18.354– Nonlinear Dynamics II: Continuum systems
Mid-term - take home.

Due: Thursday, April 23 by 12:00 in E17-412.

The above deadline is final. There will be **no extensions**. For this take-home exam you are allowed to use Acheson, your class notes and Mathematica or MATLAB. No other references are allowed. No collaboration or consultation with others is allowed.

Problem 1: Nonlinear diffusion (20 Points)

Consider a concentration field $c(t, x)$ defined on $x \geq 0$ and governed by the nonlinear diffusion equation

$$\partial_t c = D\partial_x[c^p(\partial_x c)] \tag{1}$$

where p and D are strictly positive constants. Show that the self-similar solution of the form

$$c(t, x) = \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}} F(\xi), \qquad \xi = \frac{x}{(M^p Dt)^{1/(2+p)}} \tag{2}$$

which satisfies

$$\int_0^\infty dx\, c(t, x) = M, \qquad \partial_x c(t, 0) = 0, \qquad c(t, x \to \infty) = 0$$

for some constant $M > 0$, is given by

$$F(\xi) = \left[A - \frac{p\xi^2}{2(2+p)}\right]^{1/p}, \qquad 0 < \xi < \left[\frac{2(2+p)A}{p}\right]^{1/2} \tag{3}$$

and $F \equiv 0$ otherwise. For the case $p = 1$, prove that $A = (3/8)^{1/3}$.

Problem 2: Phytoplankton-zooplankton (20 Points)

A space-dependent phytoplankton-zooplankton model can be reduced to the following equations

$$\partial_t u = \partial_{xx} u + u + u^2 - \gamma uv \tag{4}$$

$$\partial_t v = d\, \partial_{xx} v + \beta uv - v^2 \tag{5}$$

where $u(t, x)$ and $v(t, x)$ are the plankton concentrations, respectively, and β, γ, d are positive parameters. Find the regions in the (β, γ)-plane (a) in which there is a stable, homogeneous state (u_0, v_0) such that neither u_0 nor v_0 is zero and (b) in which that state may be unstable to a Turing instability. In case (b), for what values of d will the instability occur, and what is the critical wavenumber for the onset of the instability?

Problem 3: DIFFUSION DRIVEN FLOWS (30 POINTS)

Diffusion driven flow in a stratified environment is an interesting example of a counterintuitive problem in fluid mechanics that finds applications in oceanography. In such systems, diffusion can drive flow along a wall inclined at angle α to the horizontal, placed in a stratified fluid (*note*: in a stratified fluid the density varies with height due to the presence of a component, such as salt or temperature). A schematic diagram of the problem is shown in Fig. 1.

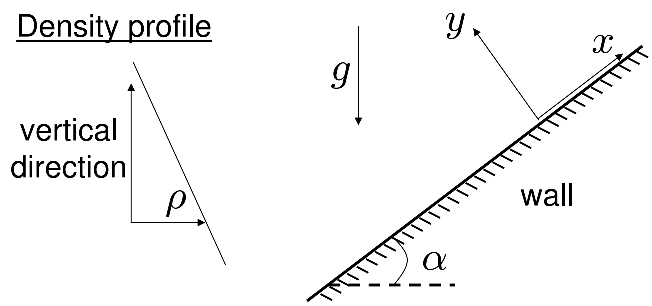

Figure 1: Setup for Problem 1: diffusion driven flow.

(i) Making the x- and y-axes parallel and perpendicular to the wall (see Fig. 1), and assuming a flow of the form $\boldsymbol{u} = (u(y), 0)$, show that the Navier-Stokes equations and the advection-diffusion equation reduce to

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{d^2 u}{dy^2} - \rho g \sin\alpha, \tag{6}$$

$$0 = -\frac{\partial p}{\partial y} - \rho g \cos\alpha, \tag{7}$$

$$u\frac{\partial \rho}{\partial x} = \kappa \left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} \right). \tag{8}$$

where κ is the diffusion coefficient of the quantity affecting the density. (Note: The advection-diffusion equation for a concentration field $c(t, \boldsymbol{x})$, sometimes also called the transport equation, is $Dc/Dt = \kappa \nabla^2 c$. For this problem, you may assume c is proportional to the density ρ, i.e. $c = a\rho$, where a is a constant).

(ii) At the inclined surface *the velocity and the normal density gradient vanish*, while far away the velocity vanishes and the density distribution reduces to its undisturbed state,

$$\rho \to \rho_0 + K(x \sin\alpha + y \cos\alpha).$$

This corresponds to a linear density gradient in the vertical direction. Note that K should be a negative number so that we have light fluid over heavy fluid and ρ_0 is a reference density far away from the wall along the line $x \sin\alpha + y\cos\alpha = 0$. Seeking a solution for the density of the form

$$\rho = \rho_0 + K(x\sin\alpha + y\cos\alpha) + \rho_0 f(y), \tag{9}$$

show that $f(y)$ satisfies

$$\frac{d^4 f}{dy^4} = -4\gamma^4 f, \tag{10}$$

where γ is given below (see hint), and determine $f(y)$ for solutions of the type $f = e^{\delta y}$.

(*Hint : You will want to eliminate pressure from your equations. To make your solution more presentable, it will help to define* $N^2 = -Kg/\rho_0$ *and* $\gamma = (N^2\rho_0 \sin^2\alpha/4\mu\kappa)^{\frac{1}{4}}$. *N is a well known quantity in fluid dynamics, called buoyancy frequency, and corresponds to the natural frequency of oscillation within a stratified fluid.* Note that δ may be a complex number, whereas you want your final result for $f(y)$ to be real).

(iii) From your solution for the density find the velocity $u(y)$. Then sketch both $u\,(y)$ and $f\,(y)$ as functions of the coordinate y away from the wall. Write a sentence about what you see in the plots. Comment on anything unphysical in the analytic solution for $u\,(y)$. In case you had trouble in (ii), assume that $u(y)$ can be found from

$$u(y) = \frac{\kappa\rho_o}{K\sin\alpha}\frac{d^2 f}{dy^2}$$

Problem 4: BENDING OF A THIN ELASTIC SHEET UNDER GRAVITY (30 POINTS)

In class we saw that, in the linearized limit of small deformations, the elastic bending energy of a two dimensional sheet with shape $y(x)$ is

$$U[y(x)] = \frac{1}{2}\left(\frac{Yh^3}{12(1-\sigma^2)}\right)\int_0^l \left(\frac{d^2 y}{dx^2}\right)^2 dx, \tag{11}$$

where l is the projected length of the beam, h is the thickness of the sheet and Y is its Young's modulus. The term $B = Yh^3/[12(1-\sigma^2)]$ is often referred to the bending modulus which measures how stiff the sheet is under bending deformations.

For large deformations, the corresponding bending energy (per unit width) is

$$U[y(s)] = \frac{1}{2}B\int_0^L \left(\frac{d\theta}{ds}\right)^2 ds, \tag{12}$$

where L is the total length of the sheet, θ is the angle that the sheet makes with the horizontal and s is the arclength along the neutral surface of the sheet. Eqn. (12) can be read in *english* as: "*The bending energy of a thin sheet per unit width equals one half of the bending modulus times the curvature* ($d\theta/ds$) *squared, integrated along its total length.*"

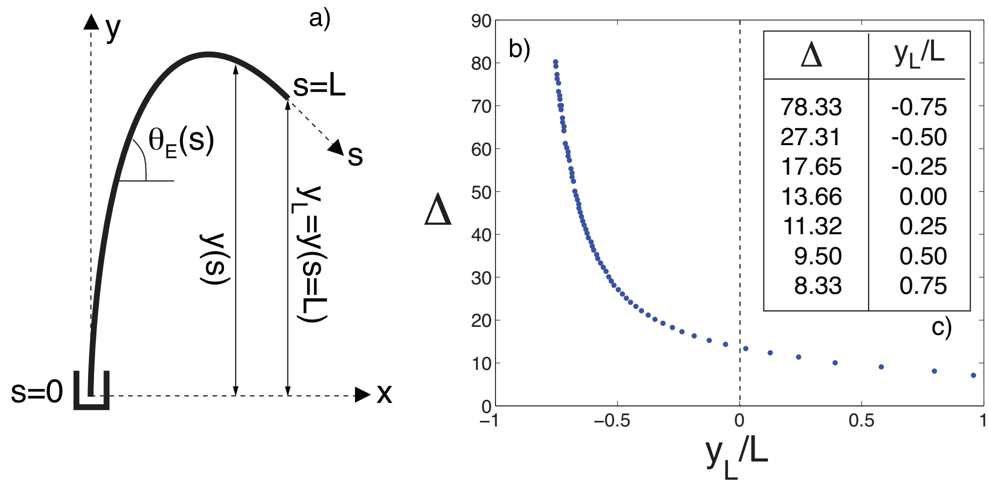

Figure 2: a) Schematic diagram of a thin sheet clamped vertically and bending under gravity. The sheet has dimensions: length L, thickness h and width/span b Note that y_L is the height of the free end of the sheet at $s = L$ b) Plot of the dimensionless parameters Δ and y_L/L for equilibrium shapes θ_E that satisfy Eqn. (15). c) Table for some values of Δ v.s. y_L/L.

Consider the following configuration. A thin sheet of thickness h, width b and total length L is held vertically at $s = 0$ and is free to bend under gravity otherwise. The boundary condition are that: 1) the sheet is vertical at the clamp $\theta(s = 0) = \pi/2$ and 2) there is no curvature at the free end $(d\theta/ds)(s = L) = 0$. Assume that the sheet has a linear density (mass per unit length) given by $\rho_l = \rho h b$, where ρ is the volumetric density. A schematic diagram of this configuration is given in Fig. 2a).

(i) Write the total energy functional, $\mathcal{E}[\theta(s)]$, for this thin strip bent under gravity. Make sure that your expression for the energy only depends on s, $\theta(s)$ and $d\theta/ds$. (*hint: the fact that* $dy = \sin\theta ds$ *i.e.* $y(s) = \int_0^s \sin[\theta(s)]ds$ *will help*).

(ii) Assuming small functional perturbations, $\delta\theta(s)$, on the equilibrium shape of the beam $\theta_E(s)$, use calculus of variations ($\delta\mathcal{E} \sim \mathcal{E}[\theta_E(s) + \delta\theta(s)] - \mathcal{E}[\theta_E(s)] \to 0$) to show that the equilibrium shape satisfies the following differential equation

$$Bb\frac{d^2\theta_E}{ds^2} = \rho_l g(L - s)\cos\theta_E \tag{13}$$

(iii) Non-dimensionalize Eqn. (13) using L and show that

$$\frac{d^2\theta_E}{d\bar{s}^2} = \Delta(1 - \bar{s})\cos\theta_E, \tag{14}$$

where $\Delta = (L/L_c)^3$ and L_c is often called the elasto-gravity lengthscale. What is L_c in terms of the physical quantities in the problem? What is its physical significance of L_c?

(iv) Determine L_c from scaling arguments and ensure that you get the same result as in (iii).

(v) Show that the differential equation in (14) is identical to

$$\frac{1}{2}\left(\frac{d\theta_E}{d\bar{s}}\right)^2 = \Delta\left[(1-\bar{s})\sin\theta_E + (\bar{y}-\bar{y}_L)\right], \tag{15}$$

where $\bar{y} = y/L$, $\bar{y}_L = y_L/L$ and y_L is the height of the free end with respect to the horizontal (see the diagram in Fig. 2a). (*hint: it will be easier to go backwards from Eqn. (15) to Eqn. (14). Extra points if you do it the other way round.*)

(vi) The differential equation that describes the equilibrium shapes of the bent sheet under gravity – Eqn. (15) – is nonlinear and therefore one has to solve it numerically to find $\theta_E(s)$ (that yields the equilibrium shapes). One of your friends has done this for you (he is good with MATLAB!) and plotted Δ as a function of y_L/L in Fig. 2b) (some points of this graph are given in the table of Fig. 2c). Develop a method to calculate the bending modulus B of a piece of paper (*hint: you may need only one of the points in the table of Fig. 2c*).

(viii) Using your technique, calculate the numerical value of B for the white paper sheet in the printers of the Athena clusters (*hint: you will need the paper density that you will find written on the label packages of the paper rims. Most likely it will be* $75g/m^2$)