§18. Vector bundles & Hodge Theory

§18.1. Vector bundles

Let M be a manifold, $\dim M = n$.

An m-dimensional (complex) <u>vector bundle</u> over M is a collection of m-dimensional complex vector spaces $(\mathcal{E}_x)_{x\in M}$ which "depend smoothly on x".

More precisely,

Defn A vector bundle is an $n+2m$ dimensional manifold $\mathcal{E}$ and a projection map $\pi : \mathcal{E} \to M$ which is onto. Each fiber
$$\mathcal{E}_x := \pi^{-1}(x) \subset \mathcal{E}, \quad x \in M,$$
should have the structure of an m-dimensional complex vector space

and we should have
local trivializations:

$\forall x_0 \in M$ $\exists$ open set $U \subset M$, $x_0 \in U$
and a C^∞ diffeomorphism

$$\varphi : \underset{\mathcal{E}}{\underset{\cap}{\pi^{-1}(U)}} \longrightarrow U \times \mathbb{C}^m$$

such that $\forall x \in U$,
φ is a linear isomorphism from
$\mathcal{E}_x$ onto the space
$\{(x, w) \mid w \in \mathbb{C}^m\}$.

If we change to a different
trivialization $\tilde{\varphi} : \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^m$
then we have the transition formula:

$$\tilde{\varphi}(x, v) = (x, \underset{}{\overset{\mathbb{C}^m}{\overset{\cup}{\tilde{w}}}}), \quad \varphi(x, v) = (x, \overset{\mathbb{C}^m}{\overset{\cup}{w}}) \Rightarrow$$

$$\Rightarrow \tilde{w} = A(x) w$$

where $A(x)$ is an invertible complex $m \times m$ matrix which is C^∞ in x, i.e.

$$A \in C^\infty(U; GL(m, \mathbb{C})).$$

Examples:

1. The trivial bundles
$\mathcal{E} = M \times \mathbb{C}^m$, $\pi(x, v) = x$

2. The tangent bundle (complexified)
$\mathcal{E} = T_{\mathbb{C}} M$, $\mathcal{E}_x = T_x M \otimes \mathbb{C}$

3. The cotangent bundle $T^*_{\mathbb{C}} M$
(will stop writing the subscript $\mathbb{C}$ from now on)

4. The bundle of densities:
$|\Omega|$, 1-dimensional;

$$\Big\}$$

for each $x \in M$,
$|\Omega|_x$ is the space of maps

$$\alpha_x : \underbrace{T_x M \times \cdots \times T_x M}_{n \text{ times}} \longrightarrow \mathbb{C}$$

which are multilinear and satisfy
for each $n \times n$ real matrix $A = (a_{jk})$
and vectors $v_1, \dots, v_n \in T_x M$

$$\alpha_x(\tilde{v}_1, \dots, \tilde{v}_n) = |\det A| \cdot \alpha_x(v_1, \dots, v_n)$$

where $\tilde{v}_j := \sum_{k=1}^{n} a_{jk} v_k$.

One can check that this does give a 1-D complex vector bundle over M.

Sections of vector bundles:

We say that a map

$x \in M \mapsto u(x) \in \mathcal{E}_x$

is a C^∞ section of the vector bundle $\mathcal{E}$ if it is C^∞ in each trivialization

$\varphi : \pi^{-1}(U) \to U \times \mathbb{C}^m$:

for $x \in U$, $\varphi(u(x)) = (x, v(x))$ for some C^∞ map $v : U \to \mathbb{C}^m$.

Denote by $C^\infty(M; \mathcal{E})$ the space of C^∞ sections.

Similarly can define $L^p_{loc}(M; \mathcal{E})$

<u>Examples</u>:

- $C^\infty(M; TM)$ = (complexified) vector fields
- $C^\infty(M; T^*M)$ = 1-forms (more later)
- Any density $u \in L^1_c(M; |\Omega|)$ can be <u>integrated</u>:

$$u \in L^1_c(M;|\Omega|) \longmapsto \int_M u \in \mathbb{C}.$$

For $M \subset \mathbb{R}^n$ open, define

$\int_M u$ to be $\int_M u(x)(e_1, \dots, e_n)\, dx$

where $e_1, \dots, e_n$ is the canonical basis of $\mathbb{R}^n$!

For general M, can use local trivializations & check that the integral does not depend on the trivialization

- If g is a Riemannian metric then $dVol_g$ is a density!

$$dVol_g(x)(v_1, \dots, v_n) = \sqrt{\det B}$$

where $B = (b_{jk})$, $b_{jk} = g(x)(v_j, v_k)$

- If $u \in C^\infty(M)$ is a function and $v \in C_c^\infty(M; |\Omega|)$ is a compactly supported density

then $u \cdot v \in C_c^\infty(M; |\Omega|)$.

Can define invariantly (no need to fix a metric)

$$(u,v) := \int_M u \cdot v$$

So it is more natural to define distributions as

$$\mathcal{D}'(M) = \text{dual space to } C_c^\infty(M; |\Omega|)$$

- If $\mathcal{E}$ is a vector bundle, define $\mathcal{D}'(M; \mathcal{E})$ = dual space to $C_c^\infty(M; \operatorname{Hom}(\mathcal{E}; |\Omega|))$

where $\forall x$, $\operatorname{Hom}(\mathcal{E}; |\Omega|)_x$ is the space of all linear maps $\mathcal{E}_x \to |\Omega|_x$

Indeed, if

$u \in C^\infty(M;\mathcal{E})$, $v \in C_c^\infty(M; \mathrm{Hom}(\mathcal{E};|\Omega|))$

we can define $\forall x \in M$ the pairing

$$\langle u, v\rangle_x = v_x(u_x) \in |\Omega|_x$$

And then define the pairing

$$(u,v) = \int_M \langle u, v\rangle \quad \text{where}$$

$$\langle u,v\rangle \in C_c^\infty(M;|\Omega|)$$

- Can define Sobolev spaces

$$H^s_{loc}(M;\mathcal{E}) \subset \mathcal{D}'(M;\mathcal{E})$$

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§18.2 Differential operators on vector bundles

First the case of trivial bundles:

if M is a manifold

and we consider trivial bundles

$\mathbb{C}^{\ell}$ "=" $M \times \mathbb{C}^{\ell}$

$\mathbb{C}^{\ell'}$ "=" $M \times \mathbb{C}^{\ell'}$

Then an operator

$$P : C^{\infty}(M; \mathbb{C}^{\ell'}) \to C^{\infty}(M; \mathbb{C}^{\ell})$$

is called a differential operator of order m if it's a matrix of differential operators: $\forall \vec{u} = (u_1, \dots, u_{\ell'}) \in C^{\infty}(M; \mathbb{C}^{\ell'})$ where the components $u_j \in C^{\infty}(M; \mathbb{C})$

We have $\displaystyle (P\vec{u})_j = \sum_{j'=1}^{\ell'} P_{jj'} u_{j'}$

where $P_{jj'} \in \mathrm{Diff}^m(M)$.

Denote by $\mathrm{Diff}^m(M; \mathbb{C}^{\ell'} \to \mathbb{C}^{\ell})$ the space of all such operators.

- Basic example:

if $M \subset \mathbb{R}^n$ is open then

$$d : C^\infty(M;\mathbb{C}) \to C^\infty(M;\mathbb{C}^n)$$

is in Diff^1

Here $df = (\partial_{x_1} f, \dots, \partial_{x_n} f)$

- The principal symbol of P

is the matrix $(\sigma_m(P_{jj'}))_{j,j'}$.

We think of it as a map from

T^*M into $\mathrm{Hom}(\mathbb{C}^{\ell'};\mathbb{C}^\ell)$

space of linear maps $\mathbb{C}^{\ell'} \to \mathbb{C}^\ell$.

Namely, if $(x,\xi) \in T^*M$ and

$\vec{v} = (v_1, \dots, v_{\ell'}) \in \mathbb{C}^{\ell'}$ then

$\sigma_m(P)(x,\xi)\cdot\vec{v} \in \mathbb{C}^\ell$ is given by

$$(\sigma_m(P)(x,\xi)\vec{v})_j = \sum_{j'=1}^{\ell'} \sigma_m(P_{jj'})\, v_{j'}.$$

Example: if $M \subset \mathbb{R}^n$ open

then $\sigma_1(d)(x,\xi) = i\xi$. $(i = \sqrt{-1})$

Here $x \in M$, $\xi \in \mathbb{R}^n$, and the ξ on the RHS is the map $t \in \mathbb{C} \mapsto t\cdot\xi \in \mathbb{C}^n$.

- Can define differential operators on sections of vector bundles: if $\mathcal{E}, \mathcal{F}$ are vector bundles over M then can define $\mathrm{Diff}^m(M; \mathcal{E} \to \mathcal{F})$ by using local trivializations.

And can define for $P \in \mathrm{Diff}^m(M; \mathcal{E} \to \mathcal{F})$ the principal symbol $\sigma_m(P)$: for $x \in M$, $\xi \in T_x^* M$,

$\sigma_m(P)(x,\xi)$ is a linear map
$\mathcal{E}_x \to \mathcal{F}_x$.

[Can write $\sigma_m(P) \in C^\infty(T^*M; \pi^* \mathrm{Hom}(\mathcal{E} \to \mathcal{F}))$
where $\mathrm{Hom}(\mathcal{E} \to \mathcal{F})$ is the bundle (over M)
of linear maps $\mathcal{E}_x \to \mathcal{F}_x$
and π^*(it) is the pullback
of that bundle by $\pi: T^*M \to M$ ...]

Elliptic Regularity II,
Elliptic Estimate, and
Fredholm mapping properties (if M compact)
still hold for operators $P \in \mathrm{Diff}^m$
on vector bundles which are
<u>elliptic</u> in the following sense:
$\forall (x,\xi) \in T_x M$, $\xi \neq 0$, the map
$\sigma_m(P)(x,\xi): \mathcal{E}_x \to \mathcal{F}_x$ is invertible (need $\dim \mathcal{E} = \dim \mathcal{F}$)

Why so? Can reduce to $M \subset \mathbb{R}^n$ open, $\mathcal{E} = \mathcal{F} = \mathbb{C}^\ell$.

Then $\underline{P} = (P_{jj'})$ is an $\ell \times \ell$ matrix of differential operators.

Can construct <u>elliptic parametrix</u> as a matrix of pseudo diff. operators:

$$Q = \left(\mathrm{Op}(q_{jj'})\right)_{j,j'=1}^{\ell}, \quad \text{i.e.}$$

$$(Q\vec{u})_j = \sum_{j'=1}^{\ell} \mathrm{Op}(q_{jj'})\, u_{j'}$$

where $\vec{u} = (u_1, \dots, u_\ell) \in \mathcal{D}'(M; \mathbb{R}^\ell)$.

The elliptic parametrix construction works similarly to the scalar case $\ell = 1$ except we start with $q^0_{jj'}(x,\xi)$ s.t. $(q^0_{jj'}(x,\xi))_{j,j'}$ = inverse of the matrix $(\sigma_m(P_{jj'})(x,\xi))_{j,j'=1}^{\ell}$

§18.3. Differential forms

Let M be a compact manifold.

Assume also it's oriented:

$\forall x \in M$ & a basis $v_1, \dots, v_n$ of $T_x M$

can decide if $(v_1, \dots, v_n)$ is positively or negatively oriented in a way which depends continuously on x.

For $0 \leq k \leq n$, define the vector bundle of (complexified) k-forms on M as

$$\Omega^k := \Lambda^k T^*_{\mathbb{C}} M$$

Here Λ^k stands for antisymmetric k-th tensor power.

That is, for each $x \in M$, Ω^k_x consists of maps $\underbrace{T_x M \times \dots \times T_x M}_{k \text{ times}} \to \mathbb{C}$

which are multilinear and change sign if we permute two arguments. Note: $\Omega^1 = T_{\mathbb{C}}^* M$.

Sections in $C^\infty(M; \Omega^k)$ are called differential k-forms.

A lot of wonderful properties: in particular,

- $\forall$ coordinate system $x : U \to V$, ($U \subset M$, $V \subset \mathbb{R}^n$)

$$x(y) = (x_1(y), \dots, x_n(y)), \quad y \in M$$

Can define the differentials

$$dx_1, \dots, dx_n \in C^\infty(U; \Omega^1)$$

and a basis of each Ω^k_x, $x \in U$, is given by $dx_{j_1} \wedge \dots \wedge dx_{j_k}$ where $1 \le j_1 < \dots < j_k \le n$.

- **Form differential:**

$$d : C^\infty(M;\Omega^k) \to C^\infty(M;\Omega^{k+1})$$

In coordinates,

$$d(f(x)\,dx_{j_1}\wedge\dots\wedge dx_{j_k})$$
$$= df\wedge dx_{j_1}\wedge\dots\wedge dx_{j_k}$$

where $df = \sum_{\ell=1}^{n} \partial_{x_\ell} f \cdot dx_\ell \in C^\infty(U;\Omega^1)$

We have $\boxed{d^2=0}$

- **Integration:** if $\omega\in C^\infty(M;\Omega^n)$, $n=\dim M$, then can define $\int_M \omega$. (This is where we use that M is oriented)

- **Stokes Thm** (M cpct, no boundary): $\forall \omega\in C^\infty(M;\Omega^{n-1})$, we have $\int_M d\omega = 0$.

· de Rham cohomology:

(should really be $\mathbb{R}$ instead of $\mathbb{C}$)

for $0 \le k \le n$, the k-th cohomology group is

$$H^k_{dR}(M;\mathbb{C}) := \frac{\{\omega \in C^\infty(M;\Omega^k) \mid d\omega = 0\}}{\{d\beta \mid \beta \in C^\infty(M;\Omega^{k-1})\}}.$$

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§18.4. Hodge Theory

Now assume we fix a Riemannian metric g on M. This defines a volume form $d\mathrm{Vol}_g$ & an inner product on each Ω^k.

The latter is defined as follows:

- $k=1 \to$ the inner product on T^*M given by g

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- $k>1$: for $\alpha_1, \dots, \alpha_k \in \Omega^1_x$
$\beta_1, \dots, \beta_k \in \Omega^1_x$

define $\langle \alpha_1 \wedge \dots \wedge \alpha_k, \beta_1 \wedge \dots \wedge \beta_k \rangle_g$
to be the determinant
of the matrix with entries
$\left(\langle \alpha_j, \beta_\ell \rangle_g \right)_{j,\ell=1}^{k}$.
(Turns out this extends to
an inner product on $\Omega^k_x \dots$)

- Now define the L^2 inner product
on $C^\infty(M; \Omega^k)$ by

$$\langle u, v \rangle_{L^2} := \int_M \langle u(x), \overline{v(x)} \rangle_g \, d\mathrm{Vol}_g(x)$$

for all $u, v \in C^\infty(M; \Omega^k)$.

- The operator $d_k : C^\infty(M; \Omega^k) \to C^\infty(M; \Omega^{k+1})$
is a differential operator of order 1:

$$d_k(f\,dx_{j_1}\wedge\cdots\wedge dx_{j_k})$$
$$=\sum_{\ell=1}^{n}\partial_{x_\ell}f\;dx_\ell\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_k}$$

this is 0 if $\ell\in\{j_1,\dots,j_k\}$

and its principal symbol

$\sigma_1(d_k)$ is given by

$$\sigma_1(d_k)(x,\xi)(\alpha)=i(\xi\wedge\alpha)$$

for all $x\in M$, $\xi\in T_x^*M=\Omega_x^1$, $\alpha\in\Omega_x^k$, $\xi\wedge\alpha$ is the wedge product, $\xi\wedge\alpha\in\Omega_x^{k+1}$.

- Now define the differential operator $\delta_k: C^\infty(M;\Omega^k)\to C^\infty(M;\Omega^{k-1})$ as the adjoint of d_{k-1}: $\delta_k=d_{k-1}^*$

That is, for all

$u \in C^\infty(M; \Omega^k)$

$v \in C^\infty(M; \Omega^{k-1})$ we have

$$\langle \delta_k u, v\rangle_{L^2} = \langle u, d_k v\rangle_{L^2}$$

i.e.

$$\int_M \langle \delta_k u(x), v(x)\rangle_g \, d\mathrm{Vol}_g(x) = \int_M \langle u(x), d_k v(x)\rangle_g \, d\mathrm{Vol}_g(x)$$

The principal symbol of δ_k should be the adjoint of the principal symbol of d_{k-1}:

if $(x, \xi) \in T^*M$ and $\alpha \in \Omega^k_x$, $\beta \in \Omega^{k-1}_x$

then $\langle \sigma_1(\delta_k)(x,\xi)\alpha, \beta\rangle_{g(x)} =$

$= -i \langle \alpha, \xi \wedge \beta\rangle_{g(x)}$.

- Consider the bundle of all differential forms

$$\Omega^\bullet := \bigoplus_{k=0}^{n} \Omega^k .$$

Then d_k, δ_k define operators

$$d, \delta : C^\infty(M; \Omega^\bullet) \to C^\infty(M; \Omega^\bullet)$$

and $d, \delta \in \mathrm{Diff}^1(M; \Omega^\bullet \to \Omega^\bullet)$

<u>Key fact</u>: the "Dirac operator"

$$d + \delta \in \mathrm{Diff}^1(M; \Omega^\bullet \to \Omega^\bullet)$$

is <u>elliptic</u> and self-adjoint.

<u>Self-adjointness</u> follows from the fact that $\delta = d^*$.

For <u>ellipticity</u>, we need to show that $\forall\, x \in M$, $\xi \in T_x^* M$, $\xi \neq 0$, the principal symbol $\sigma_1(x,\xi) = \sigma_1(d+\delta)(x,\xi)$

$$\sigma_1(x,\xi) : \Omega^\bullet_x \to \Omega^\bullet_x$$

is an invertible (linear) map.

Can assume that $|\xi|_g = 1$ & pick a system of coordinates such that $dx_1, \dots, dx_n$ is a g-orthonormal basis of $T_x^* M$ (at just one point x) and $\boxed{\xi = dx_1.}$

Then an orthonormal basis of $\Omega^\bullet_x$ is given by

$dx_A := dx_{j_1} \wedge \dots \wedge dx_{j_k}$ where $A = \{j_1, \dots, j_k\}$, $j_1 < \dots < j_k$ goes over subsets of $\{1, \dots, n\}$

We then have

$$\sigma(x,\xi)(dx_{\mathcal{A}}) = \begin{cases} i\, dx_{\{1\}\cup\mathcal{A}}, & \text{if } 1 \notin \mathcal{A} \\ -i\, dx_{\mathcal{A}\setminus\{1\}}, & \text{if } 1 \in \mathcal{A} \end{cases}$$

(the first line comes from d & the second line comes from δ recalling $\sigma = \sigma_1(d+\delta)$)

which is indeed invertible.

- The Hodge Laplacian

$$\Delta_g = (d+\delta)^2 = d\delta + \delta d \quad \text{is}$$

since $d^2 = 0 = \delta^2$

also elliptic (in $\mathrm{Diff}^2(M;\Omega^\bullet \to \Omega^\bullet)$) and self-adjoint.

In fact, $\sigma(\Delta_g)(x,\xi)$ is the multiplication by $|\xi|^2$ on $\Omega^\bullet_x$. (Note: Hodge Δ_g has the opposite sign of "$\Delta = \sum \partial_{x_j}^2$" we used before)

So $\forall s$,

$d+\delta : H^{s+1}(M;\Omega^\bullet) \to H^s(M;\Omega^\bullet)$

and $\Delta_g : H^{s+2}(M;\Omega^\bullet) \to H^s(M;\Omega^\bullet)$

are Fredholm of index 0.

What is the kernel?

<u>Lemma</u> $\operatorname{Ker}(\Delta_g) = \operatorname{Ker}(d+\delta)$

$= \{u \in C^\infty(M;\Omega^\bullet) \mid du=0, \delta u = 0\}$.

This also equals $\bigoplus_{k=0}^{n} \mathcal{H}^k$

where $\mathcal{H}^k := \left\{u \in C^\infty(M;\Omega^k) \;\middle|\; \begin{matrix} du=0 \\ \delta u = 0 \end{matrix}\right\}$

is the space of

<u>harmonic k-forms</u>

And by the Fredholm property

$\dim \mathcal{H}^k < \infty$.

Proof It's easy to see that

$\operatorname{Ker}(\Delta_g) \supset \operatorname{Ker}(d+\delta) \supset \{u : du=0, \delta u=0\}$

So we just need to show that if $u \in C^\infty(M;\Omega^\cdot)$ and $\Delta_g u = 0$ then $du=0$ and $\delta u = 0$.

Compute $0 = \langle \Delta_g u, u\rangle_{L^2}$

$= \langle d\delta u + \delta d u, u\rangle_{L^2}$ (using $\delta = d^*$)

$= \langle \delta u, \delta u\rangle_{L^2} + \langle du, du\rangle_{L^2}$

$= \|\delta u\|^2_{L^2} + \|du\|^2_{L^2}$, so $du=0$, $\delta u = 0$

as needed. $\square$

Now we consider

$d_k(C^\infty) = \{du \mid u \in C^\infty(M;\Omega^k)\} \subset C^\infty(M;\Omega^{k+1})$

$\operatorname{Ker}_{C^\infty} d_k = \{u \in C^\infty(M;\Omega^k) \mid du = 0\}$

and similarly $\delta_k(C^\infty)$, $\operatorname{Ker}_{C^\infty}\delta_k$

Theorem [Hodge decomposition]

We have

$$C^\infty(M;\Omega^k) = \mathcal{H}^k \oplus d_{k-1}(C^\infty) \oplus \delta_{k+1}(C^\infty)$$

and

$$\operatorname{Ker}_{C^\infty} d_k = \mathcal{H}^k \oplus d_{k-1}(C^\infty).$$

In particular, the de Rham cohomology group

$$H^k_{dR}(M;\mathbb{C}) = \operatorname{Ker}_{C^\infty} d_k / d_{k-1}(C^\infty)$$

is isomorphic to $\mathcal{H}^k$ (each cohomology class has a unique harmonic form)

Proof 1. The sum

$\mathcal{H}^k \oplus d_{k-1}(C^\infty) \oplus \delta_{k+1}(C^\infty)$

is direct: assume $u \in \mathcal{H}^k$

$v \in C^\infty(M; \Omega^{k-1})$

$w \in C^\infty(M; \Omega^{k+1})$

and $u + dv + \delta w = 0$.

Apply d: since $du = 0$ & $d^2 = 0$

get $d\delta w = 0$. But then

$0 = \langle d\delta w, w \rangle_{L^2} = \langle \delta w, \delta w \rangle_{L^2} \Rightarrow \delta w = 0$.

Apply δ: get $\delta d v = 0$, so

$0 = \langle \delta d v, v \rangle_{L^2} = \langle dv, dv \rangle_{L^2} \Rightarrow dv = 0$

Thus $u = 0$ as well.

2. We have ($\mathcal{H}^\cdot = \bigoplus_{k=0}^{n} \mathcal{H}^k$)

$$C^\infty(M;\Omega^\cdot) \subset \mathcal{H}^\cdot \oplus d(C^\infty) \oplus \delta(C^\infty).$$

Indeed, take $\beta \in C^\infty(M;\Omega^\cdot)$ and any $s \in \mathbb{R}$.

The operator $d+\delta : H^s(M;\Omega^\cdot) \to H^{s-1}(M;\Omega^\cdot)$ is Fredholm and self-adjoint on L^2 with kernel $\mathcal{H}^\cdot$.

Its range $(d+\delta)(H^s)$ is

$$\{\alpha \in H^{s-1}(M;\Omega^\cdot) \mid \langle \alpha, w\rangle_{L^2} = 0 \ \forall w \in \mathcal{H}^\cdot\}.$$

Thus there exists unique $u \in \mathcal{H}^\cdot$ such that

$$\beta - u \in (d+\delta)(H^s).$$

And then there exists unique $v \in H^s(M;\Omega^\cdot)$ such that

$$\beta = u + (d+\delta)v \quad \text{and} \quad v \perp_{L^2} \mathcal{H}^\cdot$$

Since u, v are unique they have to be the same for all s.

Thus $v \in H^s(M; \Omega^\cdot)$ $\forall s$ which by Sobolev embedding (Pset 8, Exercise 2(b)) shows that $v \in C^\infty(M; \Omega^\cdot)$

Now $\beta = u + dv + \delta v$ as needed.

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3. It remains to show that

$$\mathrm{Ker}_{C^\infty} d = \mathcal{H}^\cdot \oplus d(C^\infty).$$

$\supseteq$ is immediate $(d(\mathcal{H}^\cdot) = 0, d^2 = 0)$.

<u>To show</u> $\subseteq$, take any

$\beta \in \mathrm{Ker}_{C^\infty} d \subset C^\infty(M; \Omega^\cdot)$

and write the Hodge decomposition

$$\beta = u + dv + \delta w, \quad v, w \in C^\infty(M; \Omega^\cdot)$$

Since $d\beta = 0$, $du = 0$, $d^2 v = 0$, get $d\delta w = 0$

which as in Step 1 shows that

$\delta w = 0$. So $\beta = u + dv$

as needed. $\square$

Using that $H^k_{dR}(M; \mathbb{C}) \simeq \mathcal{H}^k$

we can show the de Rham version of <u>Poincaré duality</u>:

$$\dim \mathcal{H}^k = \dim \mathcal{H}^{n-k} \quad \forall k.$$

This is because there exists a 0-th order diff. operator

$$* : C^\infty(M; \Omega^k) \to C^\infty(M; \Omega^{n-k}) \quad \forall k$$

such that $\delta_k : C^\infty(M;\Omega^k) \to C^\infty(M;\Omega^{k-1})$

is given by $\delta_k = (-1)^k *^{-1} d *$

and $** = (-1)^{k(n-k)} I$ on k-forms

From here one can see that

$*: \mathcal{H}^k \to \mathcal{H}^{n-k}$ is

an isomorphism $\forall k$

(i.e. if $du = 0$, $\delta u = 0$ then
$d(*u) = 0$, $\delta(*u) = 0$.)