§5. Homogeneous distributions

§5.1. Basics

<u>Defn.</u> A function $f: \mathbb{R}^n \to \mathbb{C}$ is <u>homogeneous</u> of degree $a \in \mathbb{C}$ if
$$f(tx) = t^a f(x)$$
for all $t > 0$, $x \in \mathbb{R}^n$.

Here $t^a := \exp(a \cdot \log t)$ for a complex

Homogeneous functions show up quite often, e.g. as fundamental solutions of constant coefficient PDEs (more on that later).

But sometimes these "functions" are actually distributions.

¿How to define homogeneity for distributions?

If $f \in L^1_{loc}(\mathbb{R}^n)$ is homogeneous of degree a then $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$, $t > 0$

$$\int_{\mathbb{R}^n} f(x)\varphi(x)\,dx = \int_{\mathbb{R}^n} f(ty)\varphi(ty)t^n\,dy$$

$$= \int_{\mathbb{R}^n} t^a f(y)\,\varphi(ty)\,t^n\,dy.$$

That is, $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$, $t > 0$

$$(*) \quad (f, \varphi) = t^a (f, \varphi_t)$$

where $\varphi_t(x) = t^n \varphi(tx) \in C_c^\infty(\mathbb{R}^n)$.

Defn. We say $f \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous of degree $a \in \mathbb{C}$ if $(*)$ holds.

Examples:

1. $f(x) = 1$ is homogeneous of degree 0
2. $f(x) = \delta_0(x)$:

$$(\delta_0, \varphi_t) = \varphi_t(0) = t^n \varphi(0) = t^n (\delta_0, \varphi)$$

So δ_0 is homogeneous of degree $-n$

A couple of general properties, with proofs omitted (see Hörmander, §3.2)

1 $f \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous of degree $a \iff f$ solves Euler's Equ:

$$(x_1 \partial_{x_1} + \cdots + x_n \partial_{x_n}) f = af.$$

2 [H, Thm. 3.2.3] If $f \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree a (i.e. $(*)$ holds $\forall \varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$) and $a \notin \{-n, -n-1, -n-2, \dots\}$ then there exists unique $u \in \mathcal{D}'(\mathbb{R}^n)$ homogeneous of degree a such that $u|_{\mathbb{R}^n \setminus \{0\}} = f$.

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§5.2. Homogeneous distributions on $\mathbb{R}$

Here is a basic example:

$$x_+^a := \begin{cases} x^a, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases}$$

is homogeneous of degree a

Note: Heaviside fn $H(x) = x_+^0$.

(There is also $x_-^a = (-x)_+^a$ which has similar properties.)

But x_+^a only lies in $L^1_{loc}(\mathbb{R})$ if $\operatorname{Re} a > -1$.

We will explain how to extend x_+^a to all $a \in \mathbb{C} \setminus -\mathbb{N}$ where $-\mathbb{N} = \{-1, -2, -3, \dots\}$.

To do this, we note that

$$(\star)\quad \partial_x x_+^a = a x_+^{a-1} \text{ in } \mathcal{D}'(\mathbb{R}) \text{ for } \operatorname{Re} a > 0.$$

That is, $\forall \varphi \in C_c^\infty(\mathbb{R})$ we have

$$-\int_0^\infty x^a \varphi'(x)\, dx = \int_0^\infty a x^{a-1} \varphi(x)\, dx$$

which can be checked by integration by parts.

We use $(\star)$ to define

$$x_+^a := \frac{\partial_x x_+^{a+1}}{a+1} \quad \text{for } \operatorname{Re} a > -2, \ a \neq -1$$

which agrees with the old x_+^a when $\operatorname{Re} a > -1$

And repeat this to define $x_+^a \in \mathcal{D}'(\mathbb{R})$ for
$\mathrm{Re}\, a > -k-1$, $a \notin \{-1, \dots, -k\}$
by $$x_+^a := \frac{\partial_x^k x_+^{a+k}}{(a+k)(a+k-1)\cdots(a+1)}.$$
That is, $\forall \varphi \in C_c^\infty(\mathbb{R})$
$$(x_+^a, \varphi) := \frac{(-1)^k}{(a+k)\cdots(a+1)} \int_0^\infty x^{a+k} \varphi(x)\, dx$$
and this definition does not depend on k (in the region $\mathrm{Re}\, a > -k-1$).

A bit of complex analysis:

x_+^a depends holomorphically on a
i.e. $\forall \varphi \in C_c^\infty(\mathbb{R})$,
(x_+^a, φ) is a holomorphic function of $a \in \mathbb{C} \setminus -\mathbb{N}$.
This gives the unique analytic extension of x_+^a from $\{\mathrm{Re}\, a > -1\}$ to $\mathbb{C} \setminus -\mathbb{N}$.

What happens at $a \in -\mathbb{N}$?

Look e.g. at $a = -1$.

For $\operatorname{Re} a > -2$, $a \neq -1$ we had the formula

$$x_+^a = \frac{\partial_x x_+^{a+1}}{a+1}.$$

Now let's Taylor expand x_+^{a+1} at $a=-1$, in $\mathcal{D}'$ in x!

$$x_+^{a+1} = \underbrace{H(x)}_{\text{Heaviside fn}} + (a+1)[\log x]_+ + O(|a+1|^2)$$

$$\boxed{[\log x]_+ = \begin{cases} \log x, & x>0 \\ 0, & x<0 \end{cases}}$$

i.e. $\forall \varphi \in C_c^\infty(\mathbb{R})$,

$$\int_0^\infty x^{a+1}\varphi(x)\,dx = \int_0^\infty \varphi(x)\,dx + \int_0^\infty \varphi(x)\log x\,dx + O(|a+1|^2)$$

So then

$$\partial_x x_+^{a+1} = \delta_0(x) + (a+1)\partial_x[\log x]_+ + O(|a+1|^2).$$

Get that x_+^a is meromorphic at $a=-1$ and has the Laurent expansion

$$x_+^a = \frac{\delta_0(x)}{a+1} + \partial_x[\log x]_+ + O(|a+1|^2)$$

What is $\partial_x [\log x]_+$?

Denoting $v(x) := \partial_x [\log x]_+ \in \mathcal{D}'(\mathbb{R})$,

get $\forall \varphi \in C_c^\infty(\mathbb{R})$

$$(v, \varphi) = -\int_0^\infty \varphi'(x) \log x \, dx$$

$$= -\lim_{\varepsilon \to 0+} \int_\varepsilon^\infty \varphi'(x) \log x \, dx$$

$$\overset{(IBP)}{=} \lim_{\varepsilon \to 0+} \left[\varphi(\varepsilon) \log \varepsilon + \int_\varepsilon^\infty \frac{\varphi(x)}{x} dx \right]$$

$$= \lim_{\varepsilon \to 0+} \left[\int_\varepsilon^\infty \frac{\varphi(x)}{x} dx + \varphi(0) \log \varepsilon \right]$$

A few curious properties of v:

- $\operatorname{supp} v = [0, \infty)$
- $v|_{(0,\infty)} = \frac{1}{x}$ ← locally integrable on $(0,\infty)$ but not on $\mathbb{R}$
- $x v = H(x)$
- v is <u>not</u> homogeneous (will skip)

A similar but more well-known distribution

is $\text{P.V.} \frac{1}{x} := \partial_x \log |x|$

↑ "principal value"

We compute $\forall \varphi \in C_c^\infty(\mathbb{R})$

$$\left(\text{p.v.} \frac{1}{x}, \varphi\right) = -\int_{\mathbb{R}} \varphi'(x) \log|x| \, dx$$

$$= -\lim_{\varepsilon \to 0+} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \varphi'(x) \log|x| \, dx$$

$$\overset{\text{IBP}}{=} \lim_{\varepsilon \to 0+} \left[\left(\varphi(\varepsilon) - \varphi(-\varepsilon)\right) \log \varepsilon + \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \frac{\varphi(x)}{x} \, dx \right]$$

$$= \lim_{\varepsilon \to 0+} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \frac{\varphi(x)}{x} \, dx$$

Some properties:

- $\left(\text{p.v.} \frac{1}{x}\right)\Big|_{\mathbb{R} \setminus \{0\}} = \frac{1}{x}$
- $x \cdot \left(\text{p.v.} \frac{1}{x}\right) = 1$
- $\text{p.v.} \frac{1}{x}$ is homogeneous of degree -1