22   Boundary layer equations

Consider a flow that's translationally invariant along the z-direction. Focussing on the components in the (x, y)-plane, denoted by $\boldsymbol{u} = (u, v)$, the Navier-Stokes equations reduce to

$$\partial_t u + u \partial_x u + v \partial_y u = -\frac{1}{\rho} \partial_x p + \nu (\partial_x^2 u + \partial_y^2 u) \tag{1a}$$

$$\partial_t v + u \partial_x v + v \partial_y v = -\frac{1}{\rho} \partial_y p + \nu (\partial_x^2 v + \partial_y^2 v) \tag{1b}$$

$$\partial_x u + \partial_y v = 0 \tag{1c}$$

Assume the flow passes a very large surface of length L, positioned at $y = 0$ along the x-axis, while satisfying the boundary condition $\boldsymbol{u} = U \boldsymbol{e}_x$ as $y \to \infty$. The Reynolds-number for this problem is then given by

$$R = \frac{UL}{\nu}. \tag{2}$$

Considering the limit $R \to \infty$, we can assume inviscid flow in the bulk, so that Eq. (1) reduces to

$$\partial_t u + u \partial_x u + v \partial_y u = -\frac{1}{\rho} \partial_x p \tag{3a}$$

$$\partial_t v + u \partial_x v + v \partial_y v = -\frac{1}{\rho} \partial_y p \tag{3b}$$

$$\partial_x u + \partial_y v = 0 \tag{3c}$$

We have seen that these simplified equations lead to incorrect predictions near the boundary $y \to 0$. To obtain a simplified set of equations that captures the main physical effects near the boundary, let's define rescaled coordinates

$$x' = \frac{x}{L}, \qquad y' = \frac{y}{\delta}, \qquad u' = \frac{u}{U}, \qquad v' = \frac{v}{U}\frac{L}{\delta}, \qquad p' = \frac{p}{\rho U^2}, \qquad t' = \frac{t}{L/U} \tag{4}$$

where $\delta \ll L$ is the (still unknown) width of the boundary layer. Then, the Navier-Stokes equations (1) take the form

$$\partial_{t'} u' + u' \partial_{x'} u' + v' \partial_{y'} u' = -\partial_{x'} p' + \left(\frac{1}{R}\right) \partial_{x'}^2 u' + \left(\frac{L^2}{R\delta^2}\right) \partial_{y'}^2 u' \tag{5a}$$

$$\partial_{t'} v' + u' \partial_{x'} v' + v' \partial_{y'} v' = -\left(\frac{L^2}{\delta^2}\right) \partial_{y'} p' + \left(\frac{1}{R}\right) \partial_{x'}^2 v' + \left(\frac{L^2}{R\delta^2}\right) \partial_{y'}^2 v' \tag{5b}$$

$$\partial_{x'} u' + \partial_{y'} v' = 0 \tag{5c}$$

This suggest to identify the boundary thickness with

$$\delta = \frac{L}{R^{1/2}}, \tag{6}$$

as, for distances smaller δ, viscous effects can no longer be neglected. Inserting this into (5)

$$\partial_{t'}u' + u'\partial_{x'}u' + v'\partial_{y'}u' = -\partial_{x'}p' + \left(\frac{1}{R}\right)\partial_{x'}^2 u' + \partial_{y'}^2 u' \tag{7a}$$

$$\frac{1}{R}\left(\partial_{t'}v' + u'\partial_{x'}v' + v'\partial_{y'}v'\right) = -\partial_{y'}p' + \frac{1}{R^2}\left(\partial_{x'}^2 v' + \partial_y^2 v'\right) \tag{7b}$$

$$\partial_{x'}u' + \partial_{y'}v' = 0 \tag{7c}$$

Now letting $R \to \infty$, as we did before in the bulk, yields the *boundary layer equations*

$$\partial_{t'}u' + u'\partial_{x'}u' + v'\partial_{y'}u' = -\partial_x'p' + \partial_{y'}^2 u' \tag{8a}$$

$$0 = -\partial_{y'}p' \tag{8b}$$

$$\partial_{x'}u' + \partial_{y'}v' = 0 \tag{8c}$$

These equations still need to be complemented by boundary conditions: There can be no flow across the boundary

$$v'\big|_{y'=0} = 0 , \qquad \forall x'. \tag{9a}$$

Furthermore, assuming no-slip at the boundary

$$u'\big|_{y'=0} = 0 , \qquad \forall x' \tag{9b}$$

and finally

$$u' \to 1 \qquad \text{as} \qquad y' \to \infty \tag{9c}$$

Furthermore, symmetry along the x-axis suggests that pressure gradients along the x-direction are also small, $\partial_{x'}p' \approx 0$, so after reinstating coefficients and assuming steady flow, we end up with the simplified boundary layer equations

$$u\partial_x u + v\partial_y u = \nu\partial_y^2 u \tag{10a}$$

$$\partial_x u + \partial_y v = 0 \tag{10b}$$

Note that, unlike the bulk equations (3), these equations still contain a viscosity term – as they should.

23   Rotating flows (Acheson, pp. 278-287; Taylor's paper)

Before leaving the topic of boundary layers, we will discuss one more example, of substantial geophysical importance. It also gives an example of laminar flow where the boundary layer actually completely controls the dynamics. In the process of deriving this result we will also learn about a rather remarkable phenomenon in rotating fluid dynamics.

23.1   The Taylor-Proudman theorem

Consider a fluid rotating with angular velocity Ω. The equation of motion in the frame of reference rotating with the fluid is

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \Omega \times (\Omega \times \boldsymbol{r}) \;\; = \;\; -\frac{1}{\rho}\nabla p_\Omega + \nu \nabla^2 \boldsymbol{u} - 2\Omega \times \boldsymbol{u}, \tag{1a}$$

$$\nabla \cdot \boldsymbol{u} \;\; = \;\; 0. \tag{1b}$$

There are two additional terms: the first $\Omega \times (\Omega \times \boldsymbol{r})$ is the centrifugal acceleration, which we have discussed before. This can be thought of as an augmentation to the pressure distribution, using the identity

$$\Omega \times (\Omega \times \boldsymbol{r}) = -\frac{1}{2}\nabla(\Omega \times \boldsymbol{r})^2. \tag{2}$$

Henceforth, we will simply absorb this into the pressure by writing

$$p = p_\Omega - \frac{\rho}{2}\nabla(\Omega \times \boldsymbol{r})^2. \tag{3}$$

For the rotating earth, the effect of this force is to simply distort the shape of the object from a sphere into an oblate ellipsoid. The second term is the *Coriolis acceleration* which is velocity dependent. Hopefully you have heard about it in classical mechanics. In the exercises you are asked to show that it arises naturally when you start from the Navier-Stokes equations and transform into a rotating reference frame.

We are going to be interested in flows which are much weaker than the rotation of the system. If U is a characteristic velocity scale and L is a characteristic length scale, then the advective term is of order U^2/L whereas the coriolis force is of order ΩU. We will assume that $\Omega U \gg U^2/L$ so that the equation of motion is effectively

$$\frac{\partial \boldsymbol{u}}{\partial t} \;\; = \;\; -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} - 2\Omega \times \boldsymbol{u}, \tag{4a}$$

$$\nabla \cdot \boldsymbol{u} \;\; = \;\; 0. \tag{4b}$$

23.2   Steady, inviscid flow

Now let's consider flow at high Reynolds number. The Reynolds number is now $\Omega L^2/\nu$ within this framework. Following our previous example with boundary layers, the first step is to write down the inviscid equations (since the viscosity is small), and then the next step is to correct them with boundary layers.

Following Acheson, let's write the flow velocity as $\boldsymbol{u} = (u_I, v_I, w_I)$ and $\boldsymbol{\Omega}=(0,0,\Omega)$. The steady, inviscid flow satisfies

$$2\Omega v_I \;\; = \;\; \frac{1}{\rho}\frac{\partial p_I}{\partial x}, \tag{5a}$$

$$2\Omega u_I \;\; = \;\; -\frac{1}{\rho}\frac{\partial p_I}{\partial y}, \tag{5b}$$

$$0 \;\; = \;\; \frac{1}{\rho}\frac{\partial p_I}{\partial z}, \tag{5c}$$

$$\frac{\partial u_I}{\partial x} + \frac{\partial v_I}{\partial y} + \frac{\partial w_I}{\partial z} \;\; = \;\; 0. \tag{5d}$$

The third equation says that the pressure is independent of z. Hence, the first two equations say that u_I and v_I are independent of z. Then the last equation says that w_I is independent of z. Thus, *the entire fluid velocity is independent of z*! This result, which is remarkable, is called the *Taylor-Proudman theorem*. Proudman discovered the theorem, but Taylor discovered what is perhaps its most remarkable consequence.

23.3   Taylor Columns

In his paper "Experiments on the motion of solid bodies in rotating fluids", Taylor posed the simple question: given the above fact that slow steady motions of a rotating liquid must be two-dimensional, what happens if one attempts to make a three dimensional motion by, for example, pushing a three dimensional object through the flow with a small uniform velocity? At the beginning of his paper he points out three possibilities:

1. The motion in the liquid is never steady.

2. The motion is steady, but our assumption that u_I is small relative to the rotation velocity breaks down near the object.

3. The motion is steady and two dimensional.

He remarks that the first possibility is unlikely, since it must settle down eventually. The realistic possibilities are (2) and (3). His paper, which can be downloaded from the course page, demonstrates that actually what happens is possibility (3). This is really rather remarkable (as Taylor notes) because there is only one way that it can really happen: An entire column of fluid must move atop the object.

23.4   More on rotating flows

Above, we wrote the equations of a rotating fluid assuming that the rotation frequency dominated the characteristic hydrodynamic flows in the problem. In other words, if Ω

is the characteristic rotation frequency, L is a horizontal lengthscale, and U is a typical velocity in the rotating frame, we assumed that

$$\text{Ro} = \frac{U}{\Omega L} \ll 1. \tag{6}$$

This dimensionless number is called the *Rossby number*. The equations were

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} - 2\boldsymbol{\Omega} \times \boldsymbol{u}, \tag{7a}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{7b}$$

We will use these ideas to revisit the famous problem of the spin-down of a coffee cup that we discussed at the very beginning of class. You might recall that the problem we had was that our simple theory of how the spin-down occurred turned out to be entirely false. We shall now construct the correct theory, while learning a bit of atmospheric and oceanic physics along the way.

23.5 More on the Taylor-Proudman theorem

Let's consider the Taylor-Proudman theorem again, this time using another method. We consider the viscosity to be small so that we can use the limit of a stationary, inviscid fluid. More precisely, we consider the ratio

$$\text{E}_{\text{k}} = \frac{\nu \nabla^2 \boldsymbol{u}}{\boldsymbol{\Omega} \times \boldsymbol{u}} = \frac{\nu}{\Omega L^2} \ll 1. \tag{8}$$

This dimensionless number is called the *Ekman number*. The flow is then strictly two dimensional. A simple derivation is just to notice that

$$-\frac{1}{\rho}\nabla p = 2\boldsymbol{\Omega} \times \boldsymbol{u}. \tag{9}$$

Taking the curl of both sides, and noticing that

$$\nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) = \boldsymbol{\Omega}\nabla \cdot \boldsymbol{u} - \boldsymbol{u}\nabla \cdot \boldsymbol{\Omega} + \boldsymbol{u} \cdot \nabla\boldsymbol{\Omega} - \boldsymbol{\Omega} \cdot \nabla\boldsymbol{u} = -\Omega\frac{\partial \boldsymbol{u}}{\partial z} \tag{10}$$

where we have used the fact that the fluid velocity is divergence free. Hence we have that $\partial \boldsymbol{u}/\partial z = 0$, or that the fluid velocity is independent of z. A major consequence of this (Taylor columns) was discussed above.

Before leaving this topic, I want to make one other remark about it. Taking the dot product of the equations of inviscid flow with $\boldsymbol{u}$, we get

$$\boldsymbol{u} \cdot \nabla p = -\rho\boldsymbol{u} \cdot (2\boldsymbol{\Omega} \times \boldsymbol{u}) = 0. \tag{11}$$

This formula states that *the velocity field moves perpendicular to the pressure gradient*, which is somewhat against one's intuition. Hence, the fluid actually moves along lines of constant pressure. Pressure work is not performed either on the fluid or by the fluid. Geophysicists call this fact the *geostrophic balance*.

There is an entertaining fact that one can deduce about atmospheric flows. For an atmospheric flow, the analogue of Ω is not the earth's rotation speed ω, but instead $\Omega = \omega \sin \phi$, where ϕ is the longitude. Now, this shows that the effective Ω changes sign in the northern and southern hemisphere. What does this imply for the dynamics? When $\Omega > 0$ the velocity moves with the high pressure on the right. Conversely in the southern hemisphere, the velocity moves with the high pressure on the left. It is also true that because of this change in sign, Naval warships have to adjust their range finding when crossing over the equator. However, the myth about the bathtub vortex does not hold because one cannot throw out inertial and viscous terms in solving this problem. The Coriolis force is only important on large scales.

24 The Ekman layer

We would now like to return to our coffee cup problem, to get the right answer. To do so, we shall consider the effect of walls on the inviscid flow we calculated in the previous lecture. For starters, lets consider a jar with the top moving at angular velocity Ω_T and the bottom moving at angular velocity Ω_B. Clearly, if $\Omega_T = \Omega_B$ then our inviscid solution applies. Let's try and figure out what happens when Ω_T becomes different from Ω_B.

24.1 A small deviation

Suppose $\Omega_T = \Omega$ and $\Omega_B = \Omega + \epsilon$. Now there is no way to satisfy the no slip condition on both the top and bottom while having the whole flow spin at angular velocity Ω. Let's move first to the rotating frame, and try to compute the secondary flow that is induced. Clearly, without viscosity it is impossible to solve this problem because the Taylor-Proudman theorem states that inviscid flow is two dimensional (and so no gradient in Ω across the cylinder axis is possible). We therefore anticipate that even though the Rossby number is small, there will be boundary layers. Let's divide the flow into three regions: (1) A boundary layer at the top plate; (2) a boundary layer at the bottom plate; and (3) a central inviscid region.

In the inviscid region we would expect that the solution is (u_I, v_I, w_I), where

$$-2\Omega v_I \;=\; -\frac{1}{\rho}\frac{\partial p_I}{\partial x}, \tag{1a}$$

$$2\Omega u_I \;=\; -\frac{1}{\rho}\frac{\partial p_I}{\partial y}, \tag{1b}$$

$$0 \;=\; \frac{1}{\rho}\frac{\partial p_I}{\partial z}. \tag{1c}$$

In the same way as before, we expect the pressure gradient of the outer flow to force the boundary layer at the rotating wall. Let's consider the structure of the boundary layer at the bottom wall, $z = 0$. There the equations are

$$-2\Omega v \;=\; -\frac{1}{\rho}\frac{\partial p_I}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}, \tag{2a}$$

$$2\Omega u \;=\; -\frac{1}{\rho}\frac{\partial p_I}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}, \tag{2b}$$

$$0 \;=\; \frac{1}{\rho}\frac{\partial p_I}{\partial z} + \nu\frac{\partial^2 w}{\partial z^2}, \tag{2c}$$

$$\nabla \cdot \mathbf{u} \;=\; 0. \tag{2d}$$

Here we have made the boundary layer approximation that $\partial/\partial z \gg \partial/\partial x, \partial/\partial y$.

From the continuity equation we deduce that w is much smaller than the velocity components parallel to the boundary so that $\partial p_I/\partial z = 0$, and the equations become

$$-2\Omega(v - v_I) = \nu\frac{\partial^2 u}{\partial z^2}, \tag{3a}$$

$$2\Omega(u - u_I) = \nu\frac{\partial^2 v}{\partial z^2}. \tag{3b}$$

These are the equations we must solve. Acheson has a good trick. Multiplying the second equation by i and adding the two yields

$$\nu\frac{\partial^2 f}{\partial z^2} = 2\Omega i f, \tag{4a}$$

where

$$f = u - u_I + i(v - v_I). \tag{4b}$$

The solution is obtained by guessing $f \sim e^{\alpha z}$, which yields $\alpha^2 = 2\Omega i/\nu$. Hence,

$$f = Ae^{(1+i)z^*} + Be^{-(1+i)z^*}, \qquad z^* = z\sqrt{\Omega/\nu}. \tag{5}$$

We require that as $z^* \to \infty$, $f \to 0$. This implies that $A = 0$. We are in the frame of reference moving with the bottom plate, so the no slip boundary condition at $z = 0$ requires that $f(z = 0) = -u_I - iv_I$. Splitting f into its real an imaginary parts implies

$$u = u_I - e^{-z^*}(u_I \cos(z/\delta) + v_I \sin(z/\delta)), \tag{6}$$
$$v = v_I - e^{-z^*}(v_I \cos(z/\delta) - u_I \sin(z/\delta)). \tag{7}$$

This is the velocity profile in the boundary layer.

What about the z-component? From the divergence free condition, we have

$$\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\frac{\partial w}{\partial z^*} = \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y}\right)e^{-z^*}\sin z^*. \tag{8}$$

Integrating from $z^* = 0$ to ∞ gives

$$w = \frac{1}{2}\left(\frac{\Omega}{\nu}\right)^{-\frac{1}{2}}\left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y}\right) = \frac{\omega_I}{2}\sqrt{\frac{\nu}{\Omega}}, \tag{9}$$

where ω_I is the vorticity in the inviscid flow. Thus if $\omega_I > 0$ (i.e., the bottom boundary is moving slower than the main body of fluid) then there is flow from the boundary layer into the fluid.

24.2 Matching

Now we have these Ekman layers at the top and the bottom. What we just did assumed that the boundary was moving at frequency Ω. If it is not, but instead moving at an

angular frequency Ω_B relative to the rotating frame, then we need to change the boundary conditions a little in the rotating frame. In this case

$$w = \left(\frac{\nu}{\Omega_B}\right)^{1/2} \left(\frac{1}{2}\omega_I - \Omega_B\right). \tag{10a}$$

We could derive this, but it is intuitive since $(\omega_I - 2\Omega_B)$ is the vorticity of the interior flow relative to the moving lower boundary. Similarly, if Ω_T denotes the angular velocity of the rigid upper boundary relative to the rotating frame, then there is a small z-component of velocity up into the boundary layer

$$w = \left(\frac{\nu}{\Omega_T}\right)^{1/2} \left(\Omega_T - \frac{1}{2}\omega_I\right). \tag{10b}$$

Now in our container both are happening. Since u_I, v_I and w_I are all independent of z then so is ω_I. Thus, the only way the experiment could work is if the induced value of ω_I from both cases matches. This implies that

$$\omega_I = \Omega_T + \Omega_B. \tag{11}$$

With $\Omega_B = 0$ and $\Omega_T = \epsilon$ we have that $\omega_I = \epsilon$. Thus, the flow in the inner region has a velocity which is entirely set by the boundary layers. Note that there is no viscosity in this formula, but viscosity plays a role in determining the flow. We have completely different behaviour for $\nu = 0$ and in the limit $\nu \to 0$.

24.3   Spin-down of this apparatus

We now want to finally solve the spin-down of our coffee cup. To do so we assume the coffee cup to be a cylinder with a top and a bottom both rotating with angular velocity $\Omega + \epsilon$. At $t = 0$ the angular velocity of the boundaries is reduced to Ω. How long does it take to reach a steady state ?

We use the time dependent formula

$$\frac{\partial u_I}{\partial t} - 2\Omega v_I = -\frac{1}{\rho}\frac{\partial p_I}{\partial x}, \tag{12}$$

$$\frac{\partial v_I}{\partial t} + 2\Omega u_I = -\frac{1}{\rho}\frac{\partial p_I}{\partial y}. \tag{13}$$

Then differentiate the first equation with respect to y and the second with respect to x. Subtracting the latter from the former, and using the continuity equation, we obtain the vorticity equation

$$\frac{\partial}{\partial t}\left(\frac{\partial v_I}{\partial x} - \frac{\partial u_i}{\partial y}\right) = \frac{\partial \omega_I}{\partial t} = 2\Omega\frac{\partial w_I}{\partial z}. \tag{14}$$

Since $\partial w_I/s\partial z < 0$ we see that vorticity is decreasing in the interior.

Now ω_I is independent of z, so

$$\int_0^L \frac{\partial \omega_I}{\partial t}dz = L\frac{\partial \omega_I}{\partial t} = 2\Omega(w(L) - w(0)). \tag{15}$$

The velocity is equal and opposite at the two boundaries (flow is leaving both boundary layers), and has magnitude $(\nu/\Omega)^{\frac{1}{2}}\omega_I/2$. Thus

$$\frac{\partial \omega_I}{\partial t} = -\frac{2\sqrt{\Omega\nu}\omega_I}{L}.$$

(16)

We see that the characteristic decay time is $L/2\sqrt{\Omega\nu}$. For the coffee cup this gives us a much more realistic spin down time compared to our experiments. In real life we should note that diffusion of the no-slip condition also will play a role, and there will be competition between the two depending on the particular shape of your coffee cup. If you go look at the corresponding flow in Acheson, you can now also understand the deep reason why coffee grounds end up at the centre of your cup.