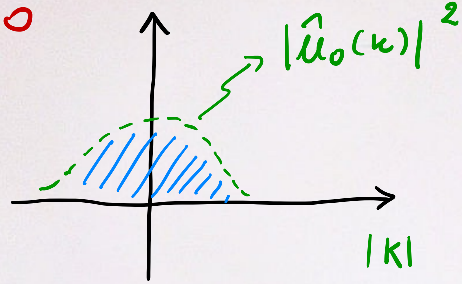
Question: Can one prove scattering?

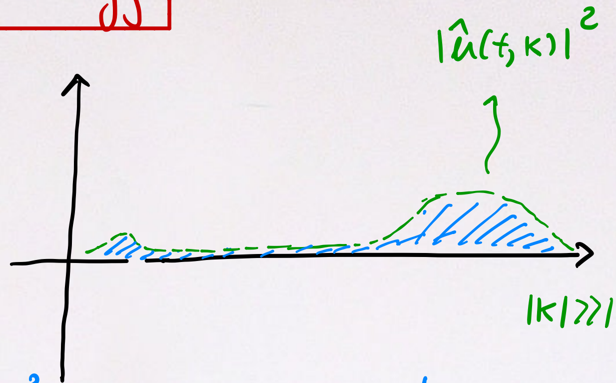
Scattering is not expected due to effects from the boundary.

So what happens when $|t| \to \infty$?

Transfer of energy

$t=0$



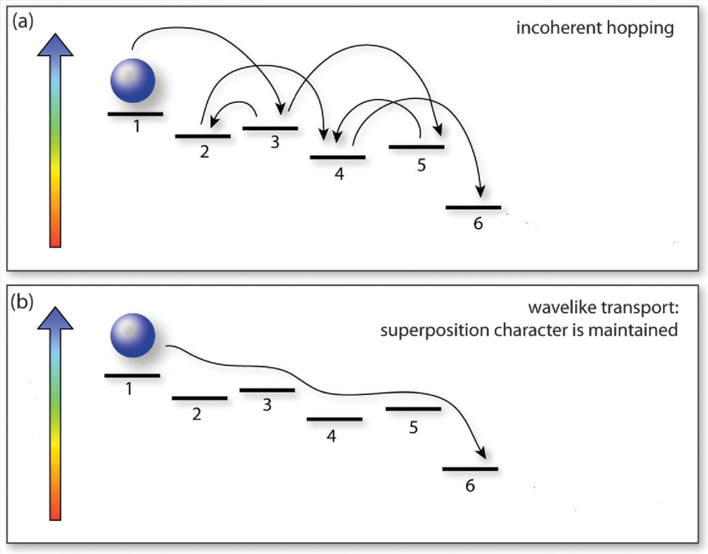


Area Subgraph $= \sum_k |\hat{u}(t,k)|^2 \approx M(u_0)$ constant!

<u>Question</u>: Does the support of $|\hat{u}(t,k)|^2$ moves to higher frequencies?

Weak turbulence, forward cascade ....

Even more interesting



If there is a migration to high frequencies is the process happening in a incoherent way

or in a coherent manner?

We are very far from understanding this for NCS.

Growth of Sobolev norms

But we can study:

$$\boxed{\sum_{k\in\mathbb{Z}} |\hat{u}(t,k)|^2 (|k|+1)^{2s} = \|u(t)\|^2_{H^s}} \quad \text{as } |t|\to\infty,$$

and check what happens when $|t|\to\infty$.

Remark: From "iteration" of local well-posedness we have an exponential (trivial) bound:

$$\boxed{\|u(t)\|_{H^s} \le C_1 \exp(C_2|t|) \quad \text{as } |t|\to\infty}$$

Some Facts

Fact 1: Complete integrability may prevent growth of Sobolev norms. (i.e 1D cubic NLS).

Fact 2: Scattering prevents growth of Sobolev norms.

Dodson: In $\mathbb{R}^2$ $\exists\, u^\pm \in H^s(\mathbb{R}^2)$ $s \geq 0$ s.t.

$$\boxed{\|u(t) - S(t)u^\pm\|_{H^s} \xrightarrow[t \to \pm\infty]{} 0}$$

Hence

$$\|u(t)\|_{H^s} \leq \|u(t) - S(t)u^\pm\|_{H^s} + \|S(t)u^\pm\|_{H^s}$$
$$\leq C + \|u^\pm\|_{H^s}$$

Some bounds from above

(1) If $u(t,x)$ is solution to the cubic defocusing NLS in $\mathbb{T}^2$ then $\forall\, s>1$

$$\boxed{\|u(t)\|_{H^s} \leq C\,|t|^{2(s-1)+\varepsilon}}$$

(Bourgain, Sohinger)

Remark: the original proof of Bourgain was only for $\mathbb{T}^2$ rational, but it is based on Strichartz estimates and it can be extended to any $\mathbb{T}^2$.

(2) Consider the NLS with nonlinearity $|u|^{p-1}u$, $3<p<5$ in generic tori $\mathbb{T}^3$. Then one has

$$\boxed{\|u(t)\|_{H^2(\mathbb{T}^3)} \le C\,(1+|t|)^{\frac{2}{5-p}+\theta(p)}}$$

For rational tori this is not there

where $\theta(p) = \min(p-3,\, 5-p)/182$

(Y. Deng - Germain)

Remarks:

- In this case "generic" means that the vector $(\omega_1, \omega_2, \omega_3)$ of the periods has a certain Diophantine property.
- Neither (1) or (2) are sharp results.

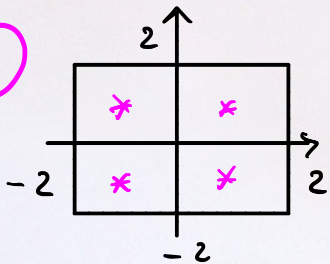
Are there solutions that grow?

(3) Fix $s>0$, $0<\delta\ll 1$, $K\gg 1$. Then for the cubic defocusing NLS in $\mathbb{T}^2$ rational, $\exists$ initial data u_0 and a time $T\gg 1$ s.t.

$$\|u_0\|_{H^s} < \delta \quad \text{and} \quad \|u(T)\|_{H^s} > K.$$

(Colliander - Keel - S - Takaoka - Tao)

(4)

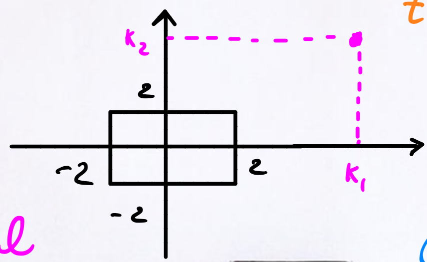


$t=0$

- - - - - -

$\mathbb{T}^2$ rational

$t = t_K$



arbitrarily large mode excited.

(Carles - Faou)

Some ideas for the proof of ③

* This is a constructive proof. Look for a solution $u(t,x)$

$\mathbb{T}^2$ = square torus:

$$\boxed{u(t,x)} = \sum_{n\in\mathbb{Z}^2} a_n(t)\, e^{i(t|n|^2 + x\cdot n)}$$

$$\Longleftrightarrow \quad \boxed{-i\partial_t a_n = -|a_n|^2 a_n + \sum_{n_1,\,n_2,\,n_3 \in \Gamma(n)} a_{n_1}\overline{a_{n_2}}\,a_{n_3}\, e^{i\omega_4 t} \qquad n\in\mathbb{Z}^2}$$

where

$$\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2$$

$$\Gamma(n) = \{(n_1,n_2,n_3) \,/\, n_1 - n_2 + n_3 = n\}$$

this is a HUGE system!

We make several reductions:

R1) Assume that (n, n_1, n_2, n_3) are in resonance:

$$\boxed{\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 0}$$

Fact: (n, n_1, n_2, n_3) are in resonance if and only if they are vertices of rectangles in $\mathbb{Z}^2$.

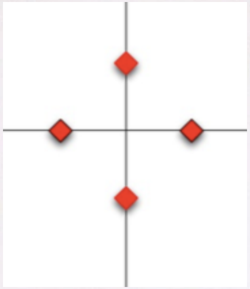
R2) Among all these rectangles we pick a special set of frequencies

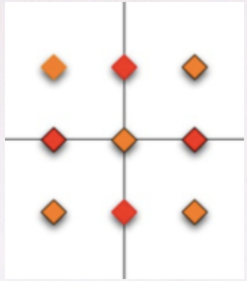
$$\boxed{\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \Lambda_N} \qquad N \gg 1$$

where the dynamics will take place.

A cartoon of Λ

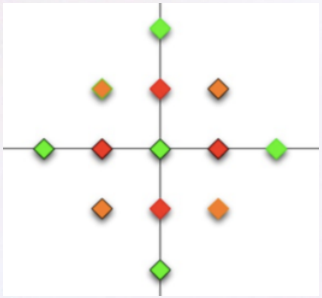
Λ_1

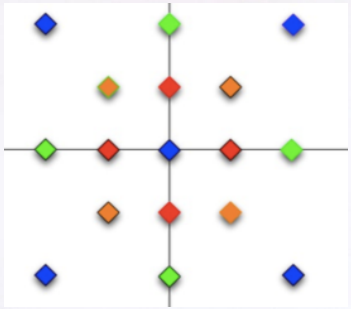




Λ_2

Λ_3





Λ_4 . . . .

Toy Model

$$\begin{cases} -i\dot{b}_j = -|b_j|^2 b_j + 2 b_{j-1}^2 \overline{b_j} + 2 b_{j+1}^2 \overline{b_j} & j = 1, \dots, N \\ b_1(t) = b_N(t) = 0 \rightsquigarrow \text{boundary data} \\ b_j(0) = \tilde{b}_j \rightsquigarrow \text{initial data} \end{cases}$$

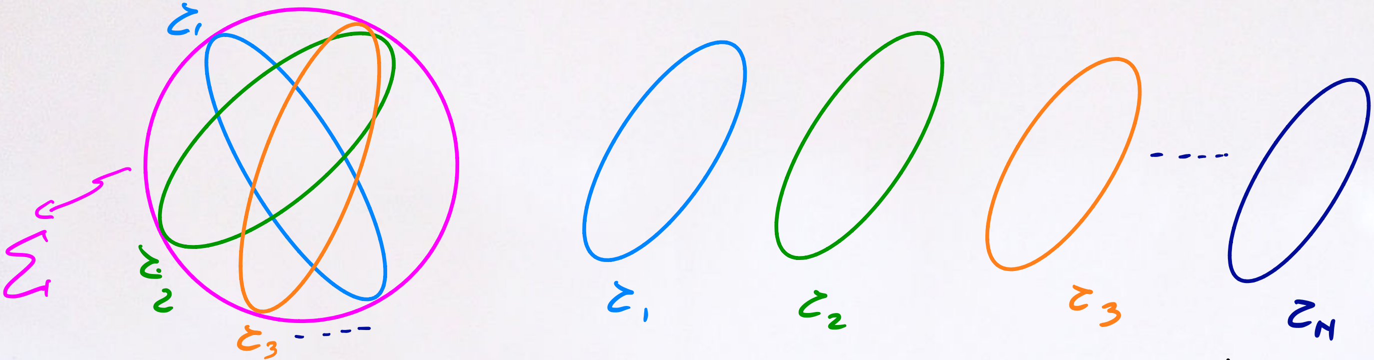
<u>Remark</u>: Although this is not the original system, one can prove that its solutions approximate well those of the original NLS.

This Toy Model conserves mass, momentum and energy.

Its dynamics take place on

$$\boxed{\Sigma_1 = \{ x \in \mathbb{C}^N \,/\, |x|^2 = 1 \}}$$

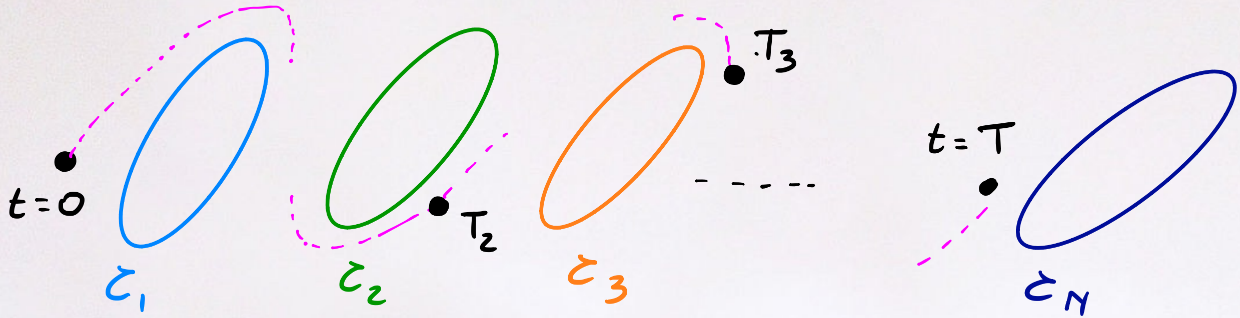
from conservation of mass.



and on Σ_1 there are $\mathcal{C}_j$, $j = 1, \dots, N$ great circles that are invariant.

the heart of the matter

Theorem:



$J=1$
(low frequency)

$J=N$
(high frequency)

(see also Guardia - Kaloshin, Haus - Procesi).

Remarks

* We do not know what happens after time T.

* In the work of Carles - Faou the procedure is different but the same set Λ of frequencies is used

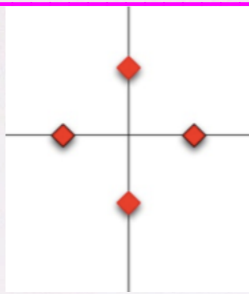
Question: What happens when π^2 is not rational?

Answer: In collaboration with B. Wilson we recently proved that indeed the dynamics in C-K-S-T-T and C-F cannot happen!

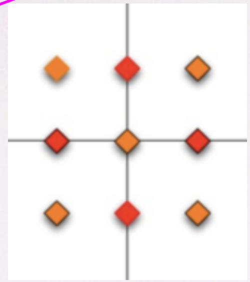
If π^2 is irrational:

$\boxed{\text{Why?}}$

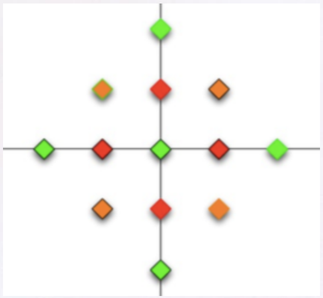
Λ_1



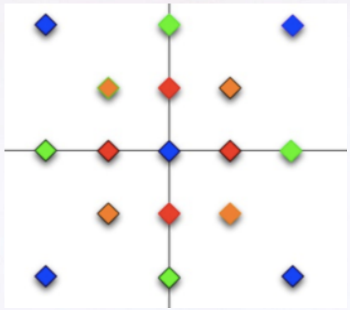
Λ_2



Λ_3



Λ_4



these configurations can't happen!

If π^2 is irrational only rectangles // to axis are allowed, or degenerate rectangles!

More on resonant set

$$\forall \; K = (k^1, k^2) \in \mathbb{Z}^2 \text{ define } \quad |K|_*^2 := \omega_1 (k^1)^2 + \omega_2 (k^2)^2 \quad \text{for } (\omega_1, \omega_2) \in \mathbb{R}_+^2$$

Π^2 rational $\Longrightarrow$ $\omega_1/\omega_2 \in \mathbb{Q}$

Π^2 irrational $\Longrightarrow$ $\omega_1/\omega_2 \in \mathbb{R} \setminus \mathbb{Q}$

For simplicity assume

Π^2 rational $\Longrightarrow$ $(\omega_1, \omega_2) = (1, 1)$ $\quad \boxed{|K|_*^2 = (k^1)^2 + (k^2)^2}$

Π^2 irrational $\Longrightarrow$ $(\omega_1, \omega_2) = (1, \sqrt{2})$ $\quad \boxed{|K|_*^2 = (k^1)^2 + \sqrt{2}\,(k^2)^2}$

The resonant set is $\boxed{\mathcal{R} = \left\{ (k_1, k_2, k_3, k_4) \middle/ \begin{array}{c} k_1 - k_2 + k_3 - k_4 = 0 \\ |k_1|_*^2 - |k_2|_*^2 + |k_3|_*^2 - |k_4|_*^2 = 0 \end{array} \right\}}$

Remark.: When the torus $\mathbb{T}^2$ is rational, that is in our case $|k|_*^2 = (k^1)^2 + (k^2)^2$, first and second components get mixed up.

When the torus $\mathbb{T}^2$ is irrational, that is in our case $|k|_*^2 = (k^1)^2 + \sqrt{2}\,(k^2)^2 \Rightarrow \mathcal{R} = R_1 \cap R_2$

$$\boxed{R_i := \left\{ (k_1, k_2, k_3, k_4) \middle/ \begin{array}{c} k_1^i - k_2^i + k_3^i - k_4^i = 0 \\ (k_1^i)^2 - (k_2^i)^2 + (k_3^i)^2 - (k_4^i)^2 = 0 \end{array} \right\}}$$

Complete decoupling by coordinates!

Conclusions

* In the irrational case the resonant set decouples into two 1D resonant sets.
(Recall that the 1D cubic NLS is integrable ⟹ no energy transfer!)

* We are not claiming that on irrational tori there is no energy transfer, but the mechanism for growth of Sobolev norms cannot be the one in C-K-S-T-T or C-F.

Research directions

1) The periodic focusing NLS
2) A direct proof of Strichartz estimates in $\mathbb{T}^d$
3) Energy transfer: polynomial bounds for Sobolev norms
4) Energy transfer: construction of solutions with growing Sobolev norms.
5) Understanding better the rational and irrational cases.
6) More numerical examples.
7) Prove more results in ANT via theorems in HA.


Thank you!