21 Classical aerofoil theory

We now know that through conformal mapping it is possible to transform a circular wing into a more realistic shape, with the bonus of also getting the corresponding inviscid, irrotational flow field. Let's consider some more realistic shapes and see what we get.

21.1 An elliptical wing

First lets rotate our cylinder by an angle α. The complex potential becomes

$$w(z) = u_0 \left(z e^{-i\alpha} + \frac{R^2}{z} e^{i\alpha} \right) - \frac{i\Gamma}{2\pi} \ln z. \tag{1}$$

Now, using the Joukowski transformation we want to turn our circular wing into an elliptical wing. The transformation stipulates that $Z = z + c^2/z$, so that

$$z = \frac{Z}{2} \pm \sqrt{\frac{Z^2}{4} - c^2} \tag{2}$$

giving us (for +) the complex potential

$$\begin{aligned} W(Z) &= u_0 e^{-i\alpha} \left(\frac{Z}{2} + \sqrt{\frac{Z^2}{4} - c^2} \right) + u_0 e^{i\alpha} \frac{R^2}{c^2} \left(\frac{Z}{2} - \sqrt{\frac{Z^2}{4} - c^2} \right) \\ & \qquad - \frac{i\Gamma}{2\pi} \ln \left(\frac{Z}{2} + \sqrt{\frac{Z^2}{4} - c^2} \right). \end{aligned} \tag{3}$$

If we choose $c = R$ the transformation is

$$Z = z + \frac{R^2}{z}, \tag{4}$$

and the ellipse collapses to a flat plate. The velocity components in the Z plane are

$$u - iv = \frac{dW}{dZ} = \frac{dw/dz}{dZ/dz} = \frac{u_0 \left(e^{-i\alpha} - \frac{R^2}{z^2} e^{i\alpha} \right) - \frac{i\Gamma}{2\pi z}}{1 - \frac{R^2}{z^2}}. \tag{5}$$

On the surface of the body we have $z = Re^{i\theta}$, so the velocities become

$$u - iv = \frac{u_0 \left(e^{-i\alpha} - e^{-2i\theta} e^{i\alpha} \right) - \frac{i\Gamma e^{-i\theta}}{2\pi R}}{1 - e^{-2i\theta}}. \tag{6}$$

At $\theta = 0, \pi$ we are in trouble because the velocities are infinite. Notably, however, this problem can be removed at $\theta = 0$ if the circulation is chosen so that the numerator vanishes

$$u - iv = \frac{e^{-i\theta} \left(u_0 (e^{i(\theta-\alpha)} - e^{-i(\theta-\alpha)}) - \frac{i\Gamma}{2\pi R} \right)}{1 - e^{-2i\theta}}. \tag{7}$$

Thus for a finite velocity at $\theta = 0$ we require

$$u_0 e^{-i\alpha} - u_0 e^{i\alpha} - \frac{i\Gamma}{2\pi R} = 0, \tag{8}$$

giving

$$\Gamma = -4\pi u_0 R \sin\alpha. \tag{9}$$

In this case flow leaves the trailing edge smoothly and parallel to the plate. Note that it is not possible to cancel out singularities at both ends simultaneously, as we have to rotate in the opposite direction to cancel out the singularity at $\theta = \pi$.

21.2 Flow past an aerofoil

What if we could now construct a mapping with a singularity just on one side ? This we can do by considering a shifted circle, that passes through $z = R$ but encloses $z = -R$. In this case we obtain an aerofoil with a rounded nose but a sharp trailing edge. The boundary of the appropriate circle is prescribed by

$$z = -\lambda + (a + \lambda)e^{i\theta}, \tag{10}$$

where θ is a parameter. First we must modify the complex potential for flow past a cylinder to take account of this new geometry. We have that

$$w(z) = u_0 \left[(z+\lambda)e^{-i\alpha} + \frac{(R+\lambda)^2}{(z+\lambda)^2} e^{i\alpha} \right] - \frac{i\Gamma}{2\pi} \ln(z+\lambda). \tag{11}$$

To find the complex potential for the aerofoil one must then substitute in $z = Z/2 + \sqrt{Z^2/4 - R^2}$. Determining the velocities as before we find that

$$u - iv = \frac{dW}{dZ} = \frac{dw/dz}{dZ/dz} = u_0 \frac{e^{-i\alpha} - \left(\frac{R+\lambda}{z+\lambda}\right)^2 - \frac{i\Gamma}{2\pi(z+\lambda)}}{1 - \frac{R^2}{z^2}}. \tag{12}$$

The value of Γ that makes the numerator zero at the trailing edge is

$$\Gamma = -4\pi u_0 (R+\lambda) \sin\alpha. \tag{13}$$

The flow is then smooth and free of singularities everywhere (because we have successfully trapped the rogue singularity inside the wing), and this is an example of the *Kutta-Joukowski condition* at work.

21.3 Blausius' lemma

Now we need to prove that $F_L = \rho u_0 + \Gamma$ and $F_D = 0$, independent of wing shape. Firstly, Blausius' lemma tells that for a steady flow with complex potential $w(z)$, if F_x and F_y are the components of the net force on the body then,

$$F_x - iF_y = \frac{i\rho}{2} \oint_C \left(\frac{dw}{dz}\right)^2 dz. \tag{14}$$

To prove this, we realise that the force on the body is due to pressure. Thus

$$F_x - iF_y = -\int p(\sin\theta + i\cos\theta)ds = -i\int pe^{-i\theta}ds. \tag{15}$$

From Bernoulli we have that $p = p_0 - \rho q^2/2$, where $q = \sqrt{u^2+v^2}$. We know $u - iv = dw/dz$, so that

$$\frac{dw}{dz} = q\cos\theta - iq\sin\theta = qe^{-i\theta}. \tag{16}$$

Thus

$$F_x - iF_y = \oint_C ie^{-i\theta}\left(\frac{\rho q^2}{2} - p_0\right)ds = \frac{i\rho}{2}\oint_C e^{-i\theta}\left(\frac{dw}{dz}e^{i\theta}\right)^2 ds. \tag{17}$$

This can be rewritten

$$F_x - iF_y = \frac{i\rho}{2}\oint\left(\frac{dw}{dz}\right)^2 e^{i\theta}ds = \frac{i\rho}{2}\oint\left(\frac{dw}{dz}\right)^2 dz, \tag{18}$$

and we have proved Blasius' lemma.

21.4 Kutta-Joukowski theorem

We now use Blasius' lemma to prove the Kutta-Joukowski lift theorem. For flow around a plane wing we can expand the complex potential in a *Laurent series*, and it must be of the form

$$\frac{dw}{dz} = u_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + ... \tag{19}$$

because the flow is uniform at infinity. Putting this back into Blausis' lemma we have that

$$F_D - iF_L = \frac{i\rho}{2}\oint_C\left(u_0 + \frac{a_1}{z} + \frac{a_2}{z^2}...\right)^2 dz. \tag{20}$$

The *residue theorem* tells us that only the term a_1/z contributes to this integral. This is basically because

$$\oint\frac{1}{z^n}dz = \oint\frac{1}{r^n e^{in\theta}}re^{i\theta}id\theta = \oint r^{-(n-1)}e^{-i(n-1)\theta}id\theta. \tag{21}$$

For the rest of the derivation see Acheson, pp. 144-145.