18.155, FALL 2021, PROBLEM SET 8

Review / helpful information:

- $\langle\xi\rangle := \sqrt{1+|\xi|^2}$. Note that $C^{-1}(1+|\xi|) \le \langle\xi\rangle \le C(1+|\xi|)$ for some global constant $C > 0$ and $\langle\xi\rangle$ is smooth in ξ.
- Plancherel Theorem: for all $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$ we have $\langle\hat\varphi, \hat\psi\rangle_{L^2(\mathbb{R}^n)} = (2\pi)^n \langle\varphi, \psi\rangle_{L^2(\mathbb{R}^n)}$.
- Sobolev space $H^s(\mathbb{R}^n)$: $u \in \mathscr{S}'(\mathbb{R}^n)$ lies in $H^s(\mathbb{R}^n)$ if and only if $\langle\xi\rangle^s \hat u(\xi) \in L^2(\mathbb{R}^n)$. Define $\|u\|_{H^s} := (2\pi)^{-n/2} \|\langle\xi\rangle^s \hat u(\xi)\|_{L^2(\mathbb{R}^n)}$.
- Note that $H^0 = L^2$ and $H^t \subset H^s$ when $t \ge s$.
- If $s \in \mathbb{N}_0$ is a nonnegative integer, then $u \in \mathcal{S}'(\mathbb{R}^n)$ lies in $H^s(\mathbb{R}^n)$ if and only if each distributional derivative $\partial^\alpha u$, $|\alpha| \le s$, lies in $L^2(\mathbb{R}^n)$.
- If $0 < s < 1$, then for each $u \in L^2(\mathbb{R}^n)$
$$u \in H^s(\mathbb{R}^n) \iff \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dxdy < \infty. \tag{1}$$
- Local Sobolev spaces: if $U \subset \mathbb{R}^n$ is open, then $H^s_{\mathrm{loc}}(U) \subset \mathcal{D}'(U)$ is defined as follows: $u \in \mathcal{D}'(U)$ lies in $H^s_{\mathrm{loc}}(U)$ if and only if $\psi u \in H^s(\mathbb{R}^n)$ for each $\psi \in C^\infty_{\mathrm{c}}(U)$. (Here ψu is in $\mathcal{E}'(U)$ which naturally embeds into $\mathcal{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$.)
- Sobolev spaces with compact support: if $U \subset \mathbb{R}^n$ is open, then $H^s_{\mathrm{c}}(U) \subset \mathcal{E}'(U)$ consists of elements of $H^s(\mathbb{R}^n)$ whose support is contained in U.
- Hölder space $C^\gamma(\mathbb{R}^n)$, $0 < \gamma < 1$: a function $u \in C^0(\mathbb{R}^n)$ lies in $C^\gamma(\mathbb{R}^n)$ if for each compact set $K \subset \mathbb{R}^n$ there exists a constant C such that for all $x, y \in K$ we have $|u(x) - u(y)| \le C|x-y|^\gamma$. The space $C^\gamma_{\mathrm{c}}(\mathbb{R}^n)$ consists of compactly supported functions in $C^\gamma(\mathbb{R}^n)$.
- Constant coefficient differential operators of order $m \in \mathbb{N}_0$ have the form $P = \sum_{|\alpha|\le m} c_\alpha D_x^\alpha$ where $c_\alpha \in \mathbb{C}$ and $D := -i\partial$. The principal symbol is $p_0(\xi) = \sum_{|\alpha|=m} c_\alpha \xi^\alpha$. We say P is elliptic if the equation $p_0(\xi) = 0$ has no solutions $\xi \in \mathbb{R}^n \setminus \{0\}$.

1. Fix $s \in \mathbb{R}$. This exercise shows that $H^{-s}(\mathbb{R}^n)$ is dual to $H^s(\mathbb{R}^n)$ with respect to the usual pairing
$$(f, g) := \int_{\mathbb{R}^n} f(x) g(x)\, dx. \tag{2}$$
(Note: since H^s is a Hilbert space, Riesz representation theorem shows that H^s is dual to itself, but this duality features the inner product $\langle\bullet,\bullet\rangle_{H^s}$ rather that (2).)

(a) Show that there exists a unique bilinear map
$$u \in H^s(\mathbb{R}^n), v \in H^{-s}(\mathbb{R}^n) \;\mapsto\; (u, v) \in \mathbb{C}$$

such that (i) for all $u, v \in \mathscr{S}(\mathbb{R}^n)$, (u, v) is given by (2) and (ii) there exists a constant C such that for all u, v we have the bound $|(u, v)| \leq C\|u\|_{H^s}\|v\|_{H^{-s}}$. A consequence of this is that each $v \in H^{-s}(\mathbb{R}^n)$ defines a bounded linear functional on $H^s(\mathbb{R}^n)$ by the rule $u \mapsto (u, v)$.

(b) Assume that $F : H^s(\mathbb{R}^n) \to \mathbb{C}$ is a bounded linear functional. Show that there exists $v \in H^{-s}(\mathbb{R}^n)$ such that $F(u) = (u, v)$ for all $u \in H^s(\mathbb{R}^n)$.

2. This exercise studies the relation between the spaces $C^k(\mathbb{R}^n)$ of k times continuously differentiable functions and the Sobolev spaces $H^s(\mathbb{R}^n)$.

(a) Show that for each $k \in \mathbb{N}_0$, the space $C^k_{\mathrm{c}}(\mathbb{R}^n)$ (where 'c' stands for 'compactly supported') embeds into $H^k(\mathbb{R}^n)$: that is, $C^k_{\mathrm{c}}(\mathbb{R}^n) \subset H^k(\mathbb{R}^n)$ and for each sequence $u_j \in C^k_{\mathrm{c}}(\mathbb{R}^n)$ converging to 0 (in a way similar to convergence in C^∞_{c} but with only k derivatives), we have $\|u_j\|_{H^k(\mathbb{R}^n)} \to 0$ as well.

(b) Show the following version of *Sobolev embedding*: if $k \in \mathbb{N}_0$ and $s > k + \frac{n}{2}$ then $H^s(\mathbb{R}^n)$ embeds into the space $\widetilde{C}^k(\mathbb{R}^n)$ of functions in $C^k(\mathbb{R}^n)$ with bounded derivatives up to order k. (Hint: for $u \in \mathscr{S}(\mathbb{R}^n)$, use Fourier inversion formula and the Cauchy–Schwarz inequality to bound the $\widetilde{C}^k$ norm of u by $\|\langle\xi\rangle^k \hat{u}(\xi)\|_{L^1}$, which is bounded in terms of $\|u\|_{H^s}$. Now, each $u \in H^s(\mathbb{R}^n)$ can be approximated by Schwartz functions, and this approximating sequence will be a Cauchy sequence in $\widetilde{C}^k$, which is a Banach space – this step is similar to the proof of the Continuous Linear Extension theorem.)

3. (Optional) This exercise extends the previous one by comparing Sobolev spaces with Hölder spaces. Assume that $0 < \gamma < 1$.

(a) Show that $C^\gamma_{\mathrm{c}}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ for each $s < \gamma$. (Hint: use (1). Note that the integral there is bounded for any $u \in L^2(\mathbb{R}^n)$ if we restrict to the region $|x - y| \geq 1$.)

(b) Show that $H^s(\mathbb{R}^n) \subset C^\gamma(\mathbb{R}^n)$ for each $s > \gamma + \frac{n}{2}$. (Hint: write each $u \in H^s(\mathbb{R}^n)$ in terms of $\hat{u}$ using the Fourier inversion formula, and use the inequality $|e^{ix\cdot\xi} - e^{iy\cdot\xi}| = |e^{i(x-y)\cdot\xi} - 1| \leq C_\gamma |x - y|^\gamma |\xi|^\gamma$.)

4. Let $U \subset \mathbb{R}^n$ be an open set. Assume that P is an elliptic constant coefficient differential operator of order m. Following Step 2 of the proof of Elliptic Regularity II in §12.2 of the lecture notes, show that for each $u \in \mathcal{D}'(U)$ such that $Pu \in H^{s-m}_{\mathrm{loc}}(U)$, we have $u \in H^s_{\mathrm{loc}}(U)$. (You do not need to reprove the existence of elliptic parametrix.)

5. For the distributions below, find out for which s they lie in $H^s(\mathbb{R}^n)$:

(a) δ_0;

(b) the indicator function of the some interval $[a, b] \subset \mathbb{R}$ (here $n = 1$).

6. (Optional) This exercise forms the basis for the theorem about restricting elements of Sobolev spaces to hypersurfaces, which is important for the study of boundary value problems. We write elements of $\mathbb{R}^n$ as (x_1, x') where $x' \in \mathbb{R}^{n-1}$, and consider the restriction operator to $\{x_1 = 0\}$,

$$T : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^{n-1}), \quad T\varphi(x') = \varphi(0, x').$$

Show that when $s > \frac{1}{2}$, there exists a constant C such that we have the bound

$$\|T\varphi\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})} \le C\|\varphi\|_{H^s(\mathbb{R}^n)} \quad \text{for all} \quad \varphi \in \mathscr{S}(\mathbb{R}^n).$$

Thus by Continuous Linear Extension T extends to a bounded operator $H^s(\mathbb{R}^n) \to H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$. (Hint: use Fourier Inversion Formula to write the Fourier transform of $T\varphi$ in terms of the integral of $\hat{\varphi}$ in the ξ_1 variable. Next, if $v \in L^2(\mathbb{R}^n)$, then we can use Cauchy–Schwartz to estimate $\int_{\mathbb{R}} \langle \xi \rangle^{-s} v(\xi_1, \xi')\, d\xi_1$ in terms of the L^2 norms of the functions $\xi_1 \mapsto (1 + |\xi_1|^2 + |\xi'|^2)^{-s/2}$ and $\xi_1 \mapsto v(\xi_1, \xi')$. It remains to show that the first of these norms is bounded by $C\langle \xi' \rangle^{\frac{1}{2}-s}$.)

7. This exercise establishes coordinate invariance of Sobolev spaces, which is key for defining Sobolev spaces on manifolds. Assume that $U, V \subset \mathbb{R}^n$ are open sets and $\Phi : U \to V$ is a C^∞ diffeomorphism. Recall the pullback operator $\Phi^* : \mathcal{E}'(V) \to \mathcal{E}'(U)$. We will show that

$$v \in H^s_{\mathrm{c}}(V) \quad \Longrightarrow \quad \Phi^* v \in H^s_{\mathrm{c}}(U) \tag{3}$$

and for each compact $K \subset V$ there exists a constant C such that $\|\Phi^* v\|_{H^s} \le C\|v\|_{H^s}$ for all $v \in H^s_{\mathrm{c}}(V)$ such that $\operatorname{supp} v \subset K$. (A similar argument shows that Φ^* maps $H^s_{\mathrm{loc}}(V)$ to $H^s_{\mathrm{loc}}(U)$ as well.)

(a) Show (3) when s is a nonnegative integer. (Hint: use the Chain Rule.)

(b) Show (3) when $0 < s < 1$. You may use the following stronger version of (1): if $A(u)$ is the square root of the right-hand side of (1) then for all $u \in L^2(\mathbb{R}^n)$

$$\|u\|_{H^s} \le C(\|u\|_{L^2} + A(u)), \qquad A(u) \le C\|u\|_{H^s}.$$

(c) (Optional) Show (3) for all $s \in \mathbb{R}$. (Hint: show that for $s \ge 0$, a function $u \in H^s(\mathbb{R}^n)$ lies in $H^{s+1}(\mathbb{R}^n)$ if and only if $\partial_{x_j} u \in H^s(\mathbb{R}^n)$ for all j, and reduce to parts (a)–(b). For $s < 0$ and $v \in H^s_{\mathrm{c}}(V)$, show that the functional $\varphi \in \mathscr{S}(\mathbb{R}^n) \mapsto (\Phi^* v, \varphi)$ is bounded in terms of the H^{-s} norm of φ and thus extends to a bounded functional on $H^{-s}(\mathbb{R}^n)$, and use Exercise 1.)