18.354 – Nonlinear Dynamics II: Continuum systems
Problem Set 3: Optimization, surface tension and elasticity
Due: Thursday, April 2 (by 1pm in class)

Problem 1: Fermat's principle

Fermat's principle states that *the time elapsed in the passage of light between two fixed points is an extremum with respect to possible paths connecting the points.*

(a) Consider a flat interface ($y = y_0$) between two optically homogeneous media. The respective light velocities within the two media are u_1 and u_2. Apply Fermat's principle to the passage of light from a point in one medium to a point in the other to obtain Snell's law,

$$\frac{\sin \phi_1}{u_1} = \frac{\sin \phi_2}{u_2}, \tag{1}$$

where ϕ_i is the angle between the normal to the interface $y = y_0$ and the light path in the corresponding medium.

(b) Now consider an optically inhomogeneous medium in which the light velocity is a continuous function of the y coordinate, $u = u(y)$. Use the calculus of variations to show that

$$\frac{1}{u\sqrt{1 + y'^2}} = K, \tag{2}$$

where K is a constant.

(c) Solve for the path y if the velocity function is $u(y) = \sqrt{y - y_0}$, where y_0 is a constant. Comment on your solution.

Problem 2: Capillary rise

Consider an infinite pool of liquid, with surface tension γ at the air interface. You know from common experience that if you put a vertical wall into the pool, the liquid will climb up the wall and forms a meniscus. The goal of this problem is to calculate how much it moves up the wall. Suppose the liquid has an angle of contact θ_c with the solid wall. Use the calculus of variations to determine $h(x)$, the shape of the interface that minimises the energy of the system (*hint: you need to consider the contribution to the energy from both surface tension and the gravitational potential*).

This problem can be solved both analytically and numerically, and you should try both. Write your total energy as in integral in x. Can you use the special form of Euler-Lagrange? You should obtain a first order ODE for $h(x)$. To find the analytic solution, you should solve this by separation of variables (you may need to look up the integral for h). Your boundary conditions are given to you physically as (i) the contact angle on the solid wall ($h'(0) = \cot\theta$), and (ii) the height goes to the resting height of the liquid far from the wall. To solve the problem numerically, you can use the ODE to relate $h'(0)$ to $h(0)$. Pick a value of θ_c, and, then use an ODE solver in MATLAB (either ode45 or ode15s) to find $h(x)$. Is the condition on $h(x)$ far away from the wall satisfied? You may also want to clean up the problem by scaling h and x by the capillary length $L_c = \sqrt{\gamma/\rho g}$. Compare your numerical solution with the analytical one.

Problem 3: LINEAR ELASTICITY AND EINSTEIN NOTATION

Using Einstein notation, the most general expression for the free energy density of a deformed isotropic body is

$$E \;=\; \frac{1}{2}\lambda e_{ii}^2 + \mu e_{ik}^2,$$

where λ and μ are called Lamè coefficients. It is convenient to replace this by another formula, decomposing the energy into a pure shear and a pure compression. Then E becomes

$$E \;=\; \mu(e_{ik} - \frac{1}{3}\delta_{ik}e_{ll})^2 + \frac{1}{2}Ke_{ll}^2.$$

This is the expression we used in class, where K and μ are respectively the modulus of compression and rigidity. Rewrite the above expressions, including the summation signs where appropriate, writing your steps explicitly and clearly.

(a) Find an expression for K in terms of λ and μ.

(b) The stress tensor σ_{ik} is related to the free energy density via $\sigma_{ik} = \partial E/\partial e_{ik}$. Show that

$$\sigma_{ik} = Ke_{ll}\delta_{ik} + 2\mu(e_{ik} - \frac{1}{3}\delta_{ik}e_{ll}).$$

(c) Show that the stress tensor, e_{ij}, can be determined by inverting the expression for the stress tension, σ_{ik}, that you found above, such that,

$$e_{ij} = \frac{\sigma_{ll}}{9K}\delta_{ij} + \frac{\sigma_{ij} - \frac{1}{3}\sigma_{ll}\delta_{ij}}{2\mu}. \tag{3}$$

Having done this, now rewrite your steps for (a) and (b) in concise form, using Einstein notation.

Hint - For (b) you will find the following relations helpful:

$$\begin{aligned}
\frac{\partial e_{ik}}{\partial e_{mn}} &= \delta_{im}\delta_{kn} \\
\frac{\partial(e_{ll})}{\partial e_{mn}} &= \frac{\partial(e_{11} + e_{22} + e_{33})}{\partial e_{mn}} = \delta_{mn} \\
\delta_{ik}^2 &= \sum_{i,k=1}^{3} \delta_{ik}\delta_{ik} = \delta_{11}\delta_{11} + \delta_{22}\delta_{22} + \delta_{33}\delta_{33} = 3
\end{aligned}$$