18.118, SPRING 2022, PROBLEM SET 1

Review/useful information:

- If X is a compact metric space, then $C^0(X)$ is the space of continuous functions on X, with the sup-norm.
- Iterate of a map $T : X \to X$: $T^n = T \circ \cdots \circ T$, n times, for $n \in \mathbb{N}_0$.
- Ergodic average: if $f : X \to \mathbb{R}$ then

$$\langle f \rangle_n = \frac{1}{n} \sum_{j=0}^{n-1} f \circ T^j.$$

- A probability measure μ on X is T-invariant if

$$\mu(T^{-1}(A)) = \mu(A) \quad \text{for all Borel sets} \quad A \subset X.$$

  Equivalently,

$$\int_X f \circ T \, d\mu = \int_X f \, d\mu \quad \text{for all} \quad f \in L^1(X, \mu).$$

- Symmetric difference: $A \triangle B := (A \setminus B) \sqcup (B \setminus A)$.
- Two measures μ, ν on X are mutually singular if there exists a Borel set $A \subset X$ such that $\mu(A) = \nu(X \setminus A) = 0$.

1. Let X be a compact metric space, $T : X \to X$ a continuous map, and μ_0 be a T-invariant probability measure on X. Show that the following two statements are equivalent, **without** using any form of the Ergodic Theorem:

(1) μ_0 is the only T-invariant probability measure;
(2) for each $f \in C^0(X)$ and $x \in X$ we have

$$\frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) \to \int_X f \, d\mu_0 \quad \text{as} \quad n \to \infty.$$

Hint: to show (1) $\Rightarrow$ (2), use Compactness Theorem (see §1.1 in lecture notes) to reduce to the statement that each weak limit of a subsequence of empirical measures of T (see §1.4 in lecture notes) is equal to μ_0; the proof of Krylov–Bogolyubov Theorem shows that each such limit has to be T-invariant. To show (2) $\Rightarrow$ (1), let μ be another T-invariant probability measure. Integrating property (2) with respect to μ, show that $\int_X f \, d\mu = \int_X f \, d\mu_0$ for all $f \in C^0(X)$, which by the uniqueness part of the Riesz Representation Theorem gives $\mu = \mu_0$.

2. Let $X := \mathbb{R}/\mathbb{Z}$ and consider the map $T : X \to X$ given by $T(x) = 2x \bmod \mathbb{Z}$. Let μ be the Lebesgue measure on X. Show that μ is T-invariant. Compute the adjoint U^* of the operator

$$U : L^2(X, \mu) \to L^2(X, \mu), \quad Uf = f \circ T$$

and compute the operators U^*U and UU^*.

3. Assume that X is a compact smooth manifold (without boundary), V is a smooth vector field on X, and $\varphi^t := e^{tV} : X \to X$ is the flow of V. Fix a probability measure μ on X such that:

- μ is φ^t-invariant, i.e. $\mu(\varphi^{-t}(A)) = \mu(A)$ for any Borel set $A \subset X$ and any $t \in \mathbb{R}$;
- μ is ergodic for φ^t: for any Borel set $A \subset X$ which is μ-almost φ^t-invariant (that is, $\mu(A \triangle \varphi^{-t}(A)) = 0$ for all $t \in \mathbb{R}$), we have $\mu(A) = 0$ or $\mu(A) = 1$.

For a function $f : X \to \mathbb{R}$, define the ergodic average

$$\langle f \rangle_T(x) := \frac{1}{T} \int_0^T f(\varphi^t(x))\, dt, \quad x \in X, \quad T > 0.$$

We will show the von Neumann Ergodic Theorem for flows: for each $f \in L^2(X, \mu)$ we have

$$\langle f \rangle_T(x) \to \int_X f\, d\mu \quad \text{as } T \to \infty \quad \text{in } L^2(X, \mu). \tag{1}$$

(The Birkhoff Ergodic Theorem holds as well, but as in the case of maps, it is more difficult to prove than the von Neumann ergodic theorem. But you are welcome to try to adapt the proof that we had in class to flows :))

(a) Define the space $S = \{Vg \mid g \in C^\infty(X)\}$ where Vg is the result of differentiating a function g along the vector field V. Show that $\langle f \rangle_T \to \int_X f\, d\mu = 0$ for all $f \in S$.

(b) (Optional) Show that the orthogonal complement of S in $L^2(X, \mu)$ consists of constant functions. Use this and part (a) to show (1).

4. Let $T : X \to X$ be a Borel measurable map of a metric space.

(a) Assume that μ, ν are two T-ergodic probability measures on X such that $\mu \neq \nu$. Show that μ, ν are mutually singular. (Hint: use the Birkhoff ergodic theorem for a function f such that $\int_X f\, d\mu \neq \int_X f\, d\nu$.)

(b) (Optional) Show that a T-invariant probability measure μ on X is ergodic if and only if it is impossible to write $\mu = \alpha\nu_1 + (1 - \alpha)\nu_2$ where $\alpha \in (0, 1)$ and $\nu_1 \neq \nu_2$ are T-invariant probability measures. (Hint: if μ is decomposed as above and is ergodic, then apply the Birkhoff ergodic theorem and then take the integral with respect to ν_1.)

5. Assume that $T : X \to X$ is a map and $x_0 \in X$ is a periodic point for T. Fix the smallest $r \geq 1$ such that $T^r(x_0) = x_0$ and let μ be the delta-measure on the corresponding orbit:

$$\mu = \frac{1}{r} \sum_{j=0}^{r-1} \delta_{T^j(x_0)}.$$

When is T mixing with respect to μ?

6. (Optional) Let $X := \mathbb{R}^2/\mathbb{Z}^2$. Assume that $A \in \mathrm{SL}(2, \mathbb{Z})$ is hyperbolic and let $T : X \to X$ be the corresponding toral automorphism. Find an asymptotic formula as $n \to \infty$ for the number of periodic points of period n, defined as

$$\mathcal{N}(n) := \#\{x \in X \mid T^n(x) = x\}.$$

Write your answer in the form $\mathcal{N}(n) = (\dots)(1 + o(1))$ as $n \to \infty$, where $(\dots)$ is as simple as possible.