# §6. Hyperbolic surfaces

A <u>hyperbolic surface</u> is a complete connected oriented 2-dimensional Riemannian manifold of constant Gauss curvature $=-1$.

Here we study geodesic flows on <u>compact</u> hyperbolic surfaces.

## §6.1. Hyperbolic geometry

We quickly review some hyperbolic geometry here. See e.g.

Ratcliffe, "Foundations of Hyperbolic Manifolds"

or

Borthwick, "Spectral Theory of Infinite-Area Hyperbolic Surfaces"

for more details

We start with the <u>hyperbolic plane</u> $\mathbb{H}^2$.

There are a few models, we will most commonly use the <u>upper half-plane model</u>:

$\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$,

metric $g = \dfrac{dx^2 + dy^2}{y^2}$.

We often write $\boxed{z = x+iy}$ and

$\mathbb{H}^2 = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, $g = \dfrac{|dz|^2}{(\operatorname{Im} z)^2}$.

<u>Note</u>: $\{y=0\}$ is not in $\mathbb{H}^2$, it is (part of) the <u>boundary at infinity</u> $\partial\mathbb{H}^2 = \dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

A direct computation shows that $(\mathbb{H}^2, g)$ has Gauss curvature $= -1$.

# Geodesics on $\mathbb{H}^2$:

We can parametrize $S\mathbb{H}^2 = \{(z,v) : z \in \mathbb{H}^2, v \in \mathbb{R}^2, |v|_{g(z)} = 1\}$

by $(x,y,\theta) \in \mathbb{R} \times (0,\infty) \times S^1$, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$
as follows: $v = (y\cos\theta, y\sin\theta)$.

Contact form for the geodesic flow:

$$\alpha = \langle v, (dx, dy) \rangle_{g(x,y)}$$
$$= y^{-1}(\cos\theta\, dx + \sin\theta\, dy).$$

Reeb vector field = generator of the geodesic flow:

$V \in C^\infty(S\mathbb{H}^2; T(S\mathbb{H}^2))$ such that

$\alpha(V) = 1$, $\iota_V d\alpha = 0$

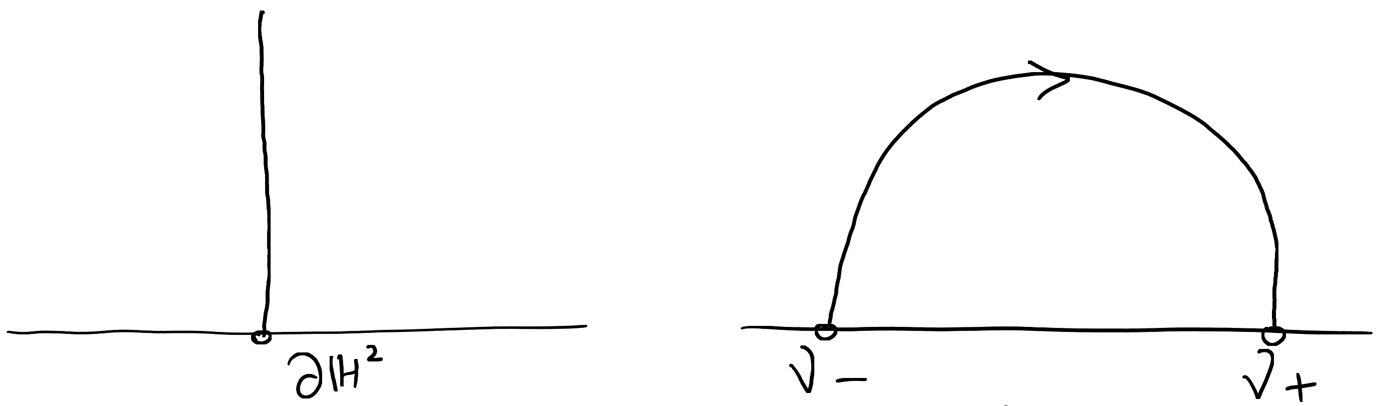
We compute

$$d\alpha = \frac{\sin\theta}{y} dx \wedge d\theta - \frac{\cos\theta}{y} dy \wedge d\theta + \frac{\cos\theta}{y^2} dx \wedge dy$$ and

$$\boxed{V = y\cos\theta\, \partial_x + y\sin\theta\, \partial_y - \cos\theta\, \partial_\theta.}$$

$\varphi^t = e^{tV}$ is the geodesic flow on $S\mathbb{H}^2$

Geodesics on $\mathbb{H}^2$ are either vertical lines or circles orthogonal to $\{y=0\}$:




Each geodesic $\gamma(t)$ has two limiting points at infinity

$$\gamma_\pm = \lim_{t\to\pm\infty} \gamma(t) \in \partial\mathbb{H}^2 = \dot{\mathbb{R}} = \mathbb{R}\cup\{\infty\}, \quad \gamma_+ \neq \gamma_-$$

(if the geodesic is a vertical line, then either $\gamma_+$ or $\gamma_-$ equals $\infty$)

Isometries on $\mathbb{H}^2$:

Let $PSL(2,\mathbb{R})$ be the quotient of $SL(2,\mathbb{R}) = \{A \ 2\times 2 \text{ real matrix}, \det A = 1\}$ by the order 2 group generated by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.
(which is central and thus normal)

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$,
define the map $\gamma_A : \mathbb{H}^2 \to \mathbb{H}^2$ by

$$\gamma_A(z) = \frac{az+b}{cz+d}, \quad z \in \mathbb{H}^2.$$

$\mathbb{H}^2 = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$

Basic properties:

- $\gamma_A$ is an orientation preserving diffeomorphism of $\mathbb{H}^2$ which is an isometry for the hyperbolic metric $g$.
- $\gamma_A$ extends to a diffeomorphism of $\partial \mathbb{H}^2 = \mathring{\mathbb{R}}$ as well
- $A \mapsto \gamma_A$ is a group homomorphism:
  $$\gamma_{AB} = \gamma_A \circ \gamma_B$$
- $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \gamma_A = \mathrm{Id}$ so we can define $\gamma_A$ for any $A \in PSL(2, \mathbb{R})$

This gives a group isomorphism
$$PSL(2, \mathbb{R}) \longrightarrow \text{orientation preserving isometries of } (\mathbb{H}^2, g)$$

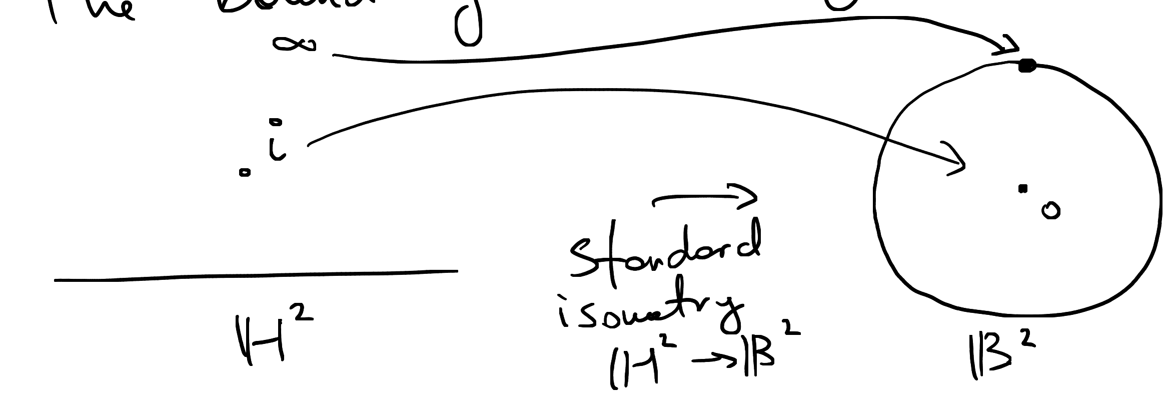
- Each $\gamma_A$ maps circles or lines to circles or lines and preserves angles (Euclidean angle = hyperbolic angle)

Note: another model of the hyperbolic plane is the ball model

$$\mathbb{B}^2 = \{ w \in \mathbb{C} : |w| < 1 \}$$

$$g = \frac{|dw|^2}{4(1-|w|^2)^2}. \quad \text{(It is isometric to } \mathbb{H}^2\text{)}$$

The boundary at infinity is $\partial \mathbb{B}^2 = S^1 \subset \mathbb{C}$

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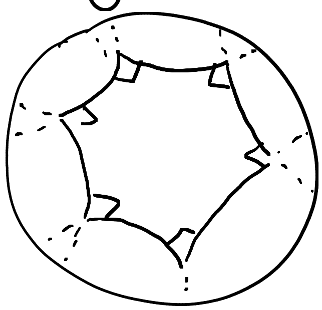
How to think of compact hyperbolic surfaces?

One way is to glue them from pairs of pants (Teichmüller theory)

One can obtain a pair of pants by starting with 2 copies of a right angled hyperbolic hexagon:



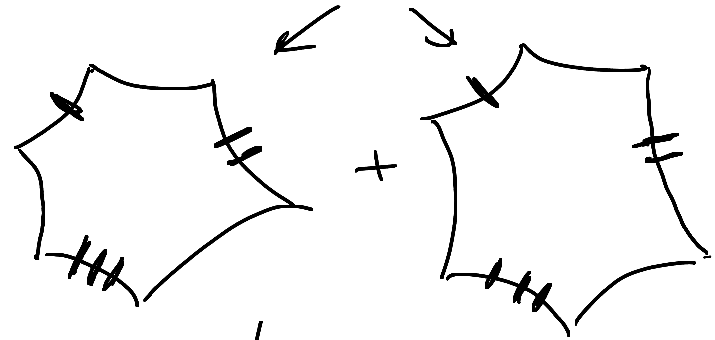
$\mathbb{B}^2$

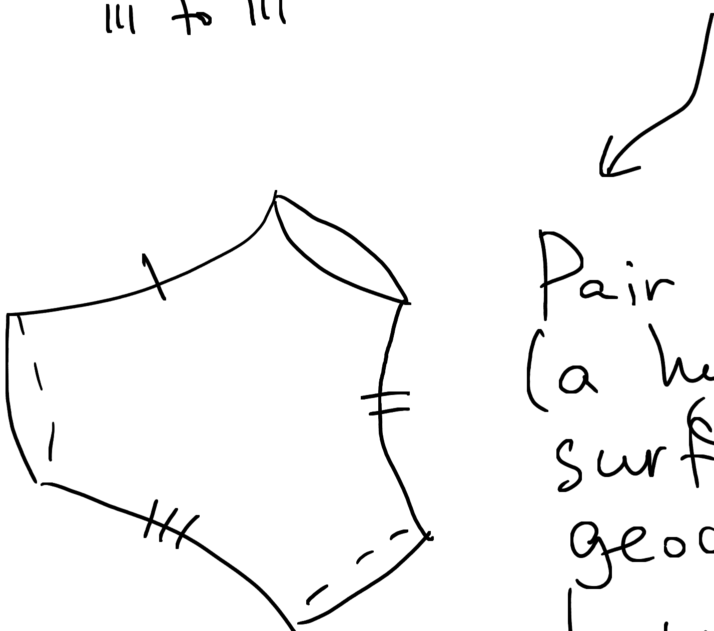
All sides are <u>geodesics</u> and all angles are $\pi/2$.

Same hexagon

**Glue:**

1 to 1
11 to 11
111 to 111

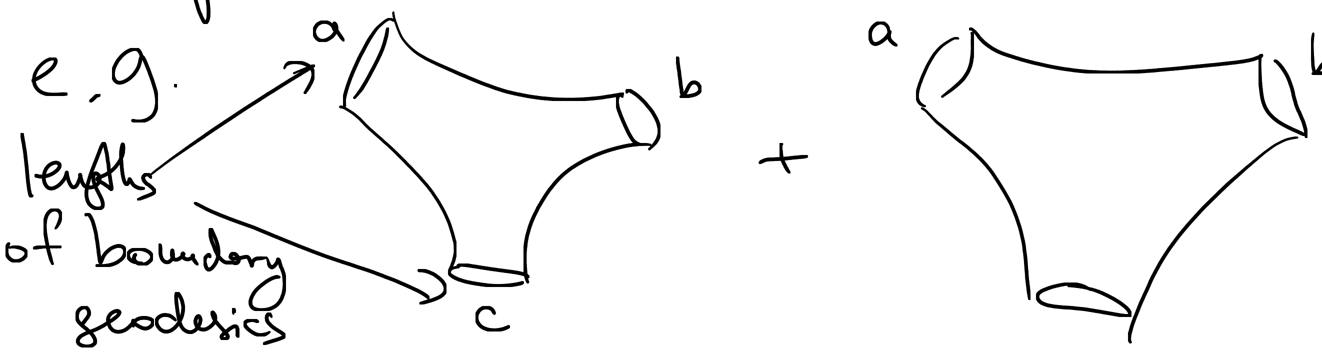




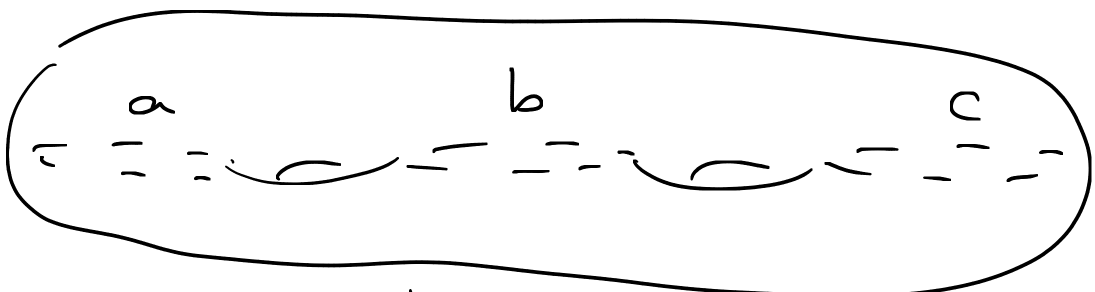
Pair of pants (a hyperbolic surface with geodesic boundary)

Can glue several pairs of pants together to get a compact surface without boundary,

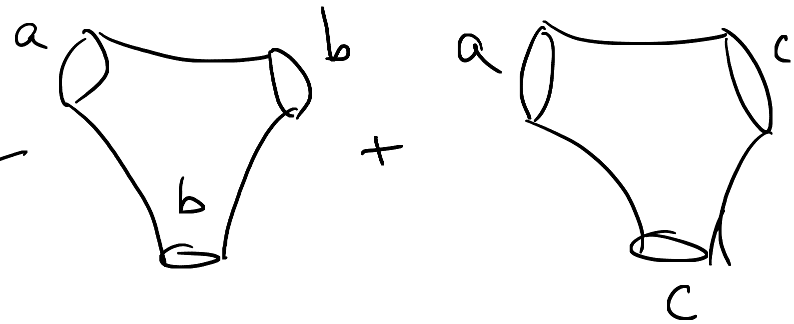
e.g.



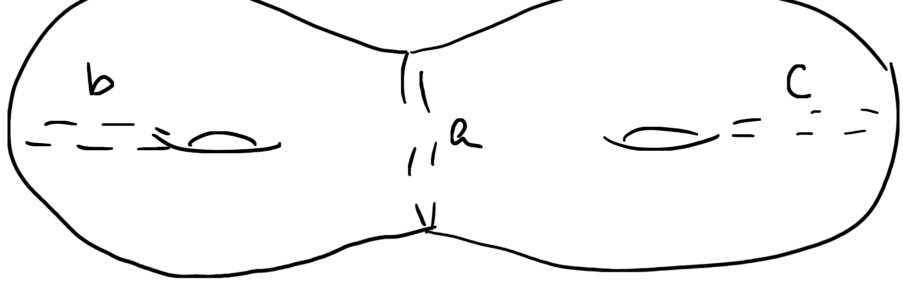

glue a to a
b to b
c to c




or, could take




glue a to a
b to b
c to c



Note: by Gauss-Bonnet Thm

every compact hyperbolic surface
(without boundary)
has genus ("# of holes") $\geq 2$
(i.e. cannot be a sphere or a torus)

Also, every compact hyperbolic surface
can be glued from pairs of pants
(that's one of the main points in Teichmüller theory)

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Another way to represent a hyperbolic surface
is as a quotient of $\mathbb{H}^2$:
if $M$ is a hyperbolic surface then

$$M \simeq \Gamma \backslash \mathbb{H}^2 \quad \text{where } \Gamma \subset PSL(2, \mathbb{R})$$

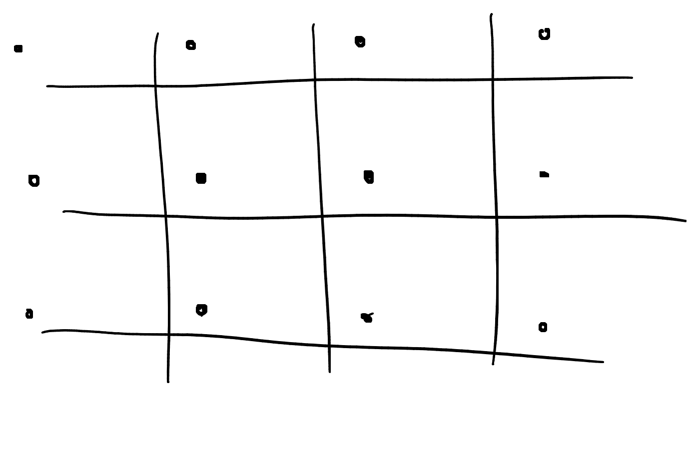
is some discrete subgroup.
This is true because each hyperbolic surface
is locally isometric to $\mathbb{H}^2$, and these local
isometries can be pieced together.
The covering map $\pi_\Gamma : \mathbb{H}^2 \to \Gamma \backslash \mathbb{H}^2 = M$
is a local diffeomorphism.

(*Note*: here $\Gamma\backslash\mathbb{H}^2$ denotes the set of $\Gamma$-orbits
$\{\gamma(z) \mid \gamma \in \Gamma\}$ where $z \in \mathbb{H}^2$.
This is similar to the representation of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

We write $\Gamma\backslash\mathbb{H}^2$ and not $\mathbb{H}^2/\Gamma$ because we defined a <u>left</u> action of $\Gamma$:
$(\gamma_1\gamma_2)(z) = \gamma_1(\gamma_2(z))$. This does not matter yet but will matter later.)

One can think of $M = \Gamma\backslash\mathbb{H}^2$ in terms of a <u>fundamental domain</u>: $D \subset \mathbb{H}^2$ such that each orbit of $\Gamma$ intersects $D$ exactly once.

E.g. for the torus $\mathbb{T}^2 = \mathbb{Z}^2\backslash\mathbb{R}^2$ get the standard tessellation by fundamental domains:
(showing <u>one</u> orbit of $\mathbb{Z}^2$)



# §6.2. Hyperbolicity of the geodesic flow

Assume that $(M,g)$ is a compact hyperbolic surface.

We will show that the geodesic flow $\varphi^t = e^{tV}: SM \circlearrowleft$ is an Anosov flow (i.e. it is hyperbolic on the entire $SM$) by explicitly constructing the stable and unstable spaces.

We first introduce a particular frame on $SM$, i.e. 3 vector fields which form a basis at each point: $\boxed{V,\ W,\ V_\perp}$

- $V$ is the generator of the geodesic flow
- $W$ is the generator of rotations in the fibers of $SM$: for $z \in M$, $v \in S_zM$

$$e^{sW}(z,v) = (z, v_s) \text{ where}$$

$v_s$ is obtained by rotating $v$ ccw by angle $s$.

Recall that $M$ is locally isometrically diffeomorphic to $\mathbb{H}^2$.
So we can use the coordinates $(x, y, \theta)$ on $S\mathbb{H}^2$ as local coordinates ~~on~~ $SM$. In these coordinates,

$$\boxed{V = y\cos\theta \cdot \partial_x + y\sin\theta \cdot \partial_y - \cos\theta \cdot \partial_\theta}$$

(computed in §6.1) and

$$\boxed{W = \partial_\theta}$$

Now, define the vector field

$$\boxed{V_\perp := [V, W]}$$

Here $[\cdot,\cdot]$ denotes Lie bracket of vector fields
vector fields: if we think of vector fields as $1^{st}$ order differential operators $C^\infty(SM)$s
then $[V,W]f = V(Wf) - W(Vf)$.
In coordinates coming from $\mathbb{H}^2$, we compute

$$\boxed{V_\perp = y\sin\theta\, \partial_x - y\cos\theta\, \partial_y - \sin\theta \cdot \partial_\theta}$$

We have the <u>commutation relations</u>

$$\boxed{[V,W]=V_\perp,\ [W,V_\perp]=V,\ [V,V_\perp]=W}$$

(these are Lie algebra relations of $sl(2,\mathbb{R})$, see Pset 3 for more...)

Note also that $V, W, V_\perp$ are linearly independent at each point and, with $\alpha$ the contact form of the geodesic flow, $\boxed{\operatorname{Ker}\alpha = \operatorname{Span}(W, V_\perp)}$.

<u>Remark</u> We did the computations above in local coordinates coming from a local isometry between $M$ and $\mathbb{H}^2$. However, the vector fields $V, W, V_\perp$ are well-defined on $M$.

If $\gamma \in PSL(2,\mathbb{R})$ acts $\gamma : \mathbb{H}^2 \circlearrowleft$ and is lifted naturally to $\tilde{\gamma} : S\mathbb{H}^2 \circlearrowleft$ then $\tilde{\gamma}$ maps each of the vector fields $V, W, V_\perp$ on $S\mathbb{H}^2$ to themselves (that is, they are $PSL(2,\mathbb{R})$-left invariant) See Pset 3 for more!

Now, we define the

<u>unstable horocyclic field</u> $U_-$

and the <u>stable horocyclic field</u> $U_+$

by $\boxed{U_+ = V_\perp + W, \quad U_- = V_\perp - W}$

Then we have the commutation relations

(*) $\boxed{[V, U_\pm] = \pm U_\pm}$

We can now prove hyperbolicity of the geodesic flow $\varphi^t = e^{tV}$:

<u>Thm</u> For each $\rho \in SM$ and $t \in \mathbb{R}$ we have

$$d\varphi^t(\rho)\, U_+(\rho) = e^{-t}\, U_+(\varphi^t(\rho)),$$
$$d\varphi^t(\rho)\, U_-(\rho) = e^{t}\, U_-(\varphi^t(\rho)).$$

Thus $\varphi^t$ is hyperbolic on the entire $SM$, with the stable/unstable spaces

$$\boxed{E_s(\rho) = \mathbb{R}\cdot U_+(\rho), \quad E_u(\rho) = \mathbb{R}\cdot U_-(\rho)}$$

<u>Proof</u> We show that

$$d\varphi^t(\rho)\,U_-(\rho) = e^t U_-(\varphi^t(\rho)),$$

with the case of $U_+$ handled similarly.

To do that we use the commutation relation (*): $\boxed{[V, U_-] = -U_-}$.

(basically we just use the definition of the Lie derivative & the formula $\mathcal{L}_V U_- = [V, U_-]$ but write out the details here)

Take any $f \in C^\infty(SM)$.

Consider the pullback operator

$$\varphi_t^* : C^\infty(SM) \circlearrowleft, \quad \varphi_t^* f := f \circ \varphi_t.$$

Since $\varphi_t$ is the flow of $V$, we have

$$\boxed{\partial_t(\varphi_t^* f) = \varphi_t^*(Vf) = V(\varphi_t^* f)}$$

where $V$ acts as a 1-st order differential operator.

Now, look at the function

$$u_t := \varphi_{-t}^* U_- \varphi_t^* f \in C^\infty(SM), \quad t \in \mathbb{R}.$$

Now, we differentiate in $t$:

$$\partial_t u_t = \partial_t (\varphi_{-t}^* U_- \varphi_t^* f)$$
$$= -\varphi_{-t}^* V U_- \varphi_t^* f + \varphi_{-t}^* U_- V \varphi_t^* f$$
$$= -\varphi_{-t}^* [V, U_-] \varphi_t^* f = \quad (\text{by } (*))$$
$$= \varphi_{-t}^* U_- \varphi_t^* f = u_t.$$

From here and the fact that $u_0 = U_- f$ (as $\varphi_0^* = \mathrm{Id}$) we see that $\boxed{u_t = e^t U_- f}$. Thus $\forall p \in SM$

$$u_t(\varphi_t(p)) = e^t (U_- f)(\varphi_t(p)) = \underline{e^t} \cdot df(\varphi_t(p)) \cdot \underline{\underline{U_-(\varphi_t(p))}}$$

and $u_t(\varphi_t(p)) = U_- \varphi_t^* f(p) =$

$$= d(f \circ \varphi_t)(p) \cdot U_-(p) =$$
$$= df(\varphi_t(p)) \cdot \underline{d\varphi_t(p)} \cdot \underline{\underline{U_-(p)}}.$$

Since $f$ is arbitrary, this gives

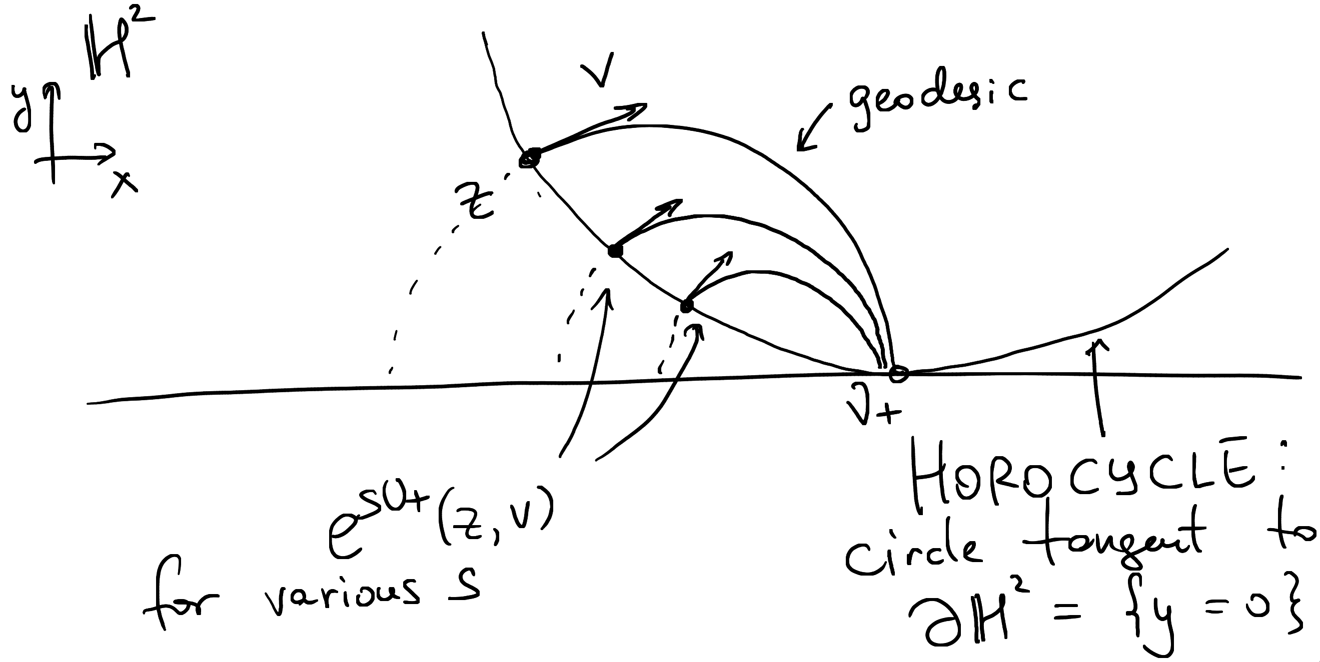
$$d\varphi_t(p) \cdot U_-(p) = e^t U_-(\varphi_t(p))$$

as needed. $\square$

Why are $U_\pm$ called <u>horocyclic</u> vector fields?

On $S\mathbb{H}^2$, their flows $e^{sU_\pm} : SM \circlearrowleft$ <u>horocyclic flows</u>:

if $(z, v) \in S\mathbb{H}^2$ then the geodesic starting at $e^{sU_+}(z,v)$ has the same limiting point $\nu_+$ at $t=\infty$ as the geodesic starting at $(z,v)$:




For $e^{sU_-}(z,v)$ we have a similar property, but now the limiting point at $t=-\infty$ is fixed.

18.118
6-18

To see why this is true, we can use the fact that $PSL(2,\mathbb{R})$ acts transitively on $\mathbb{H}^2$ to reduce to the case

$$z = i,\ v = i:$$

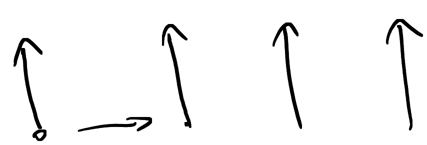
$\mathbb{H}^2$ $\quad v = i = (0,1)$

$\quad z = i = (0,1)$

In $(x,y,\theta)$ coordinates, this is $\boxed{x = 0,\ y = 1,\ \theta = \pi/2}$

We have $U_+ = V_\perp + W =$
$$= y\sin\theta \cdot \partial_x - y\cos\theta \cdot \partial_y + (1-\sin\theta)\,\partial_\theta.$$
The flow line $e^{sU_+}(i,i) = (x(s), y(s), \theta(s))$ satisfies the ODEs
$$\dot{x} = y\sin\theta,\quad \dot{y} = -y\cos\theta,\quad \dot{\theta} = 1-\sin\theta.$$
Starting with $\theta_0 = \pi/2$, we get $\boxed{\theta(s) = \pi/2}$ $\forall s$,

So then $x(s) = s,\ y(s) = 1,\ \theta(s) = \pi/2$:



all the geodesics go to $\nu_+ = \infty$ at $t \to \infty$

To finish up this subsection, we state without proof

<u>Thm</u> [Furstenberg].
Assume that $(M,g)$ is a compact hyperbolic surface. Then the <u>horocycle flows</u> $e^{sU_\pm}$ are <u>uniquely ergodic</u>: there is only one probability measure invariant under $e^{sU_+}$ (or $e^{sU_-}$) and it is the Liouville measure. In particular, <u>each</u> trajectory $\{e^{sU_+}(z,v) : s \in \mathbb{R}\}$ is <u>dense</u> in $SM$.

<u>Remarks</u> 1) This is not true for the <u>geodesic</u> flow $e^{tV}$: there are plenty of closed geodesics

2) This Thm is true in much higher generality (but we won't go there in 118)

# §6.3. Mixing of the geodesic flow

Assume that $(M,g)$ is a compact hyperbolic surface, $X = SM$,

$\varphi^t = e^{tV} : X \circlearrowleft$ is the geodesic flow.

$\mu$ = the Liouville measure, a $\varphi^t$-invariant probability measure on $X$.

(in coordinates, $\mu$ = Lebesgue measure with a $C^\infty$ positive density)

We will prove

**Thm** $\varphi^t$ is <u>mixing</u> w.r.t. $\mu$, that is

$\forall f, g \in L^2(X, \mu)$

$$\int_X f(g \circ \varphi^t)\, d\mu \xrightarrow[t\to\infty]{} \left(\int_X f\, d\mu\right)\left(\int_X g\, d\mu\right).$$

<u>Remark</u> This implies that $\forall t > 0$, the <u>map</u> $\varphi^t : X \circlearrowleft$ is <u>mixing</u> w.r.t. $\mu$

(i.e. $\forall f, g \in L^2$, $\int_X f(g \circ \varphi^{nt})\, d\mu \xrightarrow[n\to\infty]{} \left(\int_X f\, d\mu\right)\left(\int_X g\, d\mu\right)$)

and thus $\forall t > 0$, the map $\varphi^t$ is <u>ergodic</u> w.r.t. $\mu$.

By Birkhoff's ergodic theorem for the map $\varphi^1$, we see that $\forall \tilde{f} \in L^1(X,\mu)$,

$$\frac{1}{n}\sum_{j=0}^{n-1} \tilde{f}(\varphi^j(x)) \xrightarrow[n\to\infty]{} \int_X \tilde{f}\, d\mu$$

for $\mu$-almost every $x \in X$.

With a bit of work (exercise, no credit), applying this to $\tilde{f}(x) = \int_0^1 f(\varphi^s(x))\,ds$, we see that $\forall f \in L^1(X,\mu)$, for $\mu$-almost every $x \in X$

$$\frac{1}{T}\int_0^T f(\varphi^t(x))\,dt \longrightarrow \int_X f\, d\mu.$$

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We note that Thm is equivalent to saying that $\forall g \in L^2(X,\mu)$,

$$g\circ\varphi^t \xrightarrow[t\to\infty]{} \int_X g\, d\mu \quad \underline{\underline{\text{weakly}}} \text{ in } L^2(X,\mu).$$

We use the following

__Thm__ [Banach-Alaoglu] Assume $f_j \in L^2(X,\mu)$ is a sequence bounded in $L^2$ norm. Then $\exists$ subsequence $f_{j_k} \longrightarrow$ some $f \in L^2$ weakly in $L^2$.

# Proof of Banach-Alaoglu (sketch)

Let $(e_\ell)_{\ell\in\mathbb{N}}$ be an orthonormal basis of $L^2(X,\mu)$.

Since $\exists C\ \forall j\ \|f_j\|_{L^2} \le C$,

for each $\ell$ the sequence $j \mapsto \langle f_j, e_\ell\rangle_{L^2}$ is bounded.

By a diagonal argument there is a subsequence $f_{j_k}$ such that

$$\forall \ell,\quad \langle f_{j_k}, e_\ell\rangle \xrightarrow[k\to\infty]{} c_\ell$$ ← some number.

Define $S = \{$linear combinations of $e_\ell\}$ which is dense in $L^2$.

We have $\forall g\in S$, $\langle f_{j_k}, g\rangle \xrightarrow[k\to\infty]{} \sum_\ell c_\ell \langle e_\ell, g\rangle$.

From here & the fact that $\|f_{j_k}\|_{L^2}\le C$ we get that $\sum_\ell |c_\ell|^2 \le C^2$.

Put $f := \sum_\ell c_\ell e_\ell \in L^2$, then

$$\langle f_{j_k}, g\rangle \xrightarrow[k\to\infty]{} \langle f, g\rangle \quad \forall g\in S. \qquad (*)$$

Since $S$ is dense in $L^2$, we see from Lemma in §1.1 that $(*)$ holds $\forall g\in L^2$, that is $f_{j_k}\to f$ weakly in $L^2$. $\square$

Coming back to mixing for the geodesic flow on a hyperbolic surface: $\forall g \in L^2(X,\mu)$, the family $g\circ\varphi^t$ is bounded in $L^2$ norm uniformly in $t$ (as $\|g\circ\varphi^t\|_{L^2} = \|g\|_{L^2}$).

So $\forall$ sequence $t_j \to \infty$
$\exists$ subsequence $t_{j_k}$ s.t.
$g\circ\varphi^{t_{j_k}} \to$ some $h \in L^2(X,\mu)$ weakly in $L^2$.

Thus Thm (mixing) follows from

__Prop.__ Assume that $g \in L^2(X,\mu)$ and $g\circ\varphi^{t_j} \xrightarrow[j\to\infty]{} h$ weakly in $L^2$ for some $t_j \to \infty$. Then $h = \text{const}$ $\mu$-a.e.

__Proof of Prop $\Rightarrow$ Thm__ Assume $g\circ\varphi^t \underset{t\to\infty}{\not\to} \int_X g\,d\mu$ weakly in $L^2$. Then $\exists f \in L^2, \varepsilon>0$ & a sequence $t_j \to \infty$: $|\langle g\circ\varphi^{t_j}, f\rangle - \int_X g\,d\mu \cdot \langle 1, f\rangle| \geq \varepsilon$

By Banach–Alaoglu, passing to a subsequence we may assume $g\circ\varphi^{t_j} \to$ some $h \in L^2$ weakly. So $g\circ\varphi^{t_j} \to \int_X g\,d\mu$ weakly, a contradiction.

By Prop, $h = \text{const} = \int_X g\,d\mu$ (by pairing with the function 1) $\square$

We now prove the <u>Proposition.</u>

Recall the flow/stable/unstable vector fields from §6.2: $(X = SM)$

$$V,\ U_+,\ U_- \in C^\infty(X; TX)$$

linearly independent at each point.

<u>**Defn.**</u> Let $W$ be a $C^\infty$ vector field on $X$.

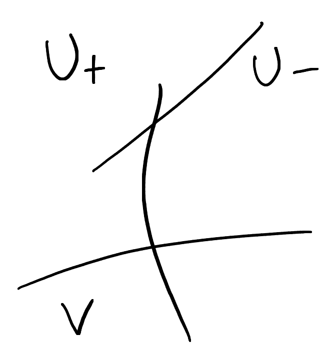
We say that $f \in L^1(X)$ is <u>$W$-invariant</u> (or invariant under the flow $e^{sW}$ of $W$)

if $\forall s \in \mathbb{R}$, $f = f \circ e^{sW}$ Lebesgue <u>almost</u> everywhere.

<u>Note:</u> $f = f \circ e^{sW}$ <u>everywhere</u> <u>$\forall s$</u> $\Leftrightarrow$

$\Leftrightarrow$ $f$ is constant on each <u>flow line</u> of $W$.

<u>Lemma</u> If $f \in L^1(X)$ is invariant under $V$, $U_+$, and $U_-$, then $f =$ constant Lebesgue almost everywhere.



## Lazy proof (using 18.155)

If $f$ is $W$-invariant, then $Wf = 0$ in the sense of distributions

(here we treat $W$ as a $1^{st}$ order differential operator)
because $\partial_s|_{s=0}(f \circ e^{sW}) = Wf$ in distributions

So we get $Vf = 0$, $U_+ f = 0$, $U_- f = 0$.

So $Pf = 0$ (in distributions)

where $P = V^2 + U_+^2 + U_-^2$

is an elliptic $2^{nd}$ order differential operator

So by elliptic regularity $f \in C^\infty(X)$.

But then $Vf = 0 = U_+ f = U_- f$ means that $df = 0 \Rightarrow f = \text{const}$ since $X = SM$ is connected. $\square$

There is a less lazy proof using Fubini's Thm but I will not give it here.

We come back to the proof of the Proposition.

Assume that $g \in L^2(X,\mu)$ and $\boxed{g \circ \varphi^{t_j} \xrightarrow[j\to\infty]{} h}$ weakly in $L^2$ for some $t_j \to \infty$.

We will show that h is invariant under $U_+$, $U_-$, and $V$. By the Lemma this will give that h is constant a.e. (Lebesgue) and finish the proof of the Proposition (and thus the Thm about mixing of the geodesic flow on a hyperbolic surface)

---

1. h is invariant under $U_+$.

Fix $s \in \mathbb{R}$, we need to show that

$$h \circ e^{sU_+} = h \quad \text{(Lebesgue almost everywhere)}$$

The map $f \mapsto f \circ e^{sU_+}$ is a bounded operator on $L^2(X,\mu)$, so we have

$$g \circ \varphi^{t_j} \circ e^{sU_+} - g \circ \varphi^{t_j} \xrightarrow[j\to\infty]{} h \circ e^{sU_+} - h$$

weakly in $L^2$

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So it suffices to show that

$$\| g\circ\varphi^t\circ e^{sU_+} - g\circ\varphi^t \|_{L^2} \xrightarrow{t\to\infty} 0. \quad (*)_+$$

(weak convergence would be enough, but we show convergence in norm)

Consider the bounded operators for $t\geq 0$

$$B_t^+ : L^2(X,\mu) \circlearrowleft,$$

$$B_t^+ g = g\circ\varphi^t\circ e^{sU_+} - g\circ\varphi^t.$$

Since pullback by $\varphi^t$ is an isometry,

$\| B_t^+ \|_{L^2 \circlearrowleft}$ is bounded uniformly in $t$.

So by the Lemma in §1.1

it suffices to show $(*)_+$ for all $g\in C^\infty(X)$

(as $C^\infty$ is dense in $L^2$)

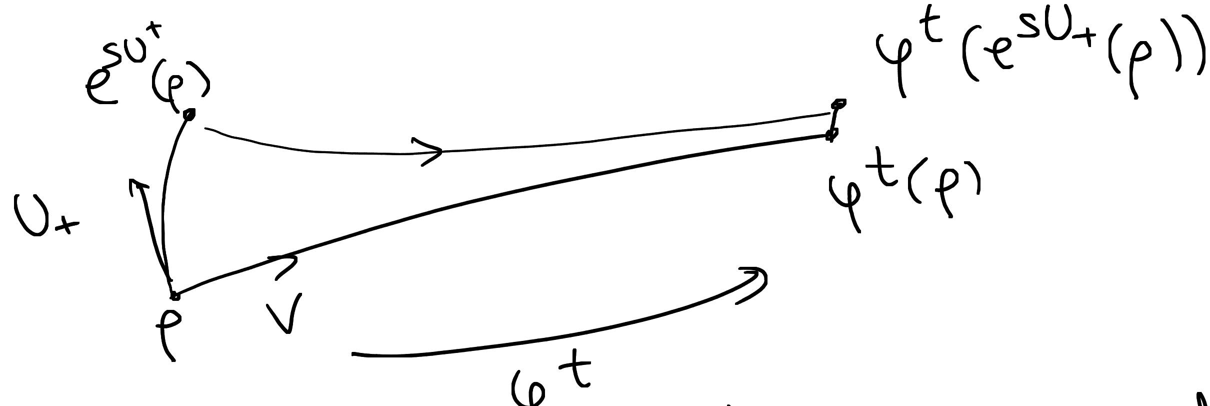
Recall from §6.2 that $\forall \rho\in X$

$$d\varphi^t(\rho)\, U_+(\rho) = e^{-t}\, U_+(\varphi^t(\rho)).$$

This implies that

$$\varphi^t(e^{sU_+}(\rho)) = e^{e^{-t}sU_+}(\varphi^t(\rho)).$$

Picture:




(flow lines of $e^{sU_+}$ are mapped to other flow lines of $e^{sU_+}$ but shrank exponentially as $t \to \infty$)

So for $g \in C^\infty(X)$ and $\rho \in X$

$$|B_t^+ g(\rho)| = |g(\varphi^t(e^{sU_+}(\rho))) - g(\varphi^t(\rho))|$$
$$= |g(e^{e^{-t}sU_+}(\varphi^t(\rho))) - g(\varphi^t(\rho))|$$
$$\leq C_g \cdot e^{-t} s \xrightarrow[t\to\infty]{} 0$$

uniformly in $\rho$ and thus in $L^2$ in $\rho$.

2. $h$ is invariant under $U_-$.

Similarly to 1, it suffices to show that $\forall g \in L^2(X,\mu)$, $\forall s \in \mathbb{R}$

$(*)_-$ $g \circ \varphi^t \circ e^{sU_-} - g \circ \varphi^t \xrightarrow[t \to \infty]{} 0$ weakly in $L^2$.

Define the space

$$I = \{ f \in L^2(X,\mu) \mid \forall s \in \mathbb{R},\ f \circ e^{sU_-} = f \}$$

= the space of $U_-$-invariant $L^2$ functions.

Then $I \subset L^2$ is a closed subspace,

so $L^2(X,\mu) = I \oplus I^\perp$.

It remains to consider 2 cases:

$\boxed{\text{Case 1:}}$ $g \in I$. Then $\forall t \in \mathbb{R}$ we have $g \circ \varphi^t \in I$ as well, that is $\forall s \in \mathbb{R}$

$g \circ \varphi^t \circ e^{sU_-} - g \circ \varphi^t = 0$, thus $(*)_-$ holds. Indeed, similarly to 1. we get $\varphi^t(e^{sU_-}(\rho)) = e^{e^t sU_-}(\varphi^t(\rho))$,

So $g \circ \varphi^t \circ e^{sU_-} - g \circ \varphi^t =$

$= (g \circ e^{e^t sU_-} - g) \circ \varphi^t = 0$ as $g \in I$.

**Case 2:** $g \in I^\perp$. In this case we will show that $\boxed{g \circ \varphi^t \xrightarrow[t\to\infty]{} 0}$ weakly in $L^2$, which also implies that

$\forall s \in \mathbb{R}$, $g \circ \varphi^t \circ e^{sU_-} \xrightarrow[t\to\infty]{} 0$ weakly in $L^2$ and gives $(*)_-$.

Since $\|g \circ \varphi^t\|_{L^2} = \|g\|_{L^2}$, using Banach-Alaoglu it suffices to show that every subsequential weak limit $= 0$:

if $t_j \to \infty$ and $\boxed{g \circ \varphi^{t_j} \xrightarrow[j\to\infty]{} q \in L^2}$ weakly in $L^2$, then $q = 0$.

Since $\|q \circ \varphi^{-t_j}\|_{L^2} = \|q\|_{L^2}$, by Banach-Alaoglu we can pass to a subsequence $(t_{j_k})$ to make

$\boxed{q \circ \varphi^{-t_{j_k}} \xrightarrow[k\to\infty]{} r \in L^2}$ weakly in $L^2$.

Arguing similarly to 1 but backwards in time we see that $r$ is $U_-$-invariant, that is $\boxed{r \in I}$.

Now compute

$$\|q\|_{L^2}^2 = \langle q, q\rangle_{L^2} = \lim_{k\to\infty} \langle g\circ\varphi^{t_{jk}}, q\rangle_{L^2}$$

$$= \lim_{k\to\infty} \langle g, q\circ\varphi^{-t_{jk}}\rangle_{L^2}$$

$$= \langle g, r\rangle_{L^2} = 0 \quad \text{as} \quad g\in I^\perp, r\in I.$$

Thus $q=0$ as needed.

---

3. $h$ is invariant under $V$.

(This part would fail for constant roof fn. suspensions of cat maps... here we effectively use that $\varphi^t$ is a contact flow)

We only use here that $h\in L^2$ is invariant under $U_+$ and $U_-$.

The key is that $U_+$ and $U_-$ do not commute: from §6.2

$$[U_+, U_-] = [V_\perp + W, V_\perp - W]$$

$$= 2[W, V_\perp] = 2V.$$

A simple way to see that $h$ is $V$-invariant is via <u>distributions</u>:

$h$ is $U_\pm$ invariant $\Rightarrow$

$\Rightarrow U_+ h = U_- h = 0$ in distributions

$\Rightarrow 2Vh = U_+ U_- h - U_- U_+ h = 0$

(also in distributions)

$\Rightarrow h$ is $V$-invariant.

---

Without using distributions, we can see that $h$ is $V$-invariant by using a commutation identity for the flows $e^{tV}, e^{sU_\pm} : X \circlearrowleft$, namely (see Pset 3)

$$\boxed{e^{aU_+} \circ e^{bU_-} = e^{b'U_-} \circ e^{a'U_+} \circ e^{tV}} \qquad (*)$$

for all $a, b, a', b', t \in \mathbb{R}$ such that

$$\boxed{ab = e^{-t/2} - 1, \quad a' = e^{-t/2} a, \quad b' = e^{t/2} b}$$

For each $t \in \mathbb{R}$ we can pick $a, b, a', b'$ so that $(*)$ holds. We have

$$h \circ e^{aU_+} \circ e^{bU_-} = h \circ e^{b'U_-} \circ e^{a'U_+} \circ e^{tV}.$$

Since $h$ is $U_+$-invariant
and $U_-$-invariant,
we get Lebesgue almost everywhere

$$h = h \circ e^{tV}$$

which (as $t$ is arbitrary)
shows that $h$ is $V$-invariant
and finishes the proof of
mixing of geodesic flows on compact
hyperbolic surfaces.