# §3. Mixing and recurrence

## §3.1. Mixing

Assume that $X$ is a metric space, $T: X \to X$ is Borel measurable, and $\mu$ is a $T$-invariant probability measure on $X$.

<u>Defn.</u> We say $T$ is <u>mixing</u> with respect to $\mu$, if for each Borel sets $A, B \subset X$ we have

$$\mu(A \cap T^{-n}(B)) \xrightarrow[n\to\infty]{} \mu(A)\mu(B)$$

<u>Picture</u>:


Want the events "$x \in A$" and "$T^n(x) \in B$" to be independent (w.r.t. $\mu$) asymptotically in the limit $n \to \infty$.

**Thm** If $T$ is mixing w.r.t. $\mu$ then $T$ is ergodic w.r.t. $\mu$.

(MIXING $\Rightarrow$ ERGODIC)

**Proof** Let $A \subset X$ be a $T$-invariant Borel set.

Then $\forall n$, we have $T^{-n}(A) = A$.

If $T$ is mixing then

$$\mu(A) = \mu(A \cap T^{-n}(A)) \xrightarrow[n\to\infty]{} \mu(A)^2.$$

Thus $\mu(A) = 0$ or $\mu(A) = 1$. $\square$

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**Thm** TFAE:

1. $T$ is mixing w.r.t. $\mu$
2. There is a dense subspace $S \subset L^2(X,\mu)$ such that for all $f, g \in S$
$$\int_X f\,(g \circ T^n)\,d\mu \xrightarrow{n\to\infty} \left(\int_X f\,d\mu\right)\left(\int_X g\,d\mu\right) \qquad (*)$$
3. $(*)$ holds for all $f, g \in L^2(X,\mu)$.

**Proof** ① $\Rightarrow$ ② :

Let $S = \operatorname{Span}\{ \mathbb{1}_A \mid A \subset X \text{ Borel} \}$

Then $S$ is dense in $L^2(X,\mu)$

(measure theory...).

To show $(*)$ for all $f, g \in S$

it suffices to show it when

$f = \mathbb{1}_A, \; g = \mathbb{1}_B, \; A, B \subset X$ Borel.

Then $\int_X f(g \circ T^n)\, d\mu = \mu(A \cap T^{-n}(B))$

and $\left(\int_X f\, d\mu\right)\left(\int_X g\, d\mu\right) = \mu(A)\mu(B)$.

② $\Rightarrow$ ③ Let $f, g \in L^2(X,\mu)$. Since

$S$ is dense in $L^2(X,\mu)$, we can take

sequences $f_k, g_k \in S$ : $f_k \xrightarrow[k\to\infty]{} f$, $g_k \xrightarrow[k\to\infty]{} g$ in $L^2(X,\mu)$.

Denote

$$I^n_{f,g} := \int_X f(g \circ T^n)\, d\mu - \left(\int_X f\, d\mu\right)\left(\int_X g\, d\mu\right).$$

Then we estimate $\forall n, k$

$$|I^n_{f,g} - I^n_{f_k,g_k}| \leq 2\|f - f_k\|_{L^2} \cdot \|g\|_{L^2} + 2\|f_k\|_{L^2} \cdot \underbrace{\|g - g_k\|_{L^2}}_{=:\alpha_k}$$

by Cauchy-Schwarz

and since $\|g \circ T^n\|_{L^2} = \|g\|_{L^2}$ $\forall g$ — $\alpha_k$ <u>indepdt. of n</u>

We have $\alpha_k \xrightarrow[k\to\infty]{} 0$.

Now, $\forall k$ we bound

$$\limsup_{n\to\infty} |I^n_{f,g}| \leq \limsup_{n\to\infty} |I^n_{f_k,g_k}| + \alpha_k$$

$$\leq \alpha_k, \quad \text{since } I^n_{f_k,g_k} \xrightarrow[n\to\infty]{} 0.$$

Taking $k \to \infty$ get $\limsup_{n\to\infty} |I^n_{f,g}| \leq 0$,

i.e. $I^n_{f,g} \xrightarrow{n\to\infty} 0$ as needed

<u>3 $\Rightarrow$ 1</u> Let $A, B \subset X$ be Borel.

Take $f := 1_A$, $g := 1_B$ in 3 to get

$$\mu(A \cap T^{-n}(B)) \xrightarrow[n\to\infty]{} \mu(A)\mu(B).$$

$\square$

__Note:__ mixing is the same as saying that $\forall f \in L^2(X,\mu)$,

$$f \circ T^n \xrightarrow[n\to\infty]{} \underbrace{\int_X f\,d\mu}_{\text{constant function}} \quad \underline{\text{weakly}} \text{ in } L^2(X,\mu)$$

where for $F_n, F \in L^2$ we say $F_n \to F$ __weakly in $L^2$__, if $\forall g \in L^2$, $\langle F_n, g\rangle \xrightarrow[n\to\infty]{} \langle F, g\rangle$.

However, we typically <u>do not have</u> $f \circ T^n \xrightarrow[n\to\infty]{} \int_X f\,d\mu$ in <u>norm</u> in $L^2$

Indeed, $\forall f \in L^2(X,\mu)$ the norm $\left\| f\circ T^n - \int_X f\,d\mu \right\|_{L^2}$ is independent of $n$.

# §3.2 Examples; cat maps

## (I) Irrational shift on the circle:

$X = \mathbb{R}/\mathbb{Z}$, $T(x) = x + r \mod \mathbb{Z}$
$r \in \mathbb{R} \setminus \mathbb{Q}$.

$\mu$ = Lebesgue measure

NOT MIXING:

take $f(x) = g(x) = e^{2\pi i x}$, then

$$\int_X f(\overline{g \circ T^n})\, dx = \int_0^1 e^{2\pi i x} e^{-2\pi i (x+nr)} dx$$

$$= e^{-2\pi i n r} \underset{n\to\infty}{\not\longrightarrow} 0 = \left(\int f dx\right)\left(\int g dx\right).$$

## (II) Expanding map on the circle:

$X = \mathbb{R}/\mathbb{Z}$, $T(x) = (2x) \mod \mathbb{Z}$

$\mu$ = Lebesgue measure.

IS MIXING:

Enough to show $\int_X f(g \circ T^n)\, d\mu \xrightarrow{n\to\infty} \int_X f d\mu \int_X g d\mu$

for all trigonometric polynomials $f, g$

which reduces to showing for $e_k(x) := e^{2\pi i k x}$, $k \in \mathbb{Z}$ that

$$\forall k, \ell \in \mathbb{Z}, \quad \langle e_k, e_\ell \circ T^n \rangle_{L^2} \to \begin{cases} 1, & \text{if } k = \ell = 0 \\ 0, & \text{otherwise.} \end{cases}$$

The latter is easy to check

since $e_\ell = e_{2^n \ell}$, so

$$\langle e_k, e_\ell \circ T^n \rangle_{L^2} = \begin{cases} 1, & \text{if } k = 2^n \ell \\ 0, & \text{otherwise.} \end{cases}$$

## III Hyperbolic toral automorphisms ("cat maps")

Now let $X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

and fix $A \in SL(2, \mathbb{Z})$,

i.e. $A$ is an integer $2\times 2$ matrix with $\det A = 1$. (note: $A^{-1} \in SL(2,\mathbb{Z})$ as well)

Define $T: X \to X$ by

$$\boxed{T(\vec{x} \bmod \mathbb{Z}^2) = (A\vec{x}) \bmod \mathbb{Z}^2}$$

for all $\vec{x} \in \mathbb{R}^2$.

Note that $T: X \circlearrowleft$ is a diffeomorphism and the Lebesgue measure $\mu$ is $T$-invariant (by the change of variables formula)

The behavior of $T$ depends on the eigenvalues of $A$.

To study these, we use

<u>Lemma</u> Let $A \in SL(2, \mathbb{R})$. Then:

- if $|\operatorname{tr} A| < 2$ then $\boxed{\text{ELLIPTIC CASE}}$
  $A$ has two non-real eigenvalues $\lambda, \bar{\lambda}$ with $|\lambda| = 1$
- if $|\operatorname{tr} A| = 2$ then $\boxed{\text{PARABOLIC CASE}}$
  $A$ has a double eigenvalue $1$ or a double eigenvalue $-1$
- if $|\operatorname{tr} A| > 2$ then $\boxed{\text{HYPERBOLIC CASE}}$
  $A$ has two real eigenvalues $\lambda, \lambda^{-1}$ with $|\lambda| > 1$

Proof The characteristic polynomial of $A$ is

$$P(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + 1$$

since $\det A = 1$.

It remains to use the quadratic formula. $\square$

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Now assume that $A \in SL(2, \mathbb{Z})$ is hyperbolic, i.e. $|\operatorname{tr} A| > 2$

Example: $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$,

in which case

$$T(x_1, x_2) = (2x_1 + x_2 \bmod \mathbb{Z},\ x_1 + x_2 \bmod \mathbb{Z})$$

Then $A$ is mixing w.r.t. the Lebesgue measure $\mu$.

Indeed, enough to show that $\forall \vec{k}, \vec{\ell} \in \mathbb{Z}^2$, we have

$$\langle e_{\vec{k}}, e_{\vec{\ell}} \circ T^n \rangle_{L^2} \to \begin{cases} 1, & \text{if } \vec{k} = \vec{\ell} = 0 \\ 0, & \text{otherwise} \end{cases}$$

where $e_{\vec{k}}(\vec{x}) = e^{2\pi i \langle \vec{k}, \vec{x} \rangle}$, $\vec{x} \in \mathbb{R}^2$

To see that, we compute

$$e_{\vec{\ell}} \circ T(\vec{x}) = e^{2\pi i \langle \vec{\ell}, A\vec{x} \rangle}$$
$$= e^{2\pi i \langle A^* \vec{\ell}, \vec{x} \rangle} = e_{A^* \vec{\ell}}$$

where $A^*$ is the transpose of $A$.

Thus $\langle e_{\vec{k}}, e_{\vec{\ell}} \circ T^n \rangle_{L^2} = \langle e_{\vec{k}}, e_{(A^*)^n \vec{\ell}} \rangle$

$$= \begin{cases} 1, & \text{if } \vec{k} = (A^*)^n \vec{\ell} \\ 0, & \text{otherwise.} \end{cases}$$

It remains to show that $\forall \vec{\ell} \in \mathbb{Z}^2 \setminus \{0\}$, we have

$$|(A^*)^n \vec{\ell}| \xrightarrow{n \to \infty} \infty.$$

If $\lambda, \lambda^{-1}$ are the eigenvalues of $A^*$ with $|\lambda| > 1$ and $E_\lambda, E_{\lambda^{-1}}$ are the corresponding eigenspaces, then we have 2 cases:

① $\vec{\ell} \notin E_{\lambda^{-1}}$. Then $|(A^*)^n \vec{\ell}| \xrightarrow{n \to \infty} \infty$

② $\vec{\ell} \in E_{\lambda^{-1}}$. Then $(A^*)^n \vec{\ell} \xrightarrow{n \to \infty} 0$ which cannot happen since this is a vector in $\mathbb{Z}^n$. $\square$

# §3.3. Poincaré recurrence



Here is another fact using invariant measures (not related to what we did before):

**Thm** [Poincaré Recurrence Thm]

Let $X$ be a metric space, $T: X \circlearrowleft$, $\mu$ a $T$-invariant probability measure on $X$. Let $A \subset X$ be a Borel set. Then for $\mu$-almost every $x \in A$ we have $T^{n_k}(x) \in A$ for some sequence $n_k \to \infty$ ("keep coming back ...")

**Proof** Enough to show:

$\forall N \geq 1$, the set $A_N := \{x \in A \mid T^n(x) \notin A \ \forall n \geq N\}$ has $\mu = 0$.

Indeed, if $x \in A \setminus \bigcup_N A_N$ then $\exists\, n_k \to \infty$: $T^{n_k}(x) \in A$

Replacing $T$ with $T^N$, we see that it suffices to consider the case $\boxed{N=1.}$

So, consider the set

$$A_1 = \{x \in A \mid T^n(x) \notin A \quad \forall n \geq 1\}.$$

We note that for each $m, k \geq 0$, $m \neq k$ we have $\boxed{T^{-m}(A_1) \cap T^{-k}(A_1) = \emptyset.}$

Indeed, assume that $m < k$ and $x \in T^{-m}(A_1)$, $x \in T^{-k}(A_1)$. Then

$$T^m(x) \in A_1 \Rightarrow T^{m+n}(x) \notin A \quad \forall n \geq 1$$

$$T^k(x) \in A_1 \Rightarrow T^k(x) \in A$$

contradiction.

Now, since $\mu$ is $T$-invariant, $\mu(T^{-m}(A_1)) = \mu(A_1)$. Now $\bigsqcup_{m \geq 0} T^{-m}(A_1) \subset X$,

so $$\underset{\textstyle \sum_{m \geq 0} \mu(A_1)}{\underset{\parallel}{\sum_{m \geq 0} \mu(T^{-m}(A_1))}} \leq \mu(X) = 1,$$

implying that $\mu(A_1) = 0$. $\square$