§12. Convex co-compact hyperbolic surfaces

We now study a particular example of a hyperbolic dynamical system: the geodesic flow on a

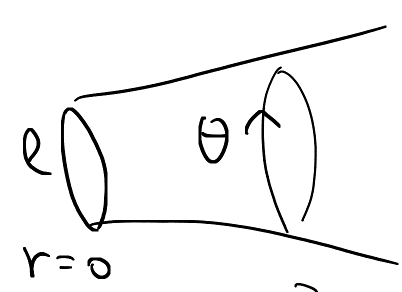
$\underbrace{\text{convex co-compact}}_{\text{CCC from now on}}$ hyperbolic surface.

§12.1. Geometry

A CCC hyperbolic surface is a noncompact complete connected oriented Riemannian manifold M of $\dim = 2$ and curvature $= -1$ whose infinite ends are funnels.

A funnel has the form $[0,\infty)_r \times (\mathbb{R}/\ell\mathbb{Z})_\theta$ with the metric

$$g = dr^2 + \cosh^2 r \cdot d\theta^2$$

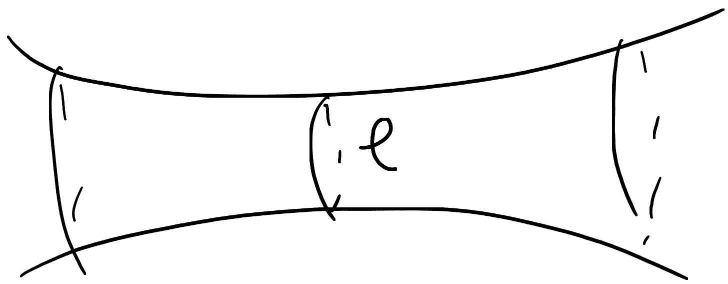


Basic example:

a hyperbolic cylinder

$M = \mathbb{R}_r \times S^1_\theta$, $S^1 = \mathbb{R}/\ell\mathbb{Z}$,

$g = dr^2 + \cosh^2 r \cdot d\theta^2$



We can also obtain a hyperbolic cylinder as the quotient of $\mathbb{H}^2$ by a subgroup Γ of $PSL(2,\mathbb{R})$.

Namely, let Γ be the group generated by $\gamma = \begin{pmatrix} e^{\ell/2} & 0 \\ 0 & e^{-\ell/2} \end{pmatrix} \in SL(2,\mathbb{R})$

The action of γ on $\mathbb{H}^2$ is

$\gamma(z) = e^{\ell} \cdot z$

A fundamental domain is given by an annulus

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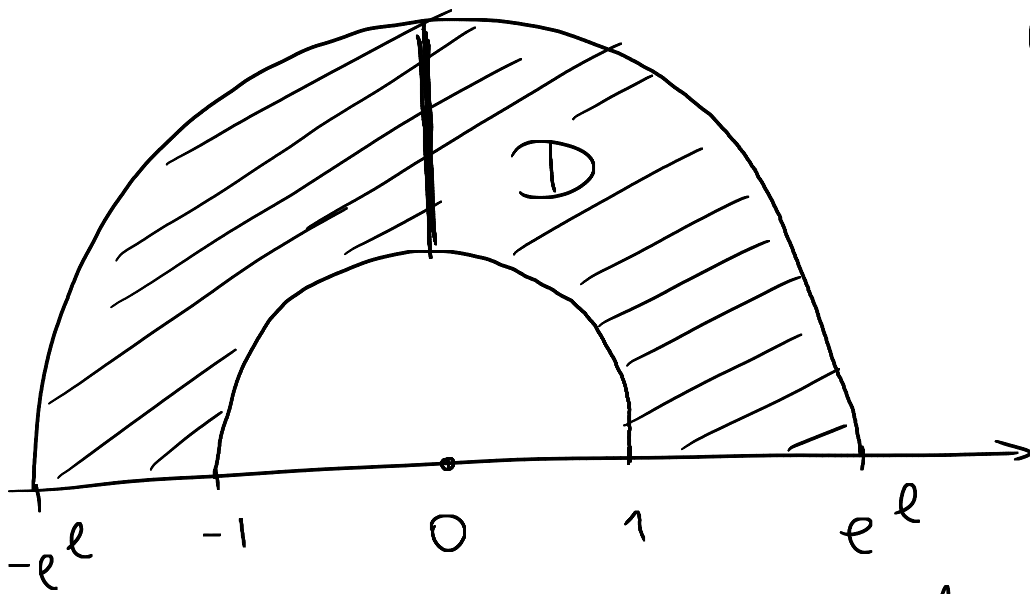
$$D = \{ z \in \mathbb{H}^2 \mid 1 \le |z| \le e^{\ell} \}:$$

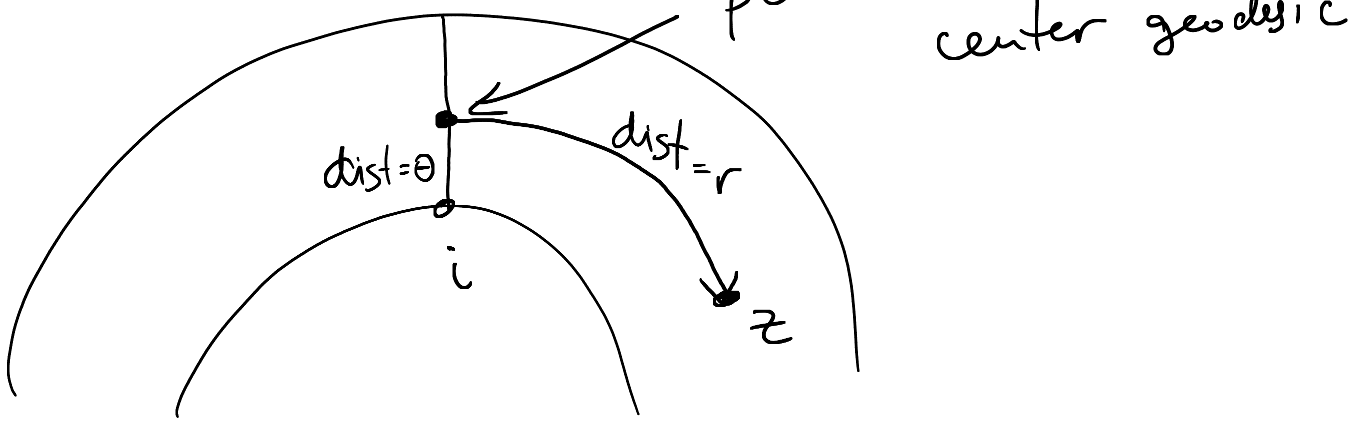
z



The vertical segment is a <u>closed geodesic</u> of length ℓ
(recall that $t \mapsto e^t \cdot i$ is a geodesic on $\mathbb{H}^2$)

To get the hyperbolic cylinder from this, glue the 2 boundary half-circles of D together.

How to map this to the model in r, θ coordinates?



This gives the following map from $\mathbb{R}_r \times S^1_\theta$ to D_z:

$$z = \exp(\theta + 2i \operatorname{arccot} e^r)$$

<u>Check:</u>

$r = 0 \to z = i e^\theta$

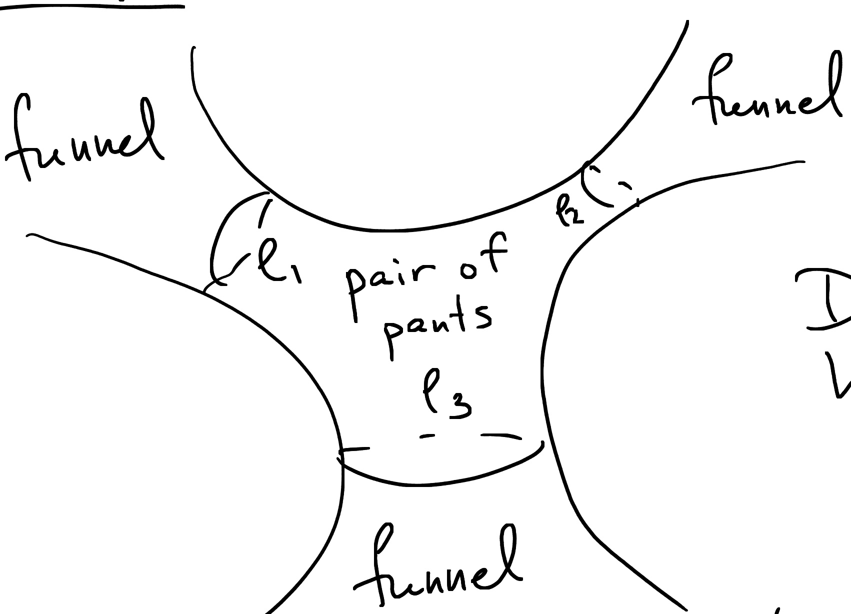
$r \to -\infty \to z = -e^\theta$

$r \to \infty \to z = e^\theta$

<u>Note:</u> the hyperbolic cylinder only has one closed geodesic (of primitive period ℓ)

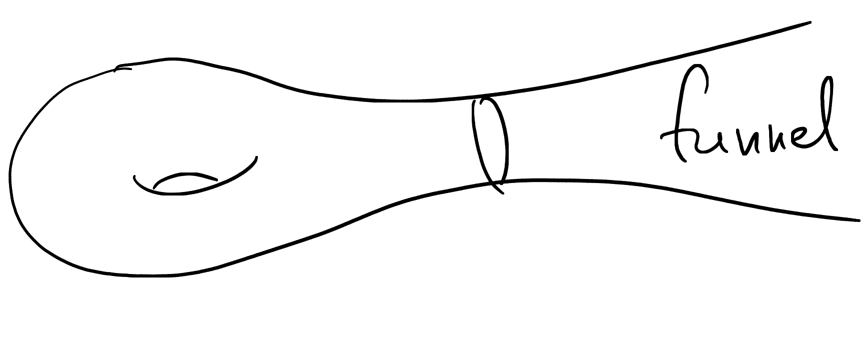
More general CCC hyperbolic surfaces are obtained by taking a compact hyperbolic surface with geodesic boundary and gluing funnel ends to each boundary geodesic: 18.118 12-5

<u>Example</u>: 3-funnel surface:



Determined by $\ell_1, \ell_2, \ell_3 > 0$

<u>Another example</u>: funneled torus:



§12.2. Schottky groups

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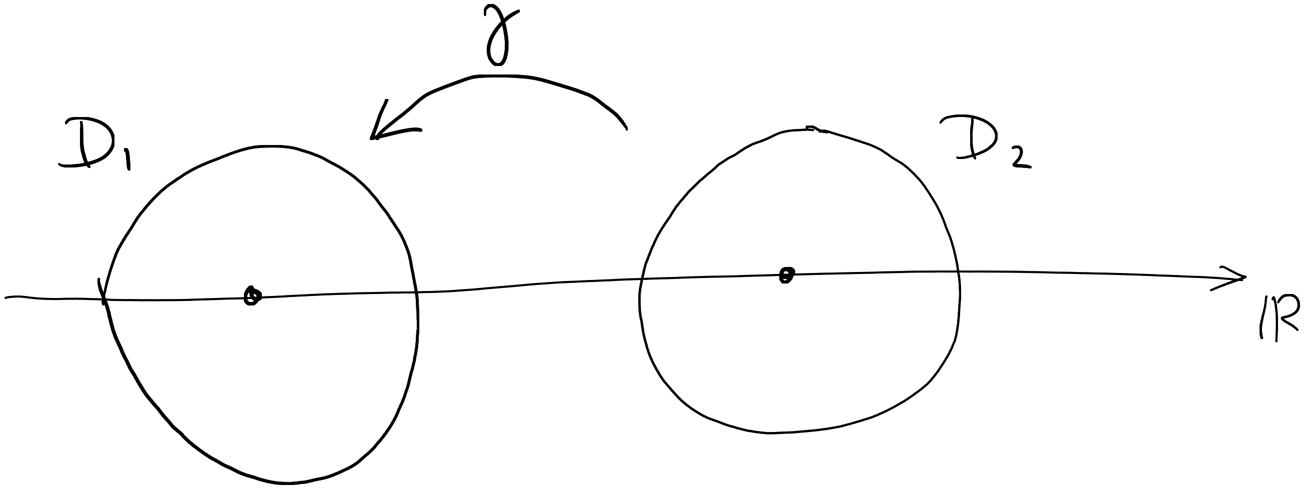
We will think of ccc hyperbolic surfaces as quotients $\Gamma \backslash \mathbb{H}^2$ where $\Gamma \subset PSL(2,\mathbb{R})$ is a (classical) <u>Schottky group</u>, as defined below.

We first give a couple of basic lemmas.

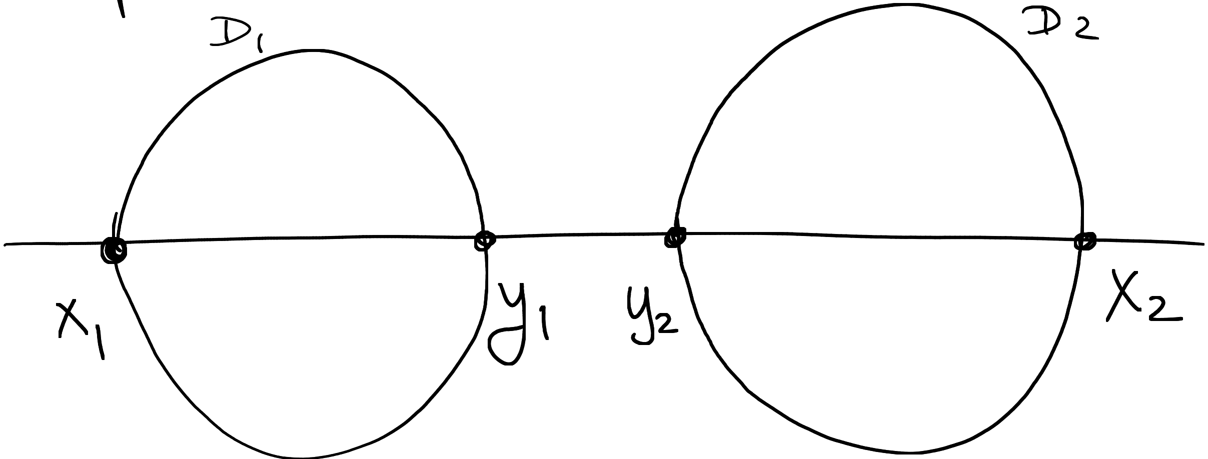
<u>Lemma 1</u> Assume that $D_1, D_2 \subset \mathbb{C}$ are nonintersecting closed disks centered on $\mathbb{R}$. Then there exists $\gamma \in SL(2,\mathbb{R})$ such that (with the action of γ on $\dot{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by Möbius transformations)

$$\gamma(\dot{\mathbb{C}} \setminus \overset{\circ}{D_2}) = D_1.$$

interior



<u>Proof</u> Let's label the endpoints of $D_1 \cap \mathbb{R}$, $D_2 \cap \mathbb{R}$:



We have $\gamma(\dot{\mathbb{C}} \setminus D_2^\circ) = D_1$ iff

$$\begin{cases} \gamma(x_2) = x_1 \\ \gamma(y_2) = y_1 \end{cases}$$

(since γ maps $\mathbb{R} \circlearrowleft$, circles to circles, and is conformal)

Making an affine change of variables on $\mathbb{C}$, we may assume that $\boxed{x_1 = 0,\ x_2 = 1}$.

Writing $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$, we get the equations

$$\frac{a+b}{c+d} = 0, \quad \frac{ay_2 + b}{cy_2 + d} = y_1, \text{ i.e.}$$

$$\begin{cases} a + b = 0 \\ ay_2 + b = cy_1y_2 + dy_1 \\ ad - bc = 1 \end{cases}$$

The 1st equation gives $b = -a$, so

$$\begin{cases} a(y_2 - 1) = cy_1y_2 + dy_1 \\ a(c+d) = 1 \end{cases}$$

We have $0 < y_1 < y_2 < 1$, in particular $y_2 \neq 1$.

So we write $a = \dfrac{cy_1y_2 + dy_1}{y_2 - 1}$.

The last equation becomes

$$\frac{cy_1y_2 + dy_1}{y_2 - 1} \cdot (c+d) = 1, \quad \text{i.e.}$$

$$y_1(cy_2 + d)(c+d) = y_2 - 1, \quad \text{i.e.}$$

$$(cy_2 + d)(c+d) = \frac{y_2 - 1}{y_1}.$$

There exists a solution (c,d), in fact a <u>curve</u> of solutions, since the range of the map $(c,d) \mapsto (cy_2+d)(c+d)$ is the whole $\mathbb{R}$.

$\square$

<u>Lemma 2</u>. Assume that $\gamma \in SL(2,\mathbb{R})$ and $\gamma \neq \pm I$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

- If γ is <u>elliptic</u> ($|a+d| < 2$) then it has 2 fixed points on $\dot{\mathbb{C}}$, which do not lie on $\dot{\mathbb{R}}$
- If γ is <u>parabolic</u> ($|a+d| = 2$) then it has 1 fixed point on $\dot{\mathbb{C}}$, which lies on $\dot{\mathbb{R}}$
- If γ is <u>hyperbolic</u> ($|a+d| > 2$) then it has 2 fixed points $x_-, x_+ \in \dot{\mathbb{C}}$ which both lie on $\dot{\mathbb{R}}$. In this case $\forall z \in \dot{\mathbb{C}} \setminus \{x_+, x_-\}$ we have (in the topology of $\dot{\mathbb{C}}$)

$$\gamma^j(z) \longrightarrow x_\pm \quad \text{as} \quad j \to \pm\infty.$$

<u>Proof</u> The fixed point equation for γ is the eqn. $\frac{az+b}{cz+d} = z$, which is quadratic:

$$cz^2 + (d-a)z - b = 0.$$

ad-bc = 1

Discriminant: $D = (d-a)^2 + 4bc = (a+d)^2 - 4.$

This explains the cases:

<u>Elliptic:</u> $D < 0$

<u>Parabolic:</u> $D = 0$

<u>Hyperbolic:</u> $D > 0$

and shows the statements about fixed points.

If γ is hyperbolic, we can conjugate it by some element of $SL(2,\mathbb{R})$ to the diagonal matrix $\tilde{\gamma} = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}$, $r > 0$.

For $\tilde{\gamma}(z) = e^r \cdot z$, we have the fixed points $x_- = 0$, $x_+ = \infty$ and $\forall z \in \dot{\mathbb{C}} \setminus \{0, \infty\}$ we have

$$e^{jr} \cdot z \longrightarrow x_\pm \quad \text{as} \quad j \to \pm\infty. \quad \square$$

<u>Lemma 3</u> Any transformation γ from Lemma 1 is hyperbolic, with fixed points $x_+ \in I_1$, $x_- \in I_2$ where $I_j = D_j \cap \mathbb{R}$.

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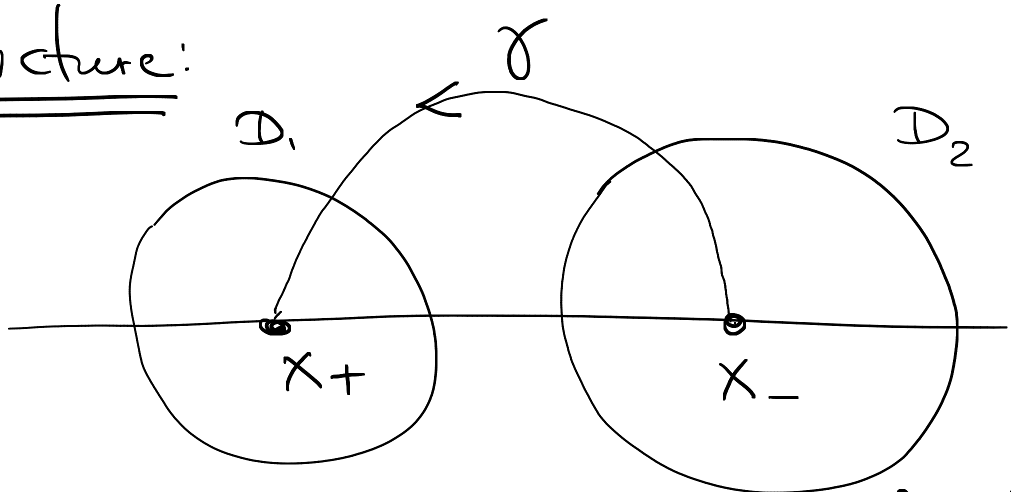
Picture:



Note: the geodesic in $\mathbb{H}^2$ from x_- to x_+ is preserved by γ

Proof Since $\gamma(\mathring{\hat{\mathbb{C}}} \setminus \mathring{D}_2) = D_1$ we have $\gamma(I_1) \subset I_1$.

Also, $\gamma^{-1}(\mathring{\hat{\mathbb{C}}} \setminus \mathring{D}_1) = D_2$, so

$$\gamma^{-1}(I_2) \subset I_2.$$

Thus (by the Intermediate Value Thm) $\exists$ fixed points of γ, $x_+ \in I_1$ and $x_- \in I_2$.

They are both real, so γ is hyperbolic and one can check that x_- is the repulsive fixed point and x_+ is the attractive one. $\square$

We are now ready to define Schottky groups. 18.118 12-12

These depend on the following date:

- $\boxed{m \geq 1,}$ an integer
- Define the alphabet

$$\mathcal{A} = \{1, \ldots, 2m\}.$$

- For $a \in \mathcal{A}$, define $\bar{a} \in \mathcal{A}$ by

$$\bar{a} := \begin{cases} a+m, & \text{if } a \leq m \\ a-m, & \text{if } a > m \end{cases}$$

- Fix nonintersecting closed disks in $\mathbb{C}$ with centers on $\mathbb{R}$,

$$\boxed{(D_a)_{a \in \mathcal{A}}}$$

- Fix group elements $\boxed{\gamma_a \in PSL(2,\mathbb{R})}$, $a \in \mathcal{A}$, such that

$$\gamma_a(\dot{\mathbb{C}} \setminus D_{\bar{a}}) = D_a,$$

$$\gamma_{\bar{a}} = \gamma_a^{-1}.$$

Define the Schottky group $\boxed{\Gamma \subset PSL(2, \mathbb{R})}$ as the group generated by $\gamma_1, \dots, \gamma_m$.

It turns out that

1. $\boxed{M = \Gamma \backslash \mathbb{H}^2}$ is a ccc surface
2. Any ccc surface is isometric to $\Gamma \backslash \mathbb{H}^2$ for some Schottky group Γ.

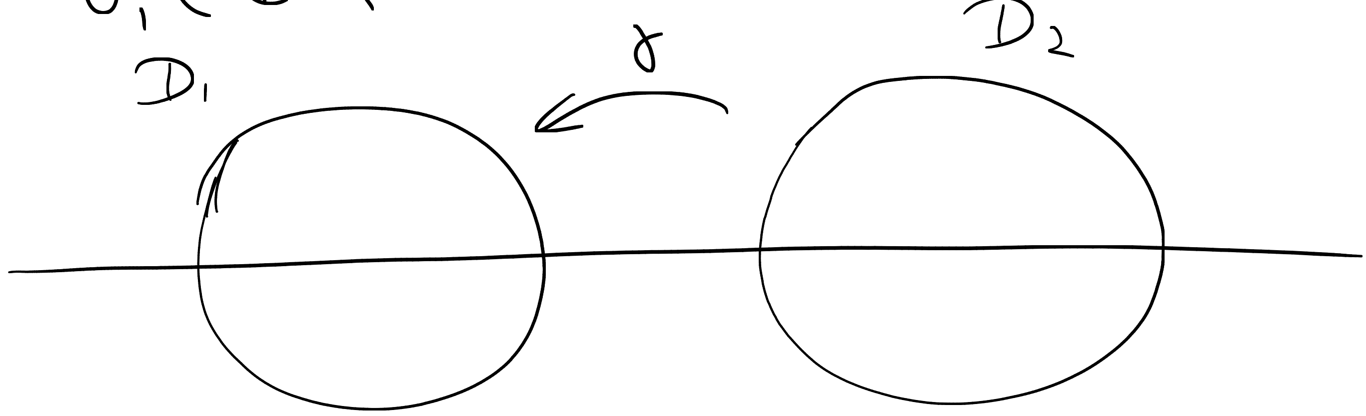
We will not give all the details of the proofs here, focusing instead on some basic properties and examples (see Borthwick, §15.1 for more details)

<u>**Example 1:**</u> $\boxed{m=1}$, i.e.

we only have <u>2 disks</u> D_1, D_2

and maps $\gamma_1, \gamma_2 = \gamma_1^{-1}$ s.t.

$\gamma_1(\hat{\mathbb{C}} \setminus \mathring{D}_2) = D_1$:



Then $\Gamma = \{\gamma_1^j \mid j \in \mathbb{Z}\}$.

We can conjugate γ_1 by an element of $SL(2,\mathbb{R})$

to $\tilde{\gamma}_1 = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}$ for some $r > 0$

Then Γ is conjugated to

$\tilde{\Gamma} = \{\tilde{\gamma}_1^j \mid j \in \mathbb{Z}\}$ and

$\Gamma \setminus \mathbb{H}^2 \simeq \tilde{\Gamma} \setminus \mathbb{H}^2$ is a <u>hyperbolic cylinder</u> with center geodesic of length r.

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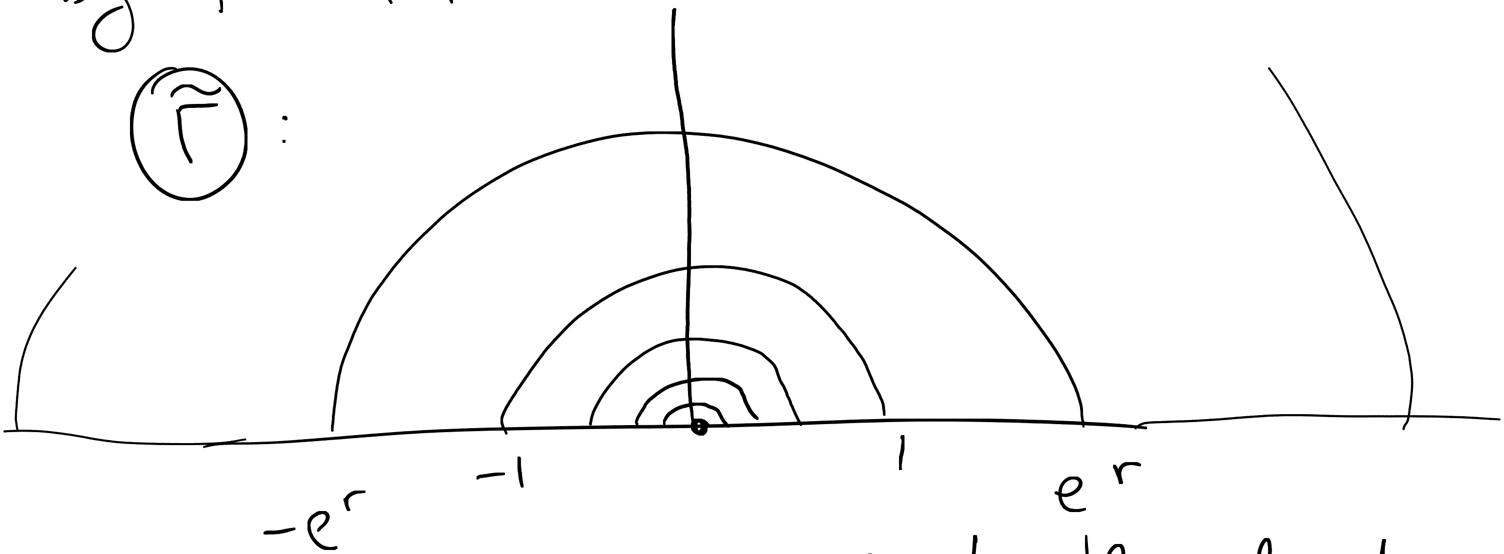
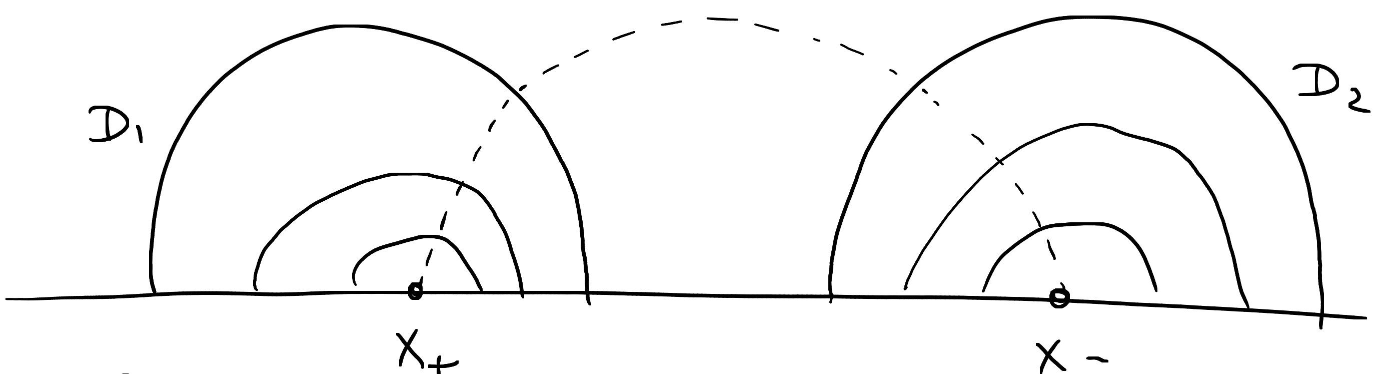
Picture of the tessellation of $\mathbb{H}^2$ by the fundamental domains:

$\tilde{\Gamma}$:



the vertical geodesic projects to the closed geodesic on the hyperbolic cylinder

Γ:



$x_\pm$ fixed points of γ_1
the geodesic from x_- to x_+ projects to the closed geodesic on the cylinder

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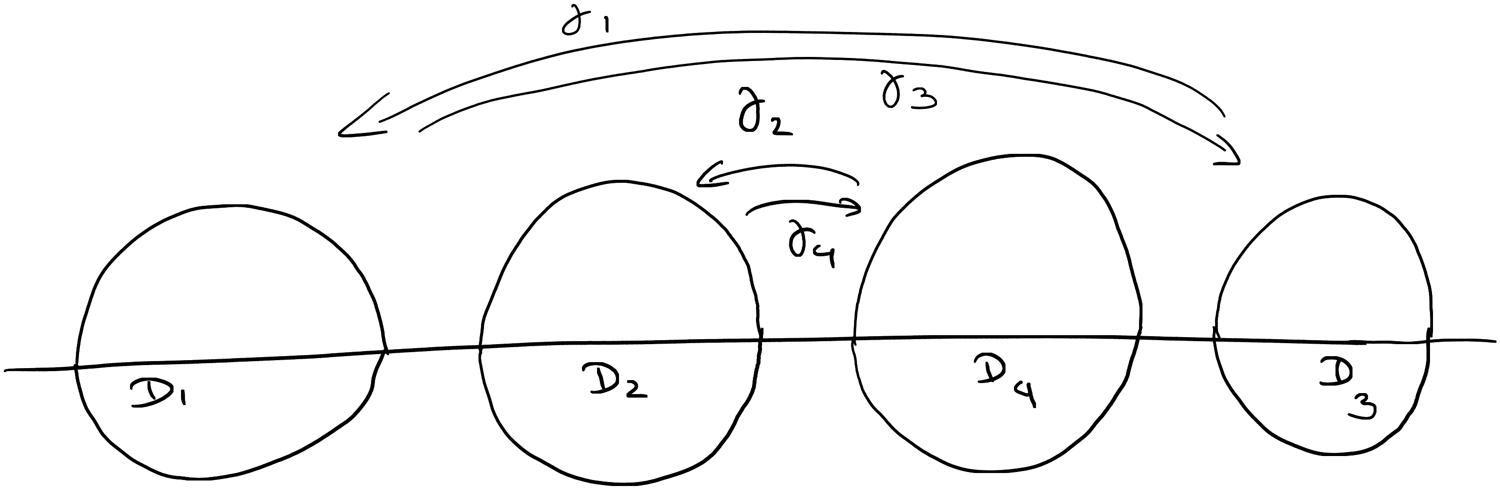
Example 2 : $\boxed{m=2}$,

with 4 disks D_1, D_2, D_3, D_4
arranged as follows:



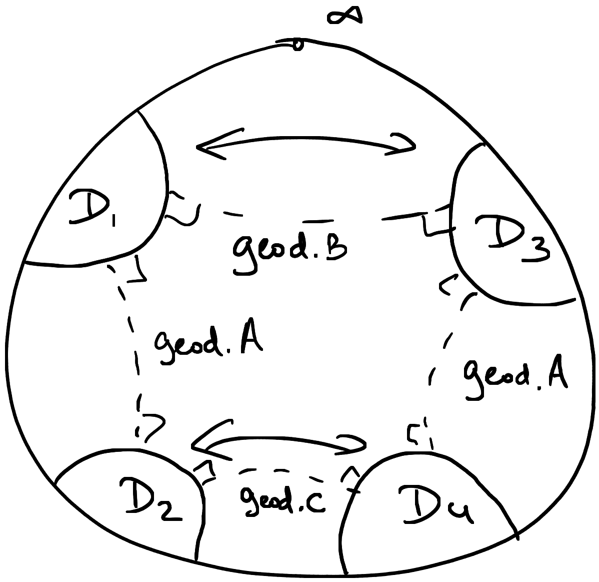
A fundamental domain of Γ is given by the complement of the disks:

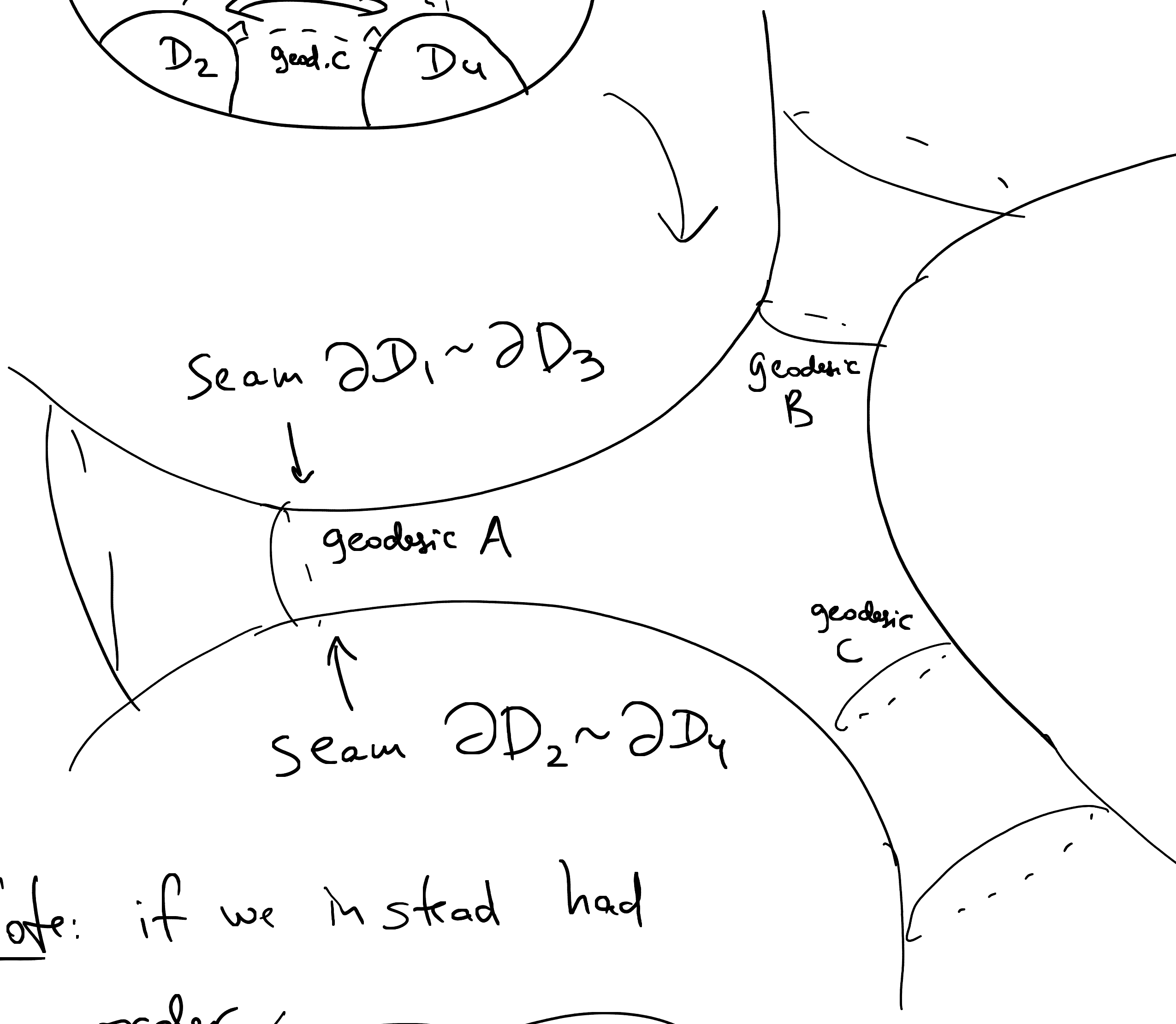
$$\Omega = \mathbb{H}^2 \setminus \bigsqcup_{a=1}^{4} D_a$$

Glue ∂D_1 with ∂D_3 via γ_1
∂D_2 with ∂D_4 via γ_2

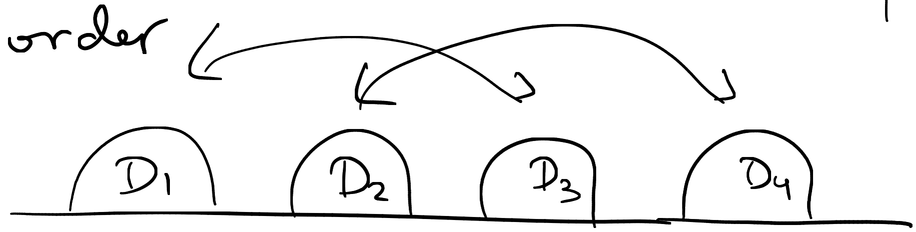
Get a 3-funnel surface:

Look at the ball model of $\mathbb{H}^2$

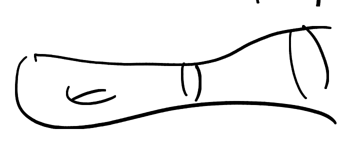




<u>Note</u>: if we instead had the order



then $\Gamma \backslash \mathbb{H}^2$ is a funneled torus



§12.3. Words and the limit set

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Let $\Gamma \subset PSL(2,\mathbb{R})$ be a Schottky group, generated by $\gamma_1, \dots, \gamma_m \in PSL(2,\mathbb{R})$

Put $\mathcal{A} = \{1, \dots, 2m\}$ as before, recall that $\gamma_{\bar{a}} = \gamma_a^{-1}$ where $\bar{a} = a \pm m$.

For $n \geq 0$, define the set of words $\mathcal{W}^n = \{ a_1 \dots a_n \in \mathcal{A}^n :$
$a_{j+1} \neq \bar{a}_j \ \forall j = 1, \dots, n-1 \}$.

($\mathcal{W}^0 = \{\emptyset\}$ consists of the empty word)

- For $\vec{a} = a_1 \dots a_n \in \mathcal{W}^n$, define the group element $\gamma_{\vec{a}} := \gamma_{a_1} \dots \gamma_{a_n} \in \Gamma$.

<u>Note</u>: the condition $a_{j+1} \neq \bar{a}_j$ makes sure that γ_a is not put next to $\gamma_{\bar{a}} = \gamma_a^{-1}$.

- For $\vec{a} = a_1 \dots a_n \in \mathcal{W}^n$, $n \geq 1$, define the disk $D_{\vec{a}} := \gamma_{a_1 \dots a_{n-1}}(D_{a_n})$.

Then $D_{\vec{a}}$ is a closed disk in $\mathbb{C}$ centered on $\mathbb{R}$ (since it is orthogonal to $\mathbb{R}$).

We have the following "tree" properties of the disks $D_{\vec{a}}$:

(1) If $\vec{a} = a_1 \dots a_n$, $n \geq 2$, then

interior

$$D_{\vec{a}} \subset \overset{\circ}{D}_{a_1 \dots a_{n-1}}.$$

Indeed, $D_{\vec{a}} = \gamma_{a_1 \dots a_{n-2}}(\gamma_{a_{n-1}}(D_{a_n}))$,

$$D_{a_1 \dots a_{n-1}} = \gamma_{a_1 \dots a_{n-2}}(D_{a_{n-1}}),$$

so it suffices to show that

$\gamma_{a_{n-1}}(D_{a_n}) \subset \overset{\circ}{D}_{a_{n-1}}$, which follows from the Schottky mapping properties:

$$a_n \neq \overline{a}_{n-1} \Rightarrow D_{a_n} \subset \dot{\mathbb{C}} \setminus D_{\overline{a_{n-1}}},$$

and $\gamma_{a_{n-1}}(\dot{\mathbb{C}} \setminus D_{\overline{a_{n-1}}}) = \overset{\circ}{D}_{a_{n-1}}$.

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② If $\vec{a}, \vec{b} \in W^n$ and $\vec{a} \neq \vec{b}$ then $D_{\vec{a}} \cap D_{\vec{b}} = \emptyset$.

Indeed, by property 1 we may assume that $\vec{a} = a_1 \dots a_{n-1} a_n$, $\vec{b} = a_1 \dots a_{n-1} b_n$,

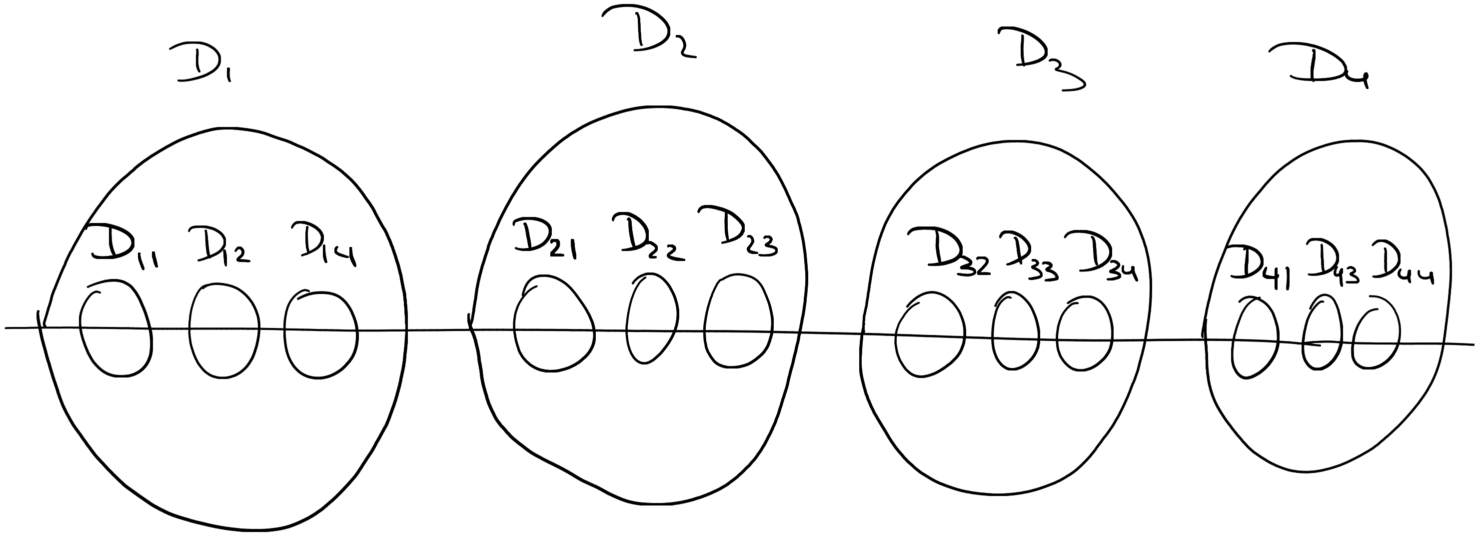
$a_n \neq b_n$.

Then $D_{\vec{a}} = \gamma_{a_1 \dots a_{n-1}}(D_{a_n})$,

$D_{\vec{b}} = \gamma_{a_1 \dots a_{n-1}}(D_{b_n})$, and

$D_{a_n} \cap D_{b_n} = \emptyset$.

Picture: (the order of sub-disks is not right ...)



Since Γ is generated by $\gamma_1, \dots, \gamma_m$,
we have $\Gamma = \{ \gamma_{\vec{a}} \mid \vec{a} \in W^n, n \geq 0 \}$

This representation is unique:

Lemma Assume that $\vec{a} \in W^n$, $\vec{b} \in W^m$ satisfy $\gamma_{\vec{a}} = \gamma_{\vec{b}}$. Then $\vec{a} = \vec{b}$.

Proof Rewriting this as $\gamma_{\vec{a}} \gamma_{\vec{b}}^{-1} = I$, enough to show that $\forall \vec{a} \in W^n$, $n \geq 1$, we have $\gamma_{\vec{a}} \neq I$.

Look at $\gamma_{\vec{a}}(\infty)$: if $\vec{a} = a_1 \dots a_n$ then
$\infty \in \dot{\mathbb{C}} \setminus D_{\bar{a}_n} \Rightarrow \gamma_{a_n}(\infty) \in D_{a_n}$.
Thus $\gamma_{\vec{a}}(\infty) \in D_{\vec{a}}$, in particular
$\gamma_{\vec{a}}(\infty) \neq \infty$, so $\vec{a} \neq I$. $\square$

Remark Lemma shows that Γ is the *free* group generated by $\gamma_1, \dots, \gamma_m$.

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Limit set:

We can define it as $\boxed{I_{\vec{a}} = D_{\vec{a}} \cap \mathbb{R}}$

$$\Lambda_\Gamma = \bigcap_{n\geq 1} \underbrace{\bigsqcup_{\vec{a}\in W^n} D_{\vec{a}}} = \bigcap_{n\geq 1} \bigsqcup_{\vec{a}\in W^n} I_{\vec{a}}$$

Note that this is a nested family of nonempty compact sets, so

$\Lambda_\Gamma \subset \mathbb{R}$ is a nonempty compact subset of $\mathbb{R}$.

Another, equivalent, definition, is:

$\forall z \in \mathbb{H}^2$,

$$\Lambda_\Gamma = \dot{\mathbb{R}} \cap \overline{\{\gamma(z) \mid \gamma \in \Gamma\}}$$

where the closure is taken in $\overline{\mathbb{H}}^2 = \mathbb{H}^2 \cup \dot{\mathbb{R}}$.

(exercise, possibly on the last pset)

This in particular implies that $\forall \gamma \in \Gamma$, γ maps Λ_Γ to itself (Γ acts on Λ_Γ)

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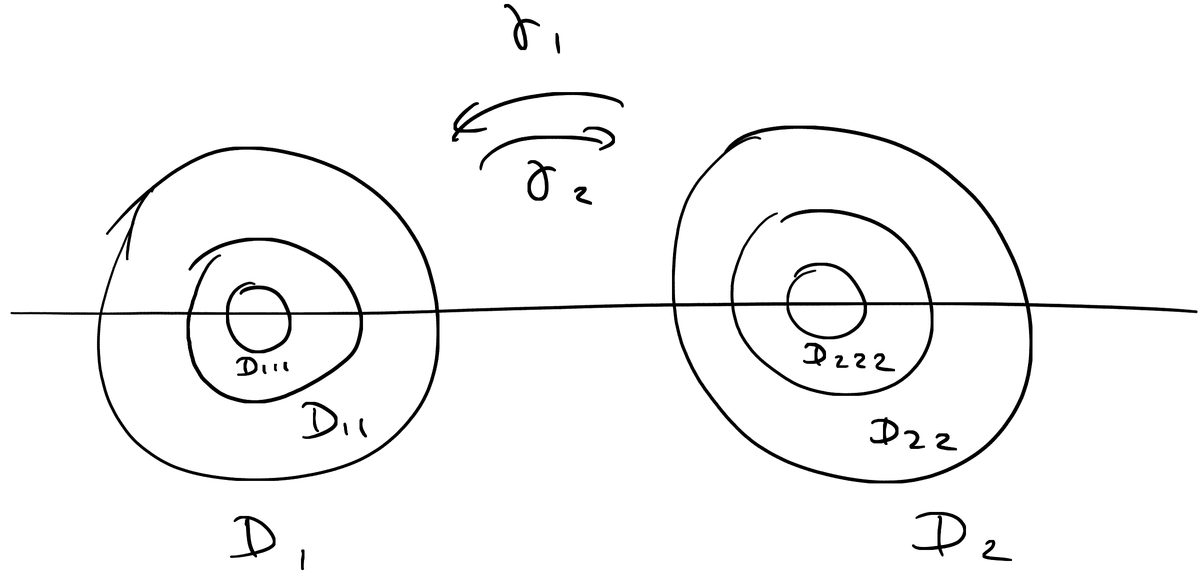
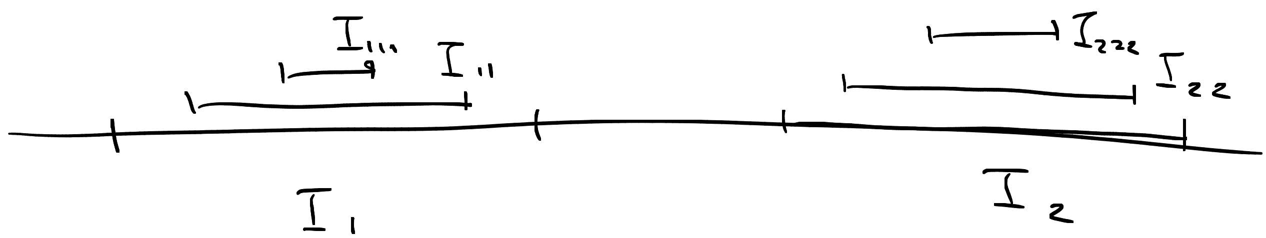
Example: hyperbolic cylinder ($m=1$)



Intervals: $I_{\vec{a}} = D_{\vec{a}} \cap \mathbb{R}$



Λ_Γ = two points, $\{x_+, x_-\}$, where $x_\pm$ are the fixed points of γ_1

Note: if $m \geq 2$ then Λ_Γ is a fractal set of dimension $\delta \in (0,1)$.
More on that later.

§12.4. Geodesic flow

Let $M = \Gamma \backslash \mathbb{H}^2$ be a ccc hyperbolic surface,

where $\Gamma \subset PSL(2, \mathbb{R})$ is a Schottky group.

Denote by $\varphi^t : SM \circlearrowleft$ the geodesic flow. Under the natural projection $\pi_\Gamma : S\mathbb{H}^2 \to SM$, φ^t lifts to the geodesic flow on $\mathbb{H}^2$.

Recall that geodesics on $\mathbb{H}^2$ are circles (or lines) orthogonal to $\mathbb{R}$.

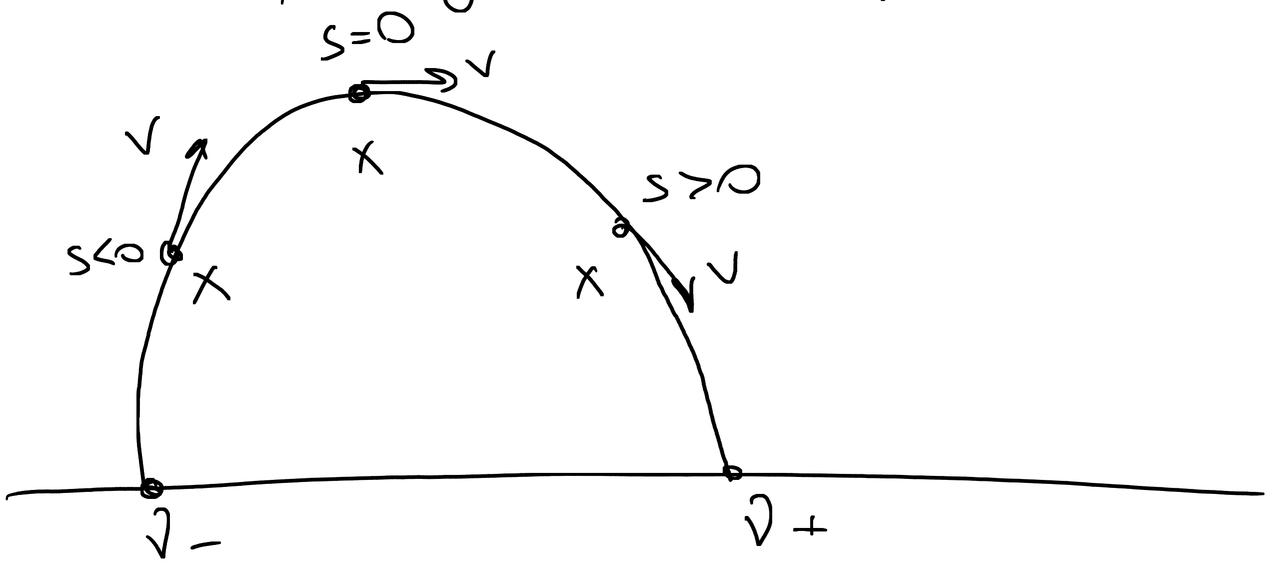
Each geodesic has limiting points at infinity, $\nu_\pm \in \mathring{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

We can introduce the following coordinates on $S\mathbb{H}^2$:

$$(x,v) \in S\mathbb{H}^2 \mapsto (\partial_+, \partial_-, s) \in (\mathring{\mathbb{R}} \times \mathring{\mathbb{R}})_\Delta \times \mathbb{R}_s$$

where $(\mathring{\mathbb{R}} \times \mathring{\mathbb{R}})_\Delta = \{(\partial_+, \partial_-) \in \mathring{\mathbb{R}} \times \mathring{\mathbb{R}} : \partial_+ \neq \partial_-\}$ and

- $\partial_\pm$ is the limit of the geodesic $\varphi^t(x,v)$ as $t \to \pm\infty$
- s is defined as follows:
  $s(\varphi^t(x,v)) = s(x,v) + t$ and
  $s(x,v) = 0 \iff x$ is the closest point to i (w.r.t. $d_{\mathbb{H}^2}$) on the geodesic $\varphi^t(x,v)$



The map $(x,v) \mapsto (\nu_+, \nu_-, s)$ is a diffeomorphism.

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Trapped geodesics

Defn. Let $\gamma : \mathbb{R} \to M$ be a geodesic (here $M = \Gamma \backslash \mathbb{H}^2$ ccc h.s. as before)

We say that γ is forward trapped, if $\exists$ compact $K_\gamma \subset M$ s.t. $\forall t \geq 0$, $\gamma(t) \in K_\gamma$.

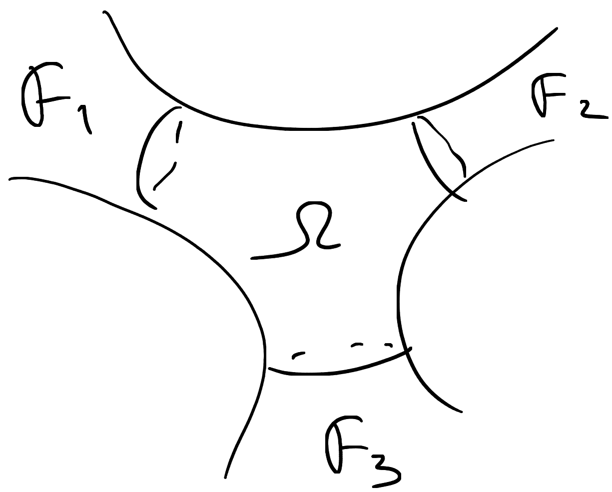
Otherwise we call forward escaping.

Recall that we can decompose M as

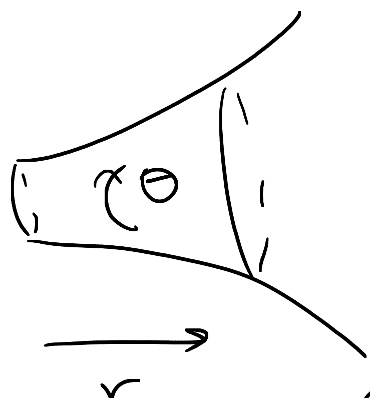
$$M = \Omega \cup \mathcal{F}_1 \cup \dots \cup \mathcal{F}_m$$

where Ω, the convex core, is a compact hyperbolic surface with boundary (for hyperbolic cylinder, Ω = circle) and $\mathcal{F}_j$'s are funnels:

e.g.



In a funnel, the metric is
$$g = dr^2 + \cosh^2 r \, d\theta^2, \quad r \geq 0, \quad \theta \in \mathbb{R}/\ell\mathbb{Z}:$$



This makes it possible to compute the geodesics in the funnel.
We will not do the computation here but will give the important consequences:

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(1) If γ is a geodesic on M and for some $t_1 < t_2$ we have

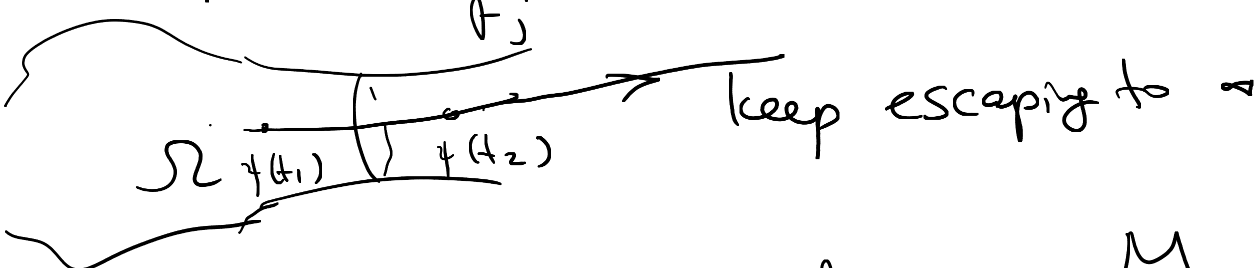
$\gamma(t_1) \in \Omega$, $\gamma(t_2) \in F_j^\circ$ $(\simeq (0,\infty)_r \times S^1_\theta)$

convex core

then $\gamma(t) \in F_j^\circ$ $\forall t \geq t_2$ and

$r(\gamma(t)) \longrightarrow \infty$ as $t \to \infty$;

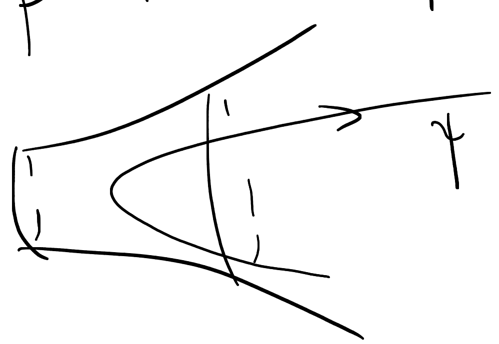
in particular, γ is forward escaping:



(2) If γ is a geodesic on M and $\gamma(t) \in F_j^\circ$ $\forall t \geq 0$, then

$r(\gamma(t)) \to \infty$ as $t \to \infty$,

in particular γ is forward escaping



Together these imply that

$\forall$ geodesic $\gamma : \mathbb{R} \to M$,

γ is forward trapped

$\Updownarrow$

$\exists T \ \forall t \geq T \ \gamma(t) \in \Omega$

Similarly we can define <u>backward trapped</u> geodesics.

We say a geodesic on M is trapped if it is both forward & backward trapped.

<u>Note:</u> any <u>closed</u> geodesic is trapped.

Denote by $\boxed{K \subset SM}$ the set of all trapped geodesics: $(x,v) \in K$

$\Leftrightarrow \varphi^t(x,v)$ is a trapped geodesic.

We note that K is <u>compact</u> (exercise, possibly in Pset 6) and φ^t-invariant.

By the stable/unstable decomposition from §6 we see that

$$\boxed{K \text{ is a } \underline{\text{hyperbolic set}} \text{ for } \varphi^t}$$

(unless M = hyperbolic cylinder)

However, $\varphi^t|_K$ is <u>not</u> an Anosov flow since K is a fractal set, not a manifold.

<u>Theorem</u> Let $\gamma : \mathbb{R} \to \mathbb{H}^2$ be a geodesic and $\gamma_\pm$ the limiting points at ∞. Let $\pi : \mathbb{H}^2 \to M = \Gamma \backslash \mathbb{H}^2$ be the projection map. Then $\pi \circ \gamma$ is forward trapped $\Leftrightarrow$

$\Leftrightarrow \gamma_+ \in \Lambda_\Gamma \leftarrow$ the limit set

Remark Similarly we have:

$\pi \circ \gamma$ is backward trapped $\iff$

$\iff \nu_- \in \Lambda_\Gamma$

And $\pi \circ \gamma$ is trapped $\iff$

$\iff \nu_+, \nu_- \in \Lambda_\Gamma$.

This gives an identification (via the diffeo. $(x,v) \mapsto (\nu_-, \nu_+, s)$)

$$\boxed{\tilde{\pi}^{-1}(K) \simeq (\Lambda_\Gamma \times \Lambda_\Gamma)_\Delta \times \mathbb{R}}$$

($\tilde{\pi} : S\mathbb{H}^2 \to SM$ projection map)

This relates the <u>limit set</u> Λ_Γ of the group Γ with the set K of <u>trapped geodesics</u> on $M = \Gamma \backslash \mathbb{H}^2$.

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Proof We only give a **sketch** of the proof.

Note that the statement stays the same, if we replace ψ by its time-shift $\psi(\cdot + T)$ for any fixed T, or if we replace it by $\gamma\circ\psi$ where $\gamma\in\Gamma$ (as $\pi\circ\gamma = \pi$)

Assume first that $\underline{\partial_+ \notin \Lambda_\Gamma}$.

We will show that ψ is forward escaping.

Replacing ψ with $\gamma\circ\psi$ for some $\gamma\in\Gamma$, we may assume that $\boxed{\partial_+ \notin I_a^\circ}$ for all $a\in\mathcal{A}=\{1,\dots,2m\}$, where $I_a = D_a\cap\mathbb{R}$.

Indeed, if $\partial_+\in I_{a_1\dots a_n}$ for some $a_1\dots a_n\in W^n$ then $\forall a_{n+1}\neq\bar{a}_n$, $\partial_+\notin I_{a_1\dots a_n a_{n+1}}$,

$\gamma_{a_1\dots a_n}^{-1}(\partial_+)\notin \bigsqcup_{a\in\mathcal{A}} I_a^\circ$ since $\gamma_{a_1\dots a_{n-1}}^{-1}(\partial_+)\in I_{a_n}$ but not in $\gamma_{a_n}\left(\bigsqcup_{a\neq\bar{a}_n} I_a\right)$.

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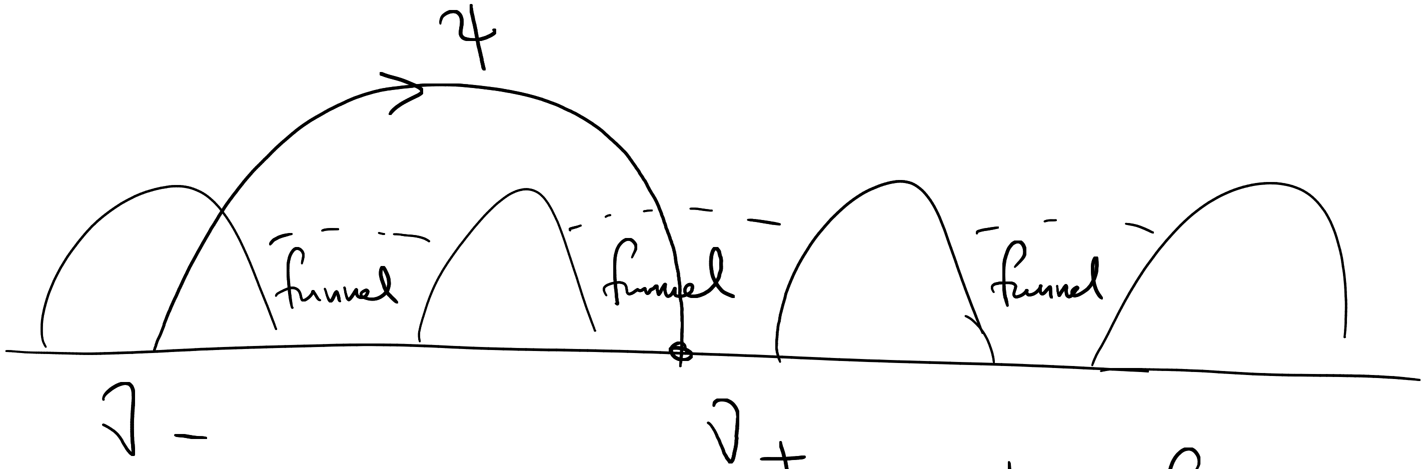
Now, if $\gamma_+ \notin \bigsqcup_{a\in\mathcal{A}} I_a^\circ$,
then γ_+ lies in the infinity of one of the funnels:



Then $\pi\circ\gamma$ escapes through that funnel.

Now, assume that $\pi\circ\gamma$ forward escapes.
Replacing γ by its time-shift,
we may assume that $\exists$ funnel F_j:
$\forall t\geq 0, \ \pi\circ\gamma(t)\in F_j$
Looking at the geodesic flow in the funnel in the (r,θ) coordinates, we see that
$\int_0^\infty \dot\theta(\pi(\gamma(t)))\,dt < \infty$.

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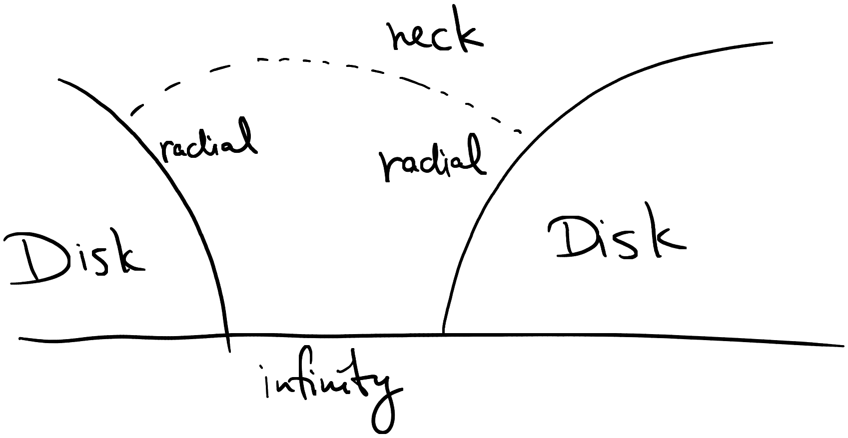
So $\Pi \circ \psi$ does not wrap around the funnel much. In particular, it can only intersect any radial geodesic $\{\theta = \text{const}\}$ finitely many times.

So then for large t, $\psi(t)$ stays in a single fundamental domain D of Γ

(the bdary of the cusp in D is made of the neck geodesic, which we cannot cross, and two radial geodesics:



Applying some element of Γ to γ, and a time shift, we may assume that $\gamma(t) \notin \bigsqcup_{a \in A} \mathring{D}_a \quad \forall t \geq 0$.

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But then $\gamma_+ \notin \bigsqcup_{a \in A} \mathring{I}_a$, which means that $\gamma_+ \in \Lambda_\Gamma$ in particular.

$\square$

Picture: