§ 8. Stable and unstable manifolds

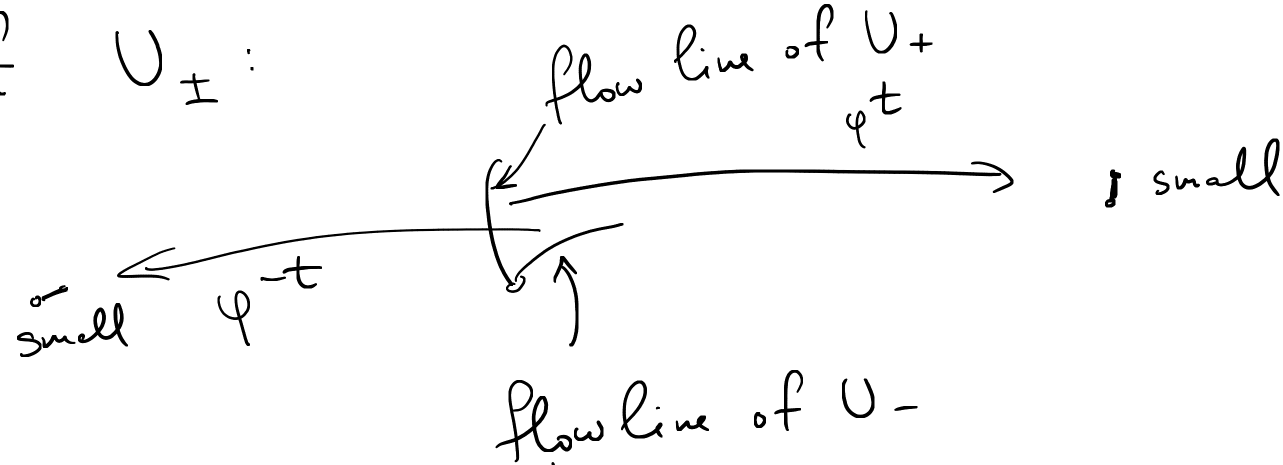
Recall from §6 that when we used Hopf's argument to show mixing of the geodesic flow on a compact hyperbolic surface, we used crucially that $\forall s \in \mathbb{R}$, $\rho \in SM$

$$d\left(\varphi^t(\rho), \varphi^t\left(e^{sU_\pm}(\rho)\right)\right) \xrightarrow[t \to \pm\infty]{} 0,$$

d = distance fn on SM

where $U_\pm$ were the stable/unstable vector fields

That is, φ^t shrinks the flow lines of $U_\pm$:



In this section we study stable/unstable manifolds for general hyperbolic <u>maps</u>, which can be used to show mixing (though it gets much harder to run Hopf's argument because $x \mapsto E_u(x), E_s(x)$ not C^1).

One can similarly define these for <u>flows</u>, but we mostly do maps here.

We first present a special case of a <u>hyperbolic fixed point</u>:

- X a manifold
- $\varphi : X \circlearrowleft$ diffeomorphism
- $x_0 \in X$ is a fixed point for φ:

$$\varphi(x_0) = x_0,$$

which is <u>hyperbolic</u>, i.e.

$$T_{x_0} X = E_u(x_0) \oplus E_s(x_0) \quad \text{and } \exists C, \theta$$

$$\|(d\varphi|_{E_s(x_0)})^n\| \leq C e^{-\theta n}, \quad n \geq 0$$

$$\|(d\varphi|_{E_u(x_0)})^{-n}\| \leq C e^{-\theta n}, \quad n \geq 0$$

In this setting we show a version of the <u>Stable/Unstable Manifold Thm</u>, also known as the <u>Hadamard-Perron Thm</u>:

Thm Assume that x_0 is a hyperbolic fixed point for φ. Then there exist C^∞ submanifolds

$$W^u(x_0), W^s(x_0) \subset X$$

which are diffeomorphic to disks and such that:

1. $W^u(x_0) \cap W^s(x_0) = \{x_0\}$
2. $T_{x_0} W^u(x_0) = E_u(x_0)$,
   $T_{x_0} W^s(x_0) = E_s(x_0)$
3. Local φ-invariance:
   $\varphi(W^s(x_0)) \subset W^s(x_0)$,
   $\varphi^{-1}(W^u(x_0)) \subset W^u(x_0)$
4. Exponential contraction: if we fix a distance function $d(\cdot,\cdot)$ on X (w.r.t. some Riemannian metric)

then $\exists\, C, \theta > 0 \quad \forall n \geq 0$

$$d(\varphi^n(x), x_0) \leq C e^{-\theta n} \quad \forall x \in W^s(x_0)$$

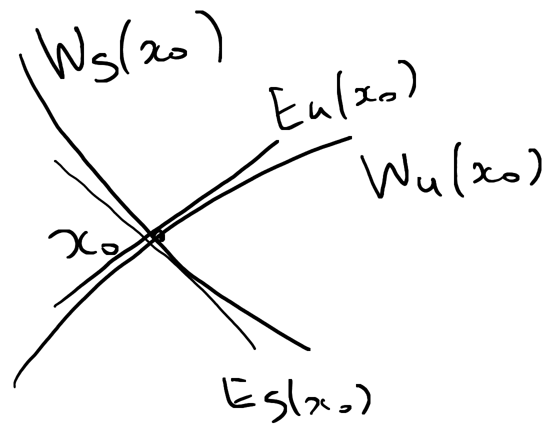
$$d(\varphi^{-n}(x), x_0) \leq C e^{-\theta n} \quad \forall x \in W^u(x_0)$$

(5) Characterization: $\exists\, \varepsilon_0 > 0$ s.t.
$\forall x \in X$,

- if $d(\varphi^n(x), \varphi^n(x_0)) \leq \varepsilon_0$ $\underline{\forall n \geq 0}$ then
$x \in \underline{W^s(x_0)}$
- if $d(\varphi^n(x), \varphi^n(x_0)) \leq \varepsilon_0$ $\underline{\forall n \leq 0}$ then
$x \in \underline{W^u(x_0)}$.

Rmk. Properties (1) & (5) imply in particular that $\exists\, \varepsilon_0 > 0$
$\forall x \in X$, if $d(\varphi^n(x), \varphi^n(x_0)) \leq \varepsilon_0$ $\forall n \in \mathbb{Z}$
then $x = x_0$.
(i.e. any trajectory other than x_0 move away from x_0 in future or in past)

Picture:



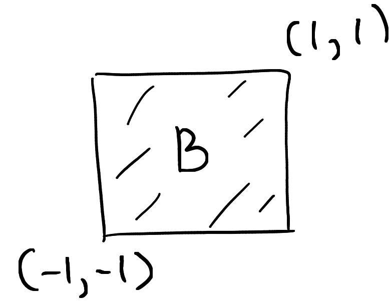
§8.1. Model case

We present the proof of the Thm above in a model 2D case. We later discuss how the proof adapts to the general case, but with fewer details.

Also, we fix arbitrary $N \geq 1$ and construct C^N stable/unstable manifolds (rather than C^∞).

We will use $N+1$ derivatives of φ (could get away with N derivatives but the proof would be harder).

Here is our <u>model setting</u>:



assume that

$B = [-1,1]^2$ and

$\varphi : U_\varphi \to V_\varphi$ is a C^{N+1} diffeomorphism,

$U_\varphi, V_\varphi \subset \mathbb{R}^2$ open sets containing B, such that:

① $\varphi(0) = 0$;

② $d\varphi(0) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$;

③ For a small constant δ to be chosen later and all multiindices α with $2 \le |\alpha| \le N+1$,

$$\sup_{V_\varphi} |\partial^\alpha \varphi| \le \delta.$$

Conditions ① + ② imply that $0 = (0,0)$ $(=: x_0)$ is a hyperbolic fixed point of φ.

Condition ③ means that φ is well-approximated by the <u>linear map</u> $(x_1, x_2) \mapsto (2x_1, \frac{x_2}{2})$.

It can be achieved by <u>rescaling</u>: for small $\delta_1 > 0$, define $\boxed{T_{\delta_1} : x \mapsto \delta_1 x}$.

Then $\forall \varphi$ satisfying ① + ②, the map $\tilde{\varphi} := T_{\delta_1}^{-1} \circ \varphi \circ T_{\delta_1}$ satisfies ① + ② + ③ if δ_1 is small enough depending on δ, N.

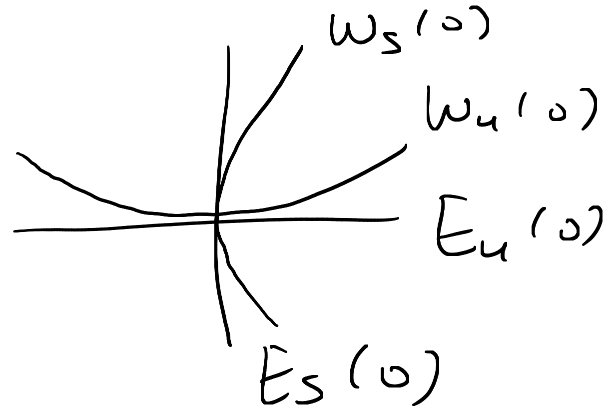
We will explain how to construct the <u>unstable</u> manifold

$$W_u = W_u(0) \subset B.$$

(The stable one is constructed similarly, using φ^{-1} instead).

Note that $E_u(0) = \mathbb{R}(1,0)$

$E_s(0) = \mathbb{R}(0,1)$:



We will construct W_u as a <u>graph</u>:

for $F: [-1,1] \to [-1,1]$, define the graph

$$G_u(F) = \{ x_2 = F(x_1), \ |x_1| \leq 1 \}$$

which is a curve inside B.

The construction of F is done in

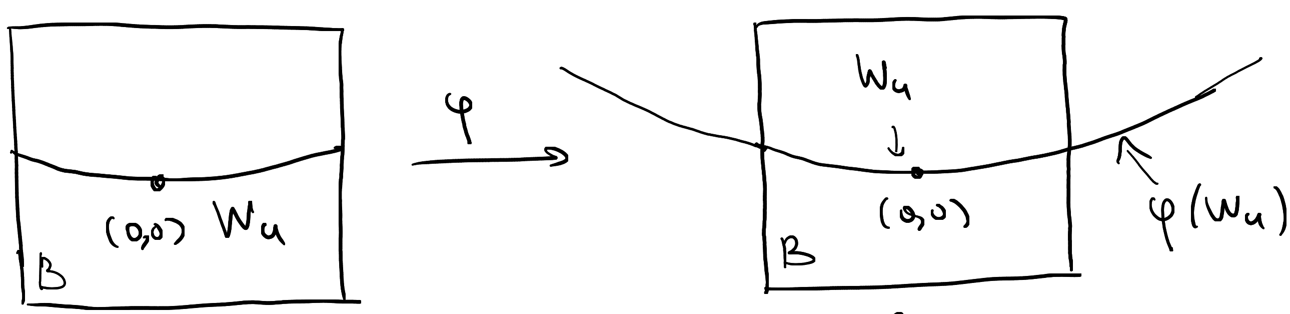
<u>Thm</u> If δ is small enough then $\exists$ a C^N function

$$F_u : [-1,1] \to [-1,1],$$

$F_u(0) = 0$, $\partial_{x_1} F_u(0) = 0$ such that, denoting $W_u := G_u(F_u)$, we have $\boxed{\varphi(W_u) \cap B = W_u}$

local φ-invariance

<u>Picture</u>:



<u>Example</u>: if $\varphi(x_1, x_2) = (2x_1, \frac{x_2}{2})$ (linear)

then we would get $F_u \equiv 0$,

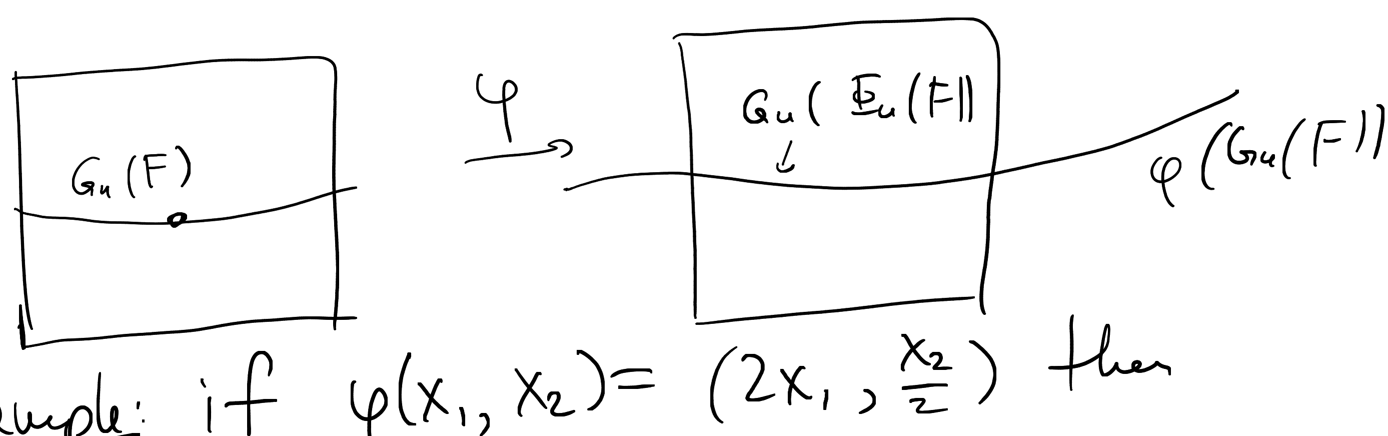
$$W_u = \{(x_1, x_2) : x_2 = 0, |x_1| \le 1\}$$

To prove the Thm, we first describe how φ acts on graphs of functions:

<u>Lemma 1</u> Assume that $F : [-1,1] \circlearrowleft$ and $F(0)=0$, $\sup |\partial_{x_1} F| \leq 1$.
Then there exists a function $\Phi_u F : [-1,1] \circlearrowleft$, $\Phi_u F(0) = 0$ s.t.

$$\varphi(G_u(F)) \cap \{|x_1| \leq 1\} = G_u(\Phi_u(F)).$$

<u>Picture</u>:



<u>Example</u>: if $\varphi(x_1, x_2) = (2x_1, \frac{x_2}{2})$ then

$$\Phi_u(F)(x_1) = \frac{1}{2} F\left(\frac{x_1}{2}\right), \quad |x_1| \leq 1.$$

<u>Proof</u> Define $G_1, G_2 : [-1,1] \to \mathbb{R}$,

$$G_1(x_1) = \varphi_1(x_1, F(x_1))$$
$$G_2(x_1) = \varphi_2(x_1, F(x_1))$$ where
$\varphi(x) = (\varphi_1(x), \varphi_2(x))$, $\varphi_1, \varphi_2 : U_\varphi \to \mathbb{R}$.

Then the image $\varphi(G_u(F))$ has the form

$$\varphi(G_u(F)) = \{(G_1(x_1), G_2(x_1)) : |x_1| \le 1\}.$$

To write this as a graph, we need to show that G_1 is invertible.

We have $G_1(0) = 0$ (as $\varphi(0,0) = 0$ and $F(0) = 0$)

$$\partial_{x_1} G_1(x_1) = \partial_{x_1}\varphi_1(x_1, F(x_1)) +$$
$$+ \partial_{x_2}\varphi_1(x_1, F(x_1))\, \partial_{x_1} F(x_1)$$
$$= 2 + O(\delta)$$ since (by 2 + 3 above)

$$\partial_{x_1}\varphi_1 = 2 + O(\delta),\ \partial_{x_2}\varphi_1 = O(\delta),\ |\partial_{x_1} F| \le 1.$$

So if δ is small enough, then | 18.118 8-11

$$\partial_{x_1} G_1(x_1) \geq \frac{3}{2}, \quad \forall x_1 \in [-1,1]$$

which shows that

G_1 is a diffeomorphism

$$[-1,1] \to G_1([-1,1]) \supset [-1,1].$$

So we can define $G_1^{-1} : [-1,1] \to [-1,1]$.

We then have

$$\varphi(G_u(F)) \cap \{|x_1| \leq 1\} = G_u(\Phi_u F)$$

where $\Phi_u F$ is defined by

$$\boxed{\Phi_u F(y_1) = G_2(G_1^{-1}(y_1))}, \quad |y_1| \leq 1. \qquad \square$$

Given Lemma 1, to show Thm we need to construct $F_u : [-1,1] \circlearrowleft$, $F_u(0)=0$, $\partial_x F_u(0)=0$ such that $\boxed{\Phi_u F_u = F_u}$.

That is, we are looking for a fixed point of the graph transform Φ_u.

To get it, we will ultimately use Contraction Mapping Principle.

To establish the contraction property for Φ_u, we will compute the derivatives of $\Phi_u F$ in terms of those of F: 18.118 8-12

<u>Lemma 2</u> Let $1 \leq k \leq N$. Assume that

$F \in C^k([-1,1];\mathbb{R})$, $F(0)=0$, $\max\limits_{1\leq j\leq k} \sup |\partial_{x_1}^j F| \leq 1$.

Then we have $\forall y_1 \in [-1,1]$,

$$\partial_{x_1}^k (\Phi_u F)(y_1) = L_k(x_1, F(x_1), \partial_{x_1} F(x_1), \dots, \partial_{x_1}^k F(x_1)),$$

$$x_1 := G_1^{-1}(y_1), \quad \text{where:}$$

- $G_1(x_1) = \varphi_1(x_1, F(x_1))$ as in Lemma 1;
- $L_k(x_1, \tau_0, \dots, \tau_k)$ is a function on the cube $Q_k = [-1,1]^{k+2}$ depending on φ but not on F;
- $L_k(x_1, \tau_0, \dots, \tau_k) = 2^{-k-1}\tau_k + O(\delta)$ with the remainder satisfying

$$\sup_{Q_k} |\partial_{x_1}^{\alpha} \partial_{\tau_0}^{\beta_0} \cdots \partial_{\tau_k}^{\beta_k} (L - 2^{-k-1}\tau_k)| \leq C_{\alpha\beta}\delta$$

$\forall \alpha, \beta_0, \dots, \beta_k$ with $\alpha + \beta_0 + k \leq N+1$.

Example: if $\varphi(x_1,x_2) = (2x_1, \frac{x_2}{2})$

then $G_1(x_1) = 2x_1$, $G_1^{-1}(y_1) = \frac{y_1}{2}$

$$\Phi_u F(y_1) = \frac{1}{2} F\left(\frac{y_1}{2}\right), \quad \text{so}$$

$$\partial_{x_1}^k (\Phi_u F)(y_1) = 2^{-k-1} \partial_{x_1}^k F\left(\frac{y_1}{2}\right)$$

That is, here $L_k(x_1, \tau_0, \dots, \tau_k) = 2^{-k-1}\tau_k$.

Proof Let us just consider the case $k=1$. The case of higher k is handled by induction, see e.g. [D, Lemma 2.2] for details

Recall that $\Phi_u F(y_1) = G_2(G_1^{-1}(y_1))$,

where $G_1(x_1) = \varphi_1(x_1, F(x_1))$

$G_2(x_1) = \varphi_2(x_1, F(x_1))$.

Denote $A(x) = d\varphi(x) = (A_{jk}(x))$,

$A_{jk}(x) = \partial_{x_k}\varphi_j(x)$, so that by (3),

$$A(x) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} + O(\delta).$$

18.118
8-14

We compute

$$\partial_{x_1}(\Phi_4 F)(y_1) = \frac{\partial_{x_1} G_2(x_1)}{\partial_{x_1} G_1(x_1)}$$

where $x_1 = G_1^{-1}(y_1)$, thus

$$\partial_{x_1}(\Phi_4 F)(y_1) = \frac{A_{21}(x_1, F(x_1)) + A_{22}(x_1, F(x_1)) \cdot \partial_{x_1} F(x_1)}{A_{11}(x_1, F(x_1)) + A_{12}(x_1, F(x_1)) \cdot \partial_{x_1} F(x_1)}$$

$$= L_1(x_1, F(x_1), \partial_{x_1} F(x_1)) \text{ where}$$

$$L_1(x_1, \tau_0, \tau_1) = \frac{A_{21}(x_1, \tau_0) + A_{22}(x_1, \tau_0)\tau_1}{A_{11}(x_1, \tau_0) + A_{12}(x_1, \tau_0)\tau_1}$$

$$= \frac{O(\delta) + (\frac{1}{2} + O(\delta))\tau_1}{2 + O(\delta) + O(\delta)\tau_1}$$

$$= \frac{1}{4}\tau_1 + O(\delta).$$

18.118
8-15

To demonstrate how the inductive step works, let's do $\underline{k=2}$:

we already know that

$$\partial_{x_1}(\Phi_u F)(y_1) = L_1(x_1, F(x_1), \partial_{x_1} F(x_1)),$$

where $x_1 = G_1^{-1}(y_1)$.

Then we differentiate again to get

$$\partial_{x_1}^2 (\Phi_u F)(y_1) = \frac{\partial_{x_1}(L_1(x_1, F(x_1), \partial_{x_1} F(x_1)))}{\partial_{x_1} G_1(x_1)} =$$

$$= \frac{(\partial_{x_1} L_1)(x_1, F(x_1), \partial_{x_1} F(x_1)) + (\partial_{\tau_0} L_1)(x_1, F(x_1), \partial_{x_1} F(x_1)) \cdot \partial_{x_1} F(x_1) + (\partial_{\tau_1} L_1)(x_1, F(x_1), \partial_{x_1} F(x_1)) \cdot \partial_{x_1}^2 F(x_1)}{A_{11}(x_1, F(x_1)) + A_{12}(x_1, F(x_1))\, \partial_{x_1} F(x_1)}$$

$$= L_2(x_1, F(x_1), \partial F(x_1), \partial_{x_1}^2 F(x_1)) \Big|_{x_1 = G_1^{-1}(y_1)}$$ where

$$L_2(x_1, \tau_0, \tau_1, \tau_2) = \frac{\partial_{x_1} L_1(x_1, \tau_0, \tau_1) + \partial_{\tau_0} L_1(x_1, \tau_0, \tau_1)\tau_1 + \partial_{\tau_1} L_1(x_1, \tau_0, \tau_1)\tau_2}{A_{11}(x_1, \tau_0) + A_{12}(x_1, \tau_0)\,\tau_1}$$

$$= \frac{1}{8}\tau_2 + O(\delta) \ldots$$

$\square$

Using Lemma 2 we can now estimate the derivatives of $\Phi_u F$ in terms of those of F.

Denote $\|F\|_{C^k} := \max\limits_{1\le j\le k} \sup\limits_{[-1,1]} |\partial^j_{x_1} F|$

which is a norm on the space of $F \in C^k([-1,1];\mathbb{R})$ such that $F(0)=0$.

Define also

$$\|F\|_{C^{k,1}} = \max\Big(\|F\|_{C^k}, \sup_{x_1 \neq \tilde{x}_1} \frac{|\partial^k_{x_1}F(x_1) - \partial^k_{x_1}F(\tilde{x}_1)|}{|x_1-\tilde{x}_1|}\Big)$$

Lipschitz norm of $\partial^k_{x_1} F$.

<u>Lemma 3</u> Let $1 \le k \le N$ and assume that $F(0)=0$ and $\|F\|_{C^k}\le 1$. Then

$$\|\Phi_u F\|_{C^k} \le \frac{1}{4}\|F\|_{C^k} + C\delta.$$

If we also have $\|F\|_{C^{k,1}} \le 1$ then

$$\|\Phi_u F\|_{C^{k,1}} \le \frac{1}{4}\|F\|_{C^{k,1}} + C\delta.$$

Here C is a constant depending only on k.

<u>**Proof**</u> We just show the first bound. The second one is similar (the Lipschitz norm of $\partial_{x_1}^k F$ is the same as the sup-norm of $\partial_{x_1}^{k+1} F$, if we knew that $F \in C^{k+1}$).

Let $y_1 \in [-1,1]$ and $x_1 = G_1^{-1}(y_1)$. Then by Lemma 2, for all $j = 1, \dots, k$

$$\partial_{x_1}^j (\Phi_u F)(y_1) = L_j(x_1, F(x_1), \dots, \partial_{x_1}^j F(x_1))$$

$$= 2^{-j-1} \partial_{x_1}^j F(x_1) + O(\delta), \text{ so}$$

$$\sup_{y_1} |\partial_{x_1}^j (\Phi_u F)(y_1)| \leq 2^{-j-1} \|F\|_{C^k} + C\delta$$

$$\leq \frac{1}{4} \|F\|_{C^k} + C\delta.$$

$\square$

We are now ready to prove a contraction property for Φ_u:

Lemma 4 Let $1 \le k \le N$ and assume that $F, \tilde{F} \in C^{k,1}([-1,1])$ satisfy
$$F(0) = \tilde{F}(0) = 0, \quad \|F\|_{C^{k,1}}, \|\tilde{F}\|_{C^{k,1}} \le 1.$$
Then
$$\|\Phi_u F - \Phi_u \tilde{F}\|_{C^k} \le \left(\tfrac{1}{4} + C\delta\right) \|F - \tilde{F}\|_{C^k}.$$

Proof Define
$$G_1(x_1) = \varphi_1(x_1, F(x_1)),$$
$$\tilde{G}_1(x_1) = \varphi_1(x_1, \tilde{F}(x_1)).$$
Take $y_1 \in [-1,1)$ and define
$$x_1 := G_1^{-1}(y_1), \quad \tilde{x}_1 := \tilde{G}_1^{-1}(y_1).$$
We first claim that
$$(1) \quad |x_1 - \tilde{x}_1| \le C\delta \|F - \tilde{F}\|_{C^2}.$$

18.118
8-19

Indeed, we have

as $\tilde{G}_1 \geq 1$

$$|x_1 - \tilde{x}_1| \leq |\tilde{G}_1(x_1) - \tilde{G}_1(\tilde{x}_1)|$$

as $G_1(x_1) = \tilde{G}_1(\tilde{x}_1) = y_1$

$$= |\tilde{G}_1(x_1) - G_1(x_1)|$$

$$= |\varphi_1(x_1, \tilde{F}(x_1)) - \varphi_1(x_1, F(x_1))|$$

$$\leq C\delta \, \|F - \tilde{F}\|_{C^1}.$$

Here we use that $\partial_{x_2}\varphi_1 = O(\delta)$ and $\sup|F - \tilde{F}| \leq \|F - \tilde{F}\|_{C^1}$ since

$$F(0) = \tilde{F}(0) = 0.$$

Next, we have ~~for~~ $j = 0, \dots, k$

$$(2) \quad |\partial_{x_1}^j F(x_1) - \partial_{x_1}^j \tilde{F}(\tilde{x}_1)| \leq (1 + C\delta)\|F - \tilde{F}\|_{C^k}.$$

Indeed, the LHS is bounded by

$$|\partial_{x_1}^j F(x_1) - \partial_{x_1}^j F(\tilde{x}_1)| + |\partial_{x_1}^j F(\tilde{x}_1) - \partial_{x_1}^j \tilde{F}(\tilde{x}_1)|$$

$$\leq |x_1 - \tilde{x}_1| + \|F - \tilde{F}\|_{C^k} \leq (1 + C\delta)\, \|F - \tilde{F}\|_{C^k}$$

here we use that $\|F\|_{C^{k,1}} \leq 1$ here we use (1)

Finally, recall that by Lemma 2,

$$\partial_{x_1}^j(\Phi_u F)(y_1) = L_j(x_1, F(x_1), \dots, \partial_{x_1}^j F(x_1))$$
$$\partial_{x_1}^j(\Phi_u \tilde F)(y_1) = L_j(\tilde x_1, F(\tilde x_1), \dots, \partial_{x_1}^j F(\tilde x_1))$$

and $L_j(x_1, \tau_0, \dots, \tau_j) = 2^{-j-1}\tau_j + O(\delta)$ with $O(\delta)$ in Lipschitz norm in $x_1, \tau_0, \dots, \tau_j$.

So $|\partial_{x_1}^j(\Phi_u F)(y_1) - \partial_{x_1}^j(\Phi_u \tilde F)(y_1)| \leq$

$$\leq 2^{-j-1}|\partial_{x_1}^j F(x_1) - \partial_{x_1}^j \tilde F(\tilde x_1)|$$
$$+ C\delta\left(|x_1 - \tilde x_1| + |F(x_1) - \tilde F(\tilde x_1)| + \dots + |\partial_{x_1}^j F(x_1) - \partial_{x_1}^j \tilde F(\tilde x_1)|\right)$$
$$\leq 2^{-j-1}\|F - \tilde F\|_{C^k} + C\delta\|F - \tilde F\|_{C^k}$$
$$\leq \left(\frac{1}{4} + C\delta\right)\|F - \tilde F\|_{C^k},$$

finishing the proof. $\square$

We are now ready to give

Proof of Thm (existence of F_u)

We need to show that $\exists\, F_u \in C^N([-1,1])$ such that $F_u(0) = \partial_{x_1} F_u(0) = 0$ and

$$\Phi_u(F_u) = F_u.$$

We will use the Contraction Mapping Principle.

It is a bit subtle because Lemma 4 above has the a priori assumption $\|F\|_{C^{N,1}} \leq 1$ but only gives contraction in the C^N norm.

Define the metric space $(\mathcal{X}, d)$ with

$$\mathcal{X} := \{F \in C^{N,1}([-1,1];\mathbb{R}) : F(0)=0,\ \|F\|_{C^{N,1}} \leq 1\}$$

$$d_{\mathcal{X}}(F,\tilde{F}) := \|F - \tilde{F}\|_{C^N}.$$

Then $(\mathcal{X}, d_{\mathcal{X}})$ is a complete metric space.

Indeed, it is the subset of the closed unit ball in the Banach space $C^N([-1,1])$ consisting of functions F such that

$$F(0)=0, \quad |\partial_{x_1}^N F(x_1) - \partial_{x_1}^N F(\tilde{x}_1)| \leq |x_1 - \tilde{x}_1| \quad \forall\, x_1, \tilde{x}_1 \in [-1,1]$$

these are closed conditions under the C^N norm

The graph transform defines a map

$$\Phi_u : \mathfrak{X} \to \mathfrak{X}$$

for δ small enough, by Lemma 3.
And for δ small enough, by Lemma 4
we see that Φ_u is a contraction w.r.t. $d_{\mathfrak{X}}$.
Thus $\exists\, F_u \in \mathfrak{X}$: $\Phi_u(F_u) = F_u$.
In fact, the contraction mapping principle gives that $\forall\, F_0 \in \mathfrak{X}$ we have

$$\Phi_u^n(F_0) \xrightarrow[n\to\infty]{} F_u \quad \text{in } C^N([-1,1]).$$

Finally, looking back at Lemma 2 we get $\forall F$, $\partial_{x_1}(\Phi_u F)(0) = \frac{1}{4}\partial_{x_1} F(0)$

Therefore, $\partial_{x_1} F_u(0) = \frac{1}{4}\partial_{x_1} F_u(0) \Rightarrow \partial_{x_1} F_u(0) = 0.$

$\square$

A similar argument gives the existence of a <u>local stable manifold</u>: if $G_s(F) = \{(F(x_2), x_2) \mid -1 \le x_2 \le 1\}$ then ~~for~~ δ small enough there exists

$$F_s \in C^N([-1,1];[-1,1]) \text{ such that}$$

$$F_s(0)=0, \quad \partial_{x_2} F_s(0) = 0, \text{ and,}$$

denoting $W_s := G_s(F_s) \subset B$, we have

$$\boxed{\varphi^{-1}(W_s) \cap B = W_s}.$$

We also have $W_u \cap W_s = \{0\}$.

Indeed, by Lemma 3 we get (for δ small)

$\|F_u\|_{C^N}, \quad \|F_s\|_{C^N} \le C\delta.$

In particular, $\|F_u\|_{C^1}, \|F_s\|_{C^1} \le C\delta$.

But if $(x_1, x_2) \in W_u \cap W_s$ then

$x_2 = F_u(x_1), \; x_1 = F_s(x_2) \Rightarrow x_1 = F_s(F_u(x_1))$

But $\sup|\partial_{x_1}(F_s \circ F_u)| < 1$, $(F_s \circ F_u)(0) = 0$,

So (by contraction mapping) we have $x_1 = 0$.

Then also $x_2 = 0$.

§8.2. Model case continued

We operate under the assumptions of §8.1 (and assume δ is small).

We will get more properties of the local unstable/stable manifolds W_u, W_s:

Thm We have: (numbering corresponds to the Thm preceding §8.1)

(4u) If $w \in W_u$ then $\forall n \geq 0$

$$|\varphi^{-n}(w)| \leq \left(\tfrac{1}{2} + C\delta\right)^n |w|.$$

In particular, $\varphi^{-n}(w) \xrightarrow[n\to\infty]{} 0$.

(4s) If $w \in W_s$ then $\forall n \geq 0$ $|\varphi^n(w)| \leq \left(\tfrac{1}{2} + C\delta\right)^n |w|$

(5u) If $w \in \overline{B} = [-1,1]^2$ and $\varphi^{-n}(w) \in \overline{B}$ $\forall n \geq 0$ then $w \in W_u$.

(5s) If $w \in \overline{B}$ and $\varphi^n(w) \in \overline{B}$ $\forall n \geq 0$ then $w \in W_s$.

Note: (4u)+(5u) give a characterization of W_u: $w \in B$ lies in W_u iff $\forall n \geq 0$, $\varphi^{-n}(w) \in B$.

Similarly for W_s:

$w \in W_s \iff \forall n \geq 0, \ \varphi^n(w) \in B$.

We only prove (4_u) & (5_u) ((4_s) + (5_s) proved similarly).

For (4_u), first note that

$$\varphi(W_u) \cap B = W_u$$

implies that $\boxed{\varphi^{-1}(W_u) \subset W_u.}$

So, for $w \in W_u$ we have

$$\varphi^{-n}(w) \in W_u \quad \forall\ n \geq 0.$$

It remains to show

Lemma 5 If $y \in W_u$ and $x := \varphi^{-1}(y)$ then $|x| \leq \left(\frac{1}{2} + C\delta\right)|y|$.

Proof Write $x = (x_1, F_u(x_1))$, $y = (y_1, F_u(y_1))$.

Then $y = \varphi(x)$, so

$$y_1 = \varphi_1(x_1, F_u(x_1))$$

Recall that $\varphi_1(x_1, x_2) = 2x_1 + O(\delta)|x|$

(since $\varphi_1(0,0) = 0$, $\partial_{x_1}\varphi_1(0,0) = 2$, $\partial_{x_2}\varphi_1(0,0) = 0$ and $\partial^2_{x_j x_k}\varphi_1 = O(\delta)$).

And we know that $\|F_u\|_{C^1} \leq C\delta$, so $|F_u(x_1)| \leq$
$\leq C\delta|x_1|$.

So $y_1 = 2x_1 + O(\delta)|x_1|$,

thus $|y_1| \geq (2 - C\delta)|x_1|$, i.e.

$$|x_1| \leq \left(\tfrac{1}{2} + C\delta\right)|y_1|.$$

Using again that $\|F_u\|_{C^1} \leq C\delta$, we get

$$|x| \leq (1+C\delta)|x_1|, \quad |y_1| \leq (1+C\delta)|y|$$

This gives $|x| \leq \left(\tfrac{1}{2} + C\delta\right)|y|$

as needed. $\square$

To show (5_u), define for $w = (w_1, w_2) \in B$ the distance to the unstable manifold:

$$d(w, W_u) := |w_2 - F_u(w_1)|.$$

Lemma 6 Assume that $w \in B$ and $\varphi(w) \in B$. Then $d(\varphi(w), W_u) \leq \left(\tfrac{1}{2} + C\delta\right) d(w, W_u)$.

Proof We write

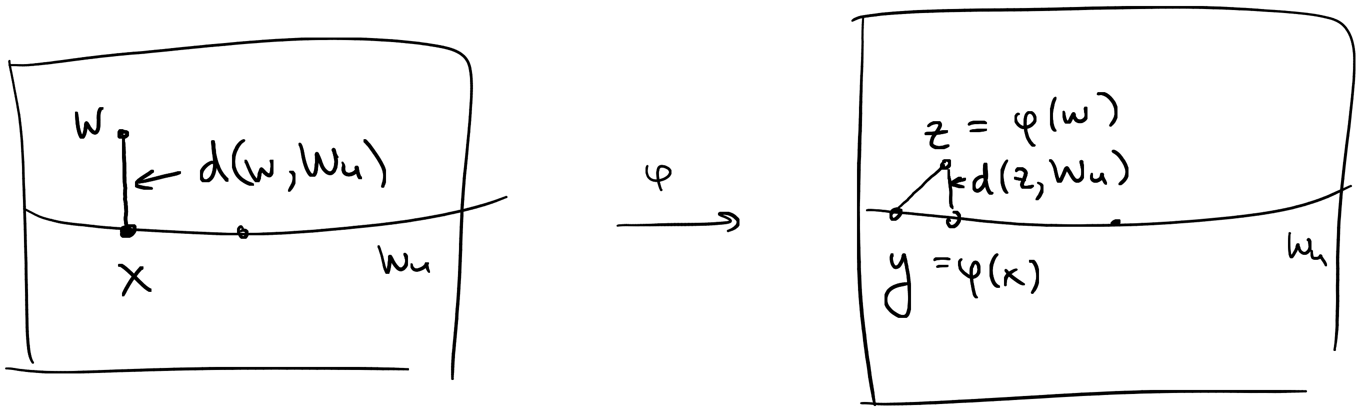
$w = (w_1, w_2)$, $\quad z := \varphi(w) = (z_1, z_2)$.

Define

$$x := (w_1, F_u(w_1)),$$

$$y := \varphi(x) = (y_1, F_u(y_1)).$$

Picture:



We have $z - y = \varphi(w) - \varphi(x) =$

$$= \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} (w - x) + O(\delta |w - x|)$$

Note that $|w - x| = d(w, W_u)$. So
And $w - x = (0, w_2 - x_2)$.

(1) $z_1 - y_1 = O(\delta) d(w, W_u)$

(2) $z_2 - F_u(y_1) = \frac{1}{2}(w_2 - F_u(w_1)) + O(\delta) d(w, W_u)$

From (1) we have $F_u(z_1) - F_u(y_1) = O(\delta) d(w, W_u)$.

Then from (2), $\underbrace{|z_2 - F_u(z_1)|}_{= d(z, W_u)} = \frac{1}{2}\underbrace{|w_2 - F_u(w_1)|}_{= d(w, W_u)} + O(\delta) d(w, W_u)$. $\square$

We can now finish the proof of S_u.

Assume that $w \in B$ and

$$w^{(n)} := \varphi^{-n}(w) \in B \quad \forall n \geq 0.$$

By Lemma 6, we have $\forall n \geq 0$,

$$d(w^{(n-1)}, W_u) = d(\varphi(w^{(n)}), W_u)$$
$$\leq \left(\frac{1}{2} + C\delta\right) d(w^{(n)}, W_u).$$

Since each $d(w^{(n)}, W_u) \leq 2$, we iterate to get $\forall n \geq 0$,

$$d(w, W_u) \leq 2\left(\frac{1}{2} + C\delta\right)^n.$$

Taking $n \to \infty$, we get

$d(w, W_u) = 0$, that is $w \in W_u$.

§8.3. The general case

We now discuss how the proof in the model case of §§ 8.1–8.2 can be adapted to more general situations.

- **More general 2D differentials:**

instead of $d\varphi(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

we could take $d\varphi(0,0) = \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}$

where $0 < |\lambda| < 1 < |\mu|$.

In Lemma 2 (which describes $\partial^k(\Phi_u F)$ in terms of F) we have

$$L_k(x_1, \tau_0, \dots, \tau_k) = \lambda \cdot \mu^{-k} \tau_k + O(\delta).$$

Indeed, in the linear case $\varphi(x_1, x_2) = (\mu x_1, \lambda x_2)$ the graph transform is given by

$$\Phi_u F(x_1) = \lambda \cdot F(\mu^{-1} x_1)$$

- General dimensions:

We can consider $d_u, d_s \geq 0$

and the "ball"

$$B = \{ (x_u, x_s) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_s} : |x_u| \leq 1, \ |x_s| \leq 1 \}$$

with a map $\varphi : U_\varphi \to V_\varphi$ diffeomorphism,

$U_\varphi, V_\varphi \subset \mathbb{R}^{d_u + d_s}$ open, $B \subset U_\varphi \cap V_\varphi$

such that:

① $\varphi(0) = 0$

② $d\varphi(0) = \begin{pmatrix} A_u & 0 \\ 0 & A_s \end{pmatrix}$ where

$\| A_u^{-1} \| \leq \lambda$, $\| A_s \| \leq \lambda$ for some

λ such that $0 < \lambda < 1$

③ $\forall \alpha$ with $2 \leq |\alpha| \leq N+1$ we have

$$\sup_{U_\varphi} | \partial^\alpha \varphi | \leq \delta.$$

We can still construct the stable/unstable manifolds (for δ small)

$$W_u = G_u(F_u), \quad W_s = G_s(F_s)$$

where $G_u(F) = \{(x_u, x_s) : x_s = F(x_u), |x_u| \le 1\}$,

$$G_s(F) = \{(x_u, x_s) : x_u = F(x_s), |x_s| \le 1\}$$

and $F_u : B_{\mathbb{R}^{d_u}}(0,1) \to B_{\mathbb{R}^{d_s}}(0,1)$

$F_s : B_{\mathbb{R}^{d_s}}(0,1) \to B_{\mathbb{R}^{d_u}}(0,1)$

are C^N maps.

The proof is largely the same, except:

- In Lemma 1, to invert (called G_1 before)

$$G_u(x_u) = \varphi_u(x_u, F(x_u)), \quad \varphi = (\varphi_u, \varphi_s), \quad x = (x_u, x_s)$$

we use Inverse Mapping Thm (or rather, its proof):

we have $\|\partial_{x_u} G_u - A_u\| = O(\delta)$

So $G_u(x_u) = \underset{\substack{\uparrow \\ \text{linear expanding map.}}}{A_u} x_u + O(\delta)_{C^1}$, So $G_u(B(0,1)) \supset B(0,1) \ldots$

- The statement & proof of Lemma 2 is more complicated, with multiindices etc. (see [D, §3.3] for details)

Note: in the linear case

$$\varphi(x_u, x_s) = (A_u x_u, A_s x_s),$$

we have $\Phi_u F(x_u) = A_s F(A_u^{-1} x_u)$

matrix inverse.

Handling general hyperbolic fixed points:

We go back to the Thm in the beginning of §8.

Assume that $\varphi: X \circlearrowleft$ is a diffeomorphism and $x_0 \in X$ is a hyperbolic fixed point for φ. We reduce to the model case (the higher dimensional version discussed just above).

For that, take <u>adapted metrics</u> (see §7.2): $|\cdot|_u$, $|\cdot|_s$ on X such that $\exists\, \lambda \in (0,1)$ with

$$\| d\varphi(x_0)^{-1}|_{E_u(x_0)} \|_u \le \lambda,$$

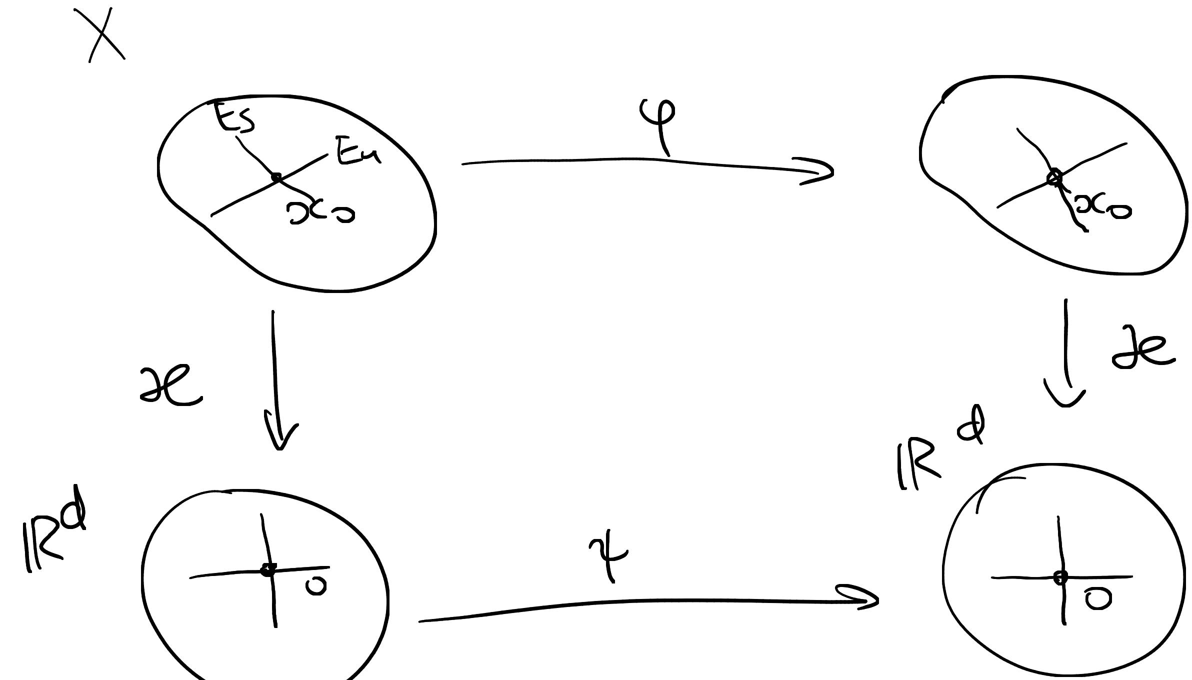
$$\| d\varphi(x_0)|_{E_s(x_0)} \|_s \le \lambda.$$

Next, choose <u>adapted coordinates</u>

$\varkappa : U \to V$, a diffeomorphism, where $U \subset X$, $V \subset \mathbb{R}^{d_u+d_s}$ are open,

- $\varkappa(x_0) = 0$
- $d\varkappa(x_0) E_u(x_0) = E_u(0) := \mathbb{R}^{d_u} \oplus 0$ and $d\varkappa(x_0)|_{E_u(x_0)}$ is an isometry from $|\cdot|_u$ to the Euclidean metric
- $d\varkappa(x_0) E_s(x_0) = E_s(0) := 0 \oplus \mathbb{R}^{d_s}$ and $d\varkappa(x_0)|_{E_s(x_0)}$ is an isometry from $|\cdot|_s$ to the Euclidean metric

Then in the chart $\varkappa$,
the map φ corresponds to
the map $\psi := \varkappa \circ \varphi \circ \varkappa^{-1}$:



Note that:

① $\psi(0) = 0$

② $d\psi(0) = \begin{pmatrix} A_u & 0 \\ 0 & A_s \end{pmatrix}$ where

$$\|A_u^{-1}\| \le \lambda, \quad \|A_s\| \le \lambda$$

We can ensure that $\partial^\alpha \psi$, $2 \le |\alpha| \le N+1$,
are small by rescaling:

take $\delta_1 > 0$ small,
define the dilation operator
$T : \mathbb{R}^d \circlearrowleft, \quad T(x) = \delta_1 \cdot x,$
and take instead of $\varkappa$ the chart

$T^{-1} \circ \varkappa$ (zooming in on x_0).
Then for δ_2 small enough
depending on $\delta, \varkappa, \varphi, N$ we have
$2 \le |\alpha| \le N \Rightarrow \sup |\partial^\alpha \psi| \le \delta.$
Then the Stable/Unstable Thm
for the model case applies to ψ
and gives local stable/unstable
manifolds $\tilde{W}^u, \tilde{W}^s \subset \mathbb{R}^d$.
Then $W^u := \varkappa^{-1}(\tilde{W}^u), \; W^s := \varkappa^{-1}(\tilde{W}^s)$
give the stable/unstable manifolds of φ
at x_0 and they satisfy all the
conditions in the Thm at the beginning of §8.

A few remarks on stable/unstable manifolds

First of all, we have a **local characterization** of $W_s(x_0)$: for $\varepsilon_0 > 0$ in the Thm (in the beginning of §8) and $\varepsilon_1 > 0$ small enough

$$W_s(x_0) \cap \underbrace{B}_{\text{metric ball in } X}(x_0, \varepsilon_1) =$$

$$= \{ x \in B(x_0, \varepsilon_1) \mid \varphi^n(x) \in B(x_0, \varepsilon_0) \ \forall n \geq 0 \}.$$

Indeed, if $x \in W_s(x_0)$, then by part ④ of Thm

$$d(\varphi^n(x), x_0) \longrightarrow 0 \quad \text{as } n \to \infty,$$

uniformly in $x \in W_s(x_0)$.

So $\exists\, n_0 : \forall n \geq n_0, \ \varphi^n(x_0) \in B(x_0, \varepsilon_0)$.

On the other hand, if $x \in B(x_0, \varepsilon_1)$ and ε_1 is small enough (depending on n_0) then $\varphi^n(x) \in B(x_0, \varepsilon_0)$ $\forall n$ with $0 \leq n \leq n_0$. This gives "$\subset$".

To show "$\supset$", assume that $x \in B(x_0, \varepsilon_1)$ and
$$\varphi^n(x) \in B(x_0, \varepsilon_0) \quad \forall n \geq 0.$$
Then $x \in W^s(x_0)$ by part 5 of the Thm

Similarly we get for <u>unstable</u> manifolds,
$$W_u(x_0) \cap B(x_0, \varepsilon_1) = \{x \in B(x_0, \varepsilon_1) \mid \varphi^n(x) \in B(x_0, \varepsilon_0) \ \forall n \leq 0\}.$$

<u>Note</u>: the whole $W_s(x_0)$ can be effectively recovered from $W_s(x_0) \cap B(x_0, \varepsilon_1)$.

Indeed, $\exists\, n_1 \geq 0$:
$$W_s(x_0) \subset \varphi^{-n}(W_s(x_0) \cap B(x_0, \varepsilon_1))$$

This is a d_s-dimensional embedded submanifold in X

Global stable/unstable manifolds:

Define for $n \geq 0$

$$W^{s,n}(x_0) = \varphi^{-n}(W^s(x_0))$$

embedded submanifold

Note that since $\varphi(W^s(x_0)) \subset W^s(x_0)$ we have $W^{s,n}(x_0) \subset W^{s,n+1}(x_0)$.

The <u>union</u> $W^{s,\infty}(x_0) = \bigcup_{n \geq 0} W^{s,n}(x_0)$ is called the <u>global</u> stable manifold of φ at x_0.

Similarly to above, we can show

$$x \in W^{s,\infty}(x_0) \iff \varphi^n(x) \longrightarrow x_0 \text{ as } n \to \infty.$$

<u>Note</u>: $W^{s,\infty}(x_0)$ is an <u>immersed</u> submanifold (image of $\mathbb{R}^{d_s}$ under an immersion)

If X is connected & φ is an Anosov map, then $W^{s,\infty}(x_0)$ is actually <u>dense</u> in X

(we won't prove this here ...)

§8.4. The even more general cases

We do not need x_0 to be a fixed point:

Thm Assume $\varphi : X \circlearrowleft$ is a diffeo & $K \subset X$ is a hyperbolic set ~~for~~ φ.

Then $\forall x \in K$ $\exists$ submanifolds

$W^s(x) \subset X$ (local (un)stable manifold at x) s.t.:

$W^u(x) \subset X$

① $W^u(x) \cap W^s(x) = \{x\}$;

② $T_x W^u(x) = E_u(x)$, $T_x W^s(x) = E_s(x)$;

③ $\varphi(W^s(x)) \subset W^s(\varphi(x))$,
$\varphi^{-1}(W^u(x)) \subset W^u(\varphi^{-1}(x))$

④ $\exists C, \theta > 0$ $\forall n \geq 0$ $\forall x \in K$

- $d(\varphi^n(x), \varphi^n(y)) \leq Ce^{-\theta n}$ $\forall y \in W^s(x)$
- $d(\varphi^{-n}(x), \varphi^{-n}(y)) \leq Ce^{-\theta n}$ $\forall y \in W^u(x)$

⑤ $\exists \varepsilon_0 > 0$ $\forall x \in K, y \in X$

- $d(\varphi^n(x), \varphi^n(y)) \leq \varepsilon_0$ $\forall n \geq 0 \Rightarrow y \in W^s(x)$
- $d(\varphi^{-n}(x), \varphi^{-n}(y)) \leq \varepsilon_0$ $\forall n \geq 0 \Rightarrow y \in W^u(x)$.

Note: (5) in particular implies that

$\forall x, y \in K$, if $d(\varphi^n(x), \varphi^n(y)) \le \varepsilon_0$ $\forall n \in \mathbb{Z}$

then $x = y$. This is important for Symbolic dynamics (we might do some of it later)

There is an analog for flows, see [D, §§4.6-4.7]

The proof of this Thm is actually quite similar to the case of hyperbolic fixed point:

- Pick adapted metrics $|\cdot|_u$, $|\cdot|_s$
- Take adapted coordinates $\forall x_0 \in K$

$\varkappa_{x_0}$: (open set in X) $\to$ (open set in $\mathbb{R}^n$) s.t.

- $\varkappa_{x_0} : x_0 \mapsto 0$
- $d\varkappa_{x_0}$ maps $E_u(x_0)$, $E_s(x_0)$ to

$$E_u(0) := \mathbb{R}^{d_u} \oplus 0, \quad E_s(0) := 0 \oplus \mathbb{R}^{d_s}$$

and is an isometry from $|\cdot|_u$ or $|\cdot|_s$ (at x_0) to the Euclidean metric.

Now, take some $x_0 \in X$ and consider its trajectory

$$x_n = \varphi^n(x), \quad n \in \mathbb{Z}.$$

Consider the maps $\varkappa_n := \varkappa_{x_n}$ and define the maps

$$\psi_n := \varkappa_{n+1} \circ \varphi \circ \varkappa_n^{-1} :$$

$$\begin{tikzcd}
x_{n-1} \arrow[r, "\varphi"] \arrow[d, "\varkappa_{n-1}"] & x_n \arrow[r, "\varphi"] \arrow[d, "\varkappa_n"] & x_{n+1} \arrow[r, "\varphi"] \arrow[d, "\varkappa_{n+1}"] & x_{n+2} \arrow[d, "\varkappa_{n+2}"] \\
0 \arrow[r, "\psi_{n-1}"] & 0 \arrow[r, "\psi_n"] & 0 \arrow[r, "\psi_{n+1}"] & 0
\end{tikzcd}$$

Note that:

- $\psi_n(0) = 0$ (as $\varphi(x_n) = x_{n+1}$)
- $d\psi_n(0) = \begin{pmatrix} A_{u,n} & 0 \\ 0 & A_{s,n} \end{pmatrix}$, $\quad \|A_{u,n}^{-1}\| \le \lambda < 1$, $\quad \|A_{s,n}\| \le \lambda < 1$
- By rescaling $\varkappa_x$ by zooming in, can make

$2 \le |\alpha| \le N+1 \Rightarrow \sup |\partial^\alpha \psi_n| \le \delta \quad \forall n$

Now we need to construct, say, $W^u(x_n)$ $\forall n$ and for that we need to construct the manifolds

$W^u_n \subset \mathbb{R}^d$, $0 \in W^u_n$,

$W^u_n = G_u(F^u_n)$, $F^u_n : B_{\mathbb{R}^{d_u}}(0,1) \to B_{\mathbb{R}^{d_s}}(0,1)$

such that in particular

$\psi_n(W^u_n) \cap B = W^u_{n+1}$ where

$B = \{(x_u, x_s) \in \mathbb{R}^d : |x_u| \leq 1, |x_s| \leq 1\}$

As before, we use the <u>graph transform</u>

$\Phi^u_n : \psi_n(G_u(F)) \cap B = G_u(\Phi^u_n F)$.

We need to find a sequence of functions $(F^u_n)_{n\in\mathbb{Z}}$ such that

$$\boxed{\Phi^u_n(F^u_n) = F^u_{n+1}.} \qquad (*)$$

For the case of a fixed point, all $\mathcal{F}_n$ were the same, so all Φ^u_n were also the same: $\Phi^u_n = \Phi^u \quad \forall n \in \mathbb{Z}$.
We used that Φ^u was a <u>contraction</u> on the functional space $\mathcal{X}$ (see the end of §8.1)
Now we know that <u>each</u> Φ^u_n is a contraction on $\mathcal{X}$.
So define the metric space
$$\mathcal{X}^{\mathbb{Z}} = \{(F_n)_{n\in\mathbb{Z}} : F_n \in \mathcal{X} \quad \forall n\}$$
with the metric $d((F_n),(\tilde{F}_n)) :=$
$$:= \sup_{n\in\mathbb{Z}} d_{\mathcal{X}}(F_n, \tilde{F}_n).$$
Then $\mathcal{X}^{\mathbb{Z}}$ is a complete metric space.
Consider the map $\Phi^{\mathbb{Z}}_u : \mathcal{X}^{\mathbb{Z}} \circlearrowleft$,
$$\Phi^{\mathbb{Z}}_u((F_n)) = (\hat{F}_n) \text{ with } \hat{F}_{n+1} = \Phi^u_n F_n.$$
Then $\Phi^{\mathbb{Z}}_u$ is a contraction on $\mathcal{X}^{\mathbb{Z}}$.
It thus has a fixed point $(F^u_n)_{n\in\mathbb{Z}}$ which gives the solution to $(*)$.
[Some technical details missing, see [D, §4.5] for more]