18.354/12.207
Spring 2014

18 Singular perturbations

The *singular perturbation* is the bogeyman of applied mathematics. The fundamental problem is to ask: when can you neglect a term in a continuous equation? The answer is not always obvious and, amongst other things, this was the reason why early attempts to understand the theory of flight failed so dramatically. Before progressing towards this, we shall begin with a few examples of singular perturbations.

18.1 Magnetisation

A magnet is composed of atoms, each of which has a molecular spin. The energetics of the interaction between the spins is that each spin produces a magnetic field which tries to align the neighbouring spins. A popular microscopic model for a magnet imagines the spins confined to a regular lattice, and then ascribes an energy

$$U = \sum_{neighbours} J_{ij} \boldsymbol{s}_i \cdot \boldsymbol{s}_j \tag{1}$$

A typical approximation is to take the sum over only the nearest neighbours of a given spin and to take the interaction constants J_{ij} to be a constant.

If one assumes that the local spins vary on a length scale much longer than a lattice spacing, then it is possible to derive a *macroscopic* analogue of the above energy. A complete derivation of this includes the effect of random thermal fluctuations and is beyond the scope of this course. For simplicity we consider just a one dimensional array of atoms for which the energy is

$$U[M(x)] = \int \left[\nu \left(\frac{dM}{dx} \right)^2 + f(M) \right] dx, \tag{2a}$$

where M is the magnitude of the local magnetisation, which depends on the average spin in a small region, and

$$f(M) = -bM^2 + cM^4. \tag{2b}$$

Physically, ν punishes gradients in magnetisation. If $b < 0$ then we have a *paramagnet*, with $M = 0$ being the minimum energy configuration. Otherwise if $b > 0$ then we have a *ferromagnet*, with minima at $\pm\sqrt{b/(2c)}$.

Using the calculus of variations the function $M(x)$ that minimises the energy satisfies

$$\nu \frac{d^2 M}{dx^2} - bM + 2cM^3 = 0. \tag{3}$$

We already know that

$$M = 0, \pm\sqrt{\frac{b}{2c}} \tag{4}$$

are three constant solutions. Now if $\nu = 0$ there is no penalty for orientation change throughout the system, and for $b > 0$ the entire system has magnetisation $\pm\sqrt{\frac{b}{2c}}$ with any orientation. If the system has boundaries then the magnetisation must match the boundary conditions, but is otherwise free to be orientated however it wants.

What happens, however, if $\nu \neq 0$? If we multiply both sides of (3) by M', then the equilibrium condition can be integrated to give

$$\frac{dM}{dx} = \frac{1}{\nu}\sqrt{bM^2 - cM^4 + k}, \tag{5}$$

where k is a constant. Rearranging this one obtains

$$\int \frac{\nu\, dM}{\sqrt{bM^2 - cM^4 + k}} = \int dx. \tag{6}$$

Solving this, subject to the appropriate boundary conditions, one finds that *domain boundaries* arise. These are transition regions in which the magnetisation flips from the value imposed at one boundary to that at the other boundary. In the limit of $\nu \to 0$, these domain boundaries become infinitely sharp.

So now you start to get an idea of the problem. If $\nu = 0$ then the orientation of the spins throughout the system is arbitrary, except at the boundaries, which are fixed. However, if ν is extremely small (e.g., 10^{-100}), we have completely different behaviour and obtain extended regions of uniform magnetisation separated by a sharp domain boundary. This entirely different behaviour arises because if ν is nonzero the entire system is forced to match the imposed boundary conditions at the edges. Setting $\nu = 0$ is therefore called a singular perturbation.

18.2 An elementary algebraic equation

As another example of a singular perturbation, consider the solution of the algebraic equation

$$bx + c = 0. \tag{7}$$

The solution is simply $x = -c/b$. Now we make a small change, and consider the equation

$$\epsilon x^2 + bx + c = 0, \tag{8}$$

Using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4\epsilon c}}{2\epsilon}. \tag{9}$$

In the limit $\epsilon \to 0$

$$x \approx -\frac{c}{b}, -\frac{2b - 2\epsilon c}{2\epsilon}. \tag{10}$$

and the latter solution can be further approximated as $-b/\epsilon$ if ϵ is very small. If this term has some physical significance then you are in trouble. You cannot simply neglect the term ϵx^2 in the original problem.

18.3 An elementary differential equation (Acheson, pp. 269-271)

Let's consider the differential equation

$$\epsilon \frac{d^2u}{dx^2} + \frac{du}{dx} = 1. \tag{11}$$

If ϵ is very small we might argue that we can neglect this term, the solution therefore being

$$u = x + C. \tag{12}$$

Alternatively, if we consider the full problem the solution is

$$u = A + x + Be^{-x/\epsilon}. \tag{13}$$

Imposing the boundary conditions $u(0) = 0, u(1) = 2$, for the full problem we determine A and B, and find that

$$u = x + \frac{1 - e^{-x/\epsilon}}{1 - e^{-1/\epsilon}} \tag{14}$$

is the exact solution. We cannot apply both these boundary conditions to our approximate solution (as it is a first order equation), so we choose the 'outer' condition $u(1) = 2$. The approximate solution satisfying the outer condition is therefore

$$u = x + 1. \tag{15}$$

In the outer region the approximate solution and the true solution are very close. However, in a region close to $x = 0$ they differ greatly. We call this the *boundary layer*. It arises because the small parameter ϵ multiplies the highest derivative in the equation, and by ignoring this term we lower the order of the system and are unable to satisfy both boundary conditions.

We need to find an approximate 'inner' solution that matches the boundary condition at $x = 0$. To do so, we change the independent variable to

$$X = \frac{x}{\epsilon}. \tag{16}$$

This enables us to zoom in on the boundary layer. With this scaling the original equation becomes

$$\frac{1}{\epsilon}\frac{d^2u}{dX^2} + \frac{1}{\epsilon}\frac{du}{dX} = 1, \tag{17}$$

so that to a first approximation the 'inner' solution satisfies

$$\frac{d^2u}{dX^2} + \frac{du}{dX} = 0. \tag{18}$$

Imposing the boundary condition at $X = 0$ gives

$$u = A(1 - e^{-X}) = A(1 - e^{-\frac{x}{\epsilon}}). \tag{19}$$

Finally, we require that as $X \to \infty$ the inner solution matches the outer solution in the limit $x \to 0$, so that $A = 1$.

We have thus been able to approximate the full solution in two parts, an inner and outer solution. Although we could solve for the full solution analytically, often this is not possible and we must resort to approximations like those used here. The inner solution is valid within a boundary layer of thickness ϵ and matches to the outer solution. Once again we see that ignoring the term multiplied by ϵ in the original problem is a *singular perturbation*; no matter how small ϵ is, there exists a region in which it has a significant affect on the solution. This idea was due to Prandtl, who first discovered it within the context of airplane flight. We will now take a bit of a digression to justify the concept of a boundary layer in fluid dynamics.

18.354/12.207
Spring 2014

19 Towards airplane flight

There are two forces, lift and drag, experienced when an airplane wing moves through the air. At the start of the 20th century, however, fluid dynamicists were unable to correctly predict them. In fact, the lift and drag were determined to be identically zero, for all wing shapes (despite the fact that the Wright brothers had successfully built airplanes!). To proceed further into airplane flight, and how this problem was resolved, we shall first need some fluid mechanical preliminaries.

19.1 The Euler equations

We have already derived the Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{\nabla p}{\rho} + \nu\nabla^2\boldsymbol{u}\ , \qquad \nabla\cdot\boldsymbol{u} = 0. \tag{1}$$

The ratio of the viscous forces to inertial forces is

$$\frac{\nu\nabla^2\boldsymbol{u}}{\boldsymbol{u}\cdot\nabla\boldsymbol{u}} \sim \frac{\nu U/L^2}{U^2/L} = \frac{\nu}{UL} = \frac{1}{Re} \tag{2}$$

where Re is called the *Reynolds number*. For an airplane typical values are U=400mph, L=5m and ν=0.1cm^2/s, giving $Re = 10^8$. Thus, the inertial forces are eight orders of magnitude larger than viscous forces, so it is seems very reasonable that we can neglect them. Doing so, we are left with the *Euler equations* for an inviscid fluid:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{\nabla p}{\rho}\ , \qquad \nabla\cdot\boldsymbol{u} = 0. \tag{3}$$

We shall first convince ourselves that these equations predict both the lift force (F_l) and the drag force (F_d) to be zero, independent of the shape of the wing.

The Euler equations can be written in an interesting form using the vector identity

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = (\nabla\times\boldsymbol{u})\times\boldsymbol{u} + \frac{\nabla(u^2)}{2} \tag{4}$$

as

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla\times\boldsymbol{u})\times\boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right). \tag{5}$$

where u is the magnitude of $\boldsymbol{u}$. The quantity

$$\boldsymbol{\omega} = \nabla\times\boldsymbol{u} \tag{6}$$

is called the *vorticity*. In 2D it represents the average angular velocity of the fluid around any point, and has little to do with global rotations of the fluid (e.g., shear flows have non-zero vorticity.) If the flow is steady ($\partial \boldsymbol{u}/\partial t = \mathbf{0}$) and *irrotational* ($\boldsymbol{\omega} = \mathbf{0}$) then we have *Bernoulli's law*:

$$H = \frac{p}{\rho} + \frac{u^2}{2} = constant. \tag{7}$$

The constancy of H in irrotational flow is a famous result, and has many simple qualitative consequences. It requires that the pressure in a fluid is smaller when the velocity is larger, and is a statement of conservation of energy when viscous dissipation can safely be ignored.

What are the consequences of $\boldsymbol{\omega} = \mathbf{0}$? The most important is that it is possible to describe the flow by a velocity potential;

$$\boldsymbol{u} = \nabla\phi. \tag{8}$$

Now the wonderful thing is that if the flow is also incompressible, then from $\nabla \cdot \boldsymbol{u} = 0$ we obtain

$$\nabla^2\phi = 0, \tag{9}$$

and we have reduced the problem to solving Laplace's equation. We can determine the pressure from the Bernoulli relation $p/\rho = H - (\nabla\phi)^2/2$. Solutions of these equations are typically called *ideal flows*, because viscosity has been neglected.

19.2 Kelvin's Theorem

When a plane is at rest we can reasonably argue that flow is initially irrotational ($\boldsymbol{\omega} = \mathbf{0}$). When the system becomes non-steady, and the plane accelerates, what happens? To answer this, consider the *circulation* around a closed loop moving with the flow:

$$\Gamma = \int_{C(t)} \boldsymbol{u} \cdot d\boldsymbol{l}. \tag{10a}$$

By Stoke's theorem

$$\Gamma(t) = \int \boldsymbol{\omega} \cdot \boldsymbol{n} dA. \tag{10b}$$

Thus if $\boldsymbol{\omega} = \mathbf{0}$, $\Gamma = 0$. What is the time evolution of Γ? We know that

$$\frac{D\Gamma}{Dt} = \frac{d}{dt}\int_{C(t)} \boldsymbol{u} \cdot d\boldsymbol{l} = \int \left[\frac{D\boldsymbol{u}}{Dt} \cdot d\boldsymbol{l} + \boldsymbol{u} \cdot \frac{D(d\boldsymbol{l})}{Dt}\right]. \tag{11}$$

The first term concerns changes in the velocity field and is identically zero since

$$\int \frac{D\boldsymbol{u}}{Dt} \cdot d\boldsymbol{l} = \frac{1}{\rho}\int -\nabla p \cdot d\boldsymbol{l} = \frac{1}{\rho}\int (\nabla \times \nabla p) \cdot \boldsymbol{n} dA = 0. \tag{12}$$

The second term relates to stretching of the loop and is zero for essentially geometric reasons. One has that

$$\frac{D(d\boldsymbol{l})}{Dt} = d\boldsymbol{u} \tag{13}$$

since one end of the spatial vector $\boldsymbol{dl}$ moves with velocity $\boldsymbol{u}$ and the other with velocity $\boldsymbol{u} + d\boldsymbol{u}$. Thus

$$\int \boldsymbol{u} \cdot \frac{D(d\boldsymbol{l})}{Dt} = \int \boldsymbol{u} \cdot d\boldsymbol{u} = \int \frac{d(u^2)}{2} = 0, \tag{14}$$

because the integration is around a closed loop and u^2 has the same value at the start and end of it. Thus

$$\frac{D\Gamma}{Dt} = 0, \tag{15}$$

so the circulation around a loop remains constant. If it is initially zero then it remains zero for all subsequent times.

One of the most interesting things about this theorem is its historical origin: Kelvin viewed it as the basis of his vortex theory of the atom. Read Acheson (p. 168) for an interesting discussion of this.

20 Towards airplane flight

In the limit where the flow is irrotational we just need to find solutions to Laplace's equation to find solutions to the Euler equations. Let's write down a couple of these to gain some intuition: our aim being to acquire techniques to begin to think about airplane flight.

20.1 Point source

We know from electrostatics that a solution of Laplace's equation is just

$$\phi = -\frac{c}{4\pi r}, \tag{16}$$

where c would be the charge in an electrostatic problem. What does this solution correspond to for us? The velocity field

$$\boldsymbol{u} = \frac{c}{4\pi r^3}\boldsymbol{r} \tag{17}$$

is a radial source or sink of fluid.

20.2 Uniform flow

Another trivial solution is simply uniform flow

$$\phi = \boldsymbol{U} \cdot \boldsymbol{r}, \tag{18}$$

which works for any constant velocity vector $\boldsymbol{U}$.

20.3 Vortex Solutions

We can also guess solutions by separation of variables $\phi = f(\theta)g(r)$, where θ and r denote cylindrical coordinates. Laplace's equation in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0. \tag{19}$$

Plugging this Ansatz into the equation gives that

$$\frac{r}{g}\frac{d}{dr}\left(r\frac{dg}{dr}\right) + \frac{1}{f}\frac{d^2 f}{d\theta^2} = 0. \tag{20}$$

Each term in (20) must be a constant, i.e.

$$\frac{d^2 f}{d\theta^2} = -fk^2, \tag{21a}$$

$$r\frac{d}{dr}\left(r\frac{dg}{dr}\right) = gk^2. \tag{21b}$$

For $k \neq 0$,

$$f = C\sin(k\theta) + D\cos(k\theta), \tag{22a}$$

with continuity of $\boldsymbol{u}$ requiring k to be an integer. However, if $k = 0$ then

$$f = C + D\theta. \tag{22b}$$

Turning to the radial part we guess that $g = r^{\alpha}$. The radial equation then requires that $\alpha = \pm k$, giving $g(r) = Ar^k + Br^{-k}$. If $k = 0$ then $g(r) = A + B\ln r$. So, the most general solution is

$$\phi(r,\theta) = (A_0 + B_0 \ln r)(C_0 + D_0\theta) + \sum_{k=1}^{\infty}(C_k \sin k\theta + D_k \cos k\theta)(A_k r^k + B_k r^{-k}) \tag{23}$$

The corresponding velocity field is

$$\boldsymbol{u} = \nabla\phi = \frac{\partial \phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{\boldsymbol{\theta}}. \tag{24}$$

Putting in our general solution with $k = 0$, we get

$$\boldsymbol{u} = \frac{B_0(C_0 + D_0\theta)}{r}\hat{\mathbf{r}} + \frac{D_0}{r}(A_0 + B_0\ln r)\hat{\boldsymbol{\theta}} \tag{25}$$

Setting $B_0 = 0$ to obtain a continuous flow field, we get a flow with no radial component, and angular component

$$u_\theta = \frac{D}{r}, \tag{26}$$

which is irrotational by virtue of its construction. This is called a *point vortex* solution. If we consider the circulation about a loop containing the origin

$$\int \boldsymbol{u} \cdot d\mathbf{l} = \int_0^{2\pi} u_\theta r d\theta = 2\pi D = \Gamma \tag{27}$$

Thus $D = \Gamma/2\pi$, where Γ is the circulation about the point vortex.

20.4 Flow around a cylindrical wing

Okay, so this isn't the true shape of an airplane wing, but it's a good place to start. Let's see if we can calculate the lift and drag on a wing of length ℓ and radius $R \ll \ell$ moving with velocity u_0. In the frame of reference of the wing, the boundary conditions are

$$\phi \to u_0 x \quad \text{as} \quad r \to \infty, \tag{28a}$$

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{at} \quad r = R. \tag{28b}$$

Using the general solution found above, the first boundary condition requires that

$$\phi = \left(u_0 r + \frac{D_1 B_1}{r}\right)\cos\theta + A_0(C_0 + D_0\theta). \tag{29}$$

The second boundary condition gives us

$$\phi = D_0\theta + u_0 \cos\theta \left(r + \frac{R^2}{r} \right), \tag{30}$$

where we have set $C_0 = 0$ since $\nabla\phi$ is all that matters. Physically we can see that $D_0 = \Gamma/2\pi$, where Γ is the circulation about the wing (check this by integrating around a circular loop containing the wing).

20.5 Forces on the circular wing

The lift and drag forces on the wing (length ℓ) are respectively given by

$$F_L = \ell \int_0^{2\pi} p(R,\theta)\ R\sin\theta d\theta, \tag{31a}$$

$$F_D = \ell \int_0^{2\pi} p(R,\theta)\ R\cos\theta d\theta. \tag{31b}$$

We can determine the pressure distribution from Bernoulli's Law

$$p = p_0 - \frac{\rho}{2}(\nabla\phi)^2\big|_{r=R} = p_0 - \frac{\rho}{2}\left(2u_0 \sin\theta - \frac{\Gamma}{2\pi R} \right)^2. \tag{32}$$

Putting this into the above relations we find that

$$F_D = 0. \tag{33}$$

This result is known as *D'Alembert's paradox*. Furthermore, the lift on the wing is linearly proportional to the circulation about the wing;

$$F_L = \Gamma\rho\, u_0\, \ell. \tag{34}$$

Thus, the lift on the wing is zero (unless it is spinning, so that $\Gamma \neq 0$!) so there really is a problem. What could be wrong? Firstly, airplane wings are not circular and maybe if consider an alternative shape we would find lift. This will be our next avenue of investigation. We could also be worried about the fact that our problem is 2D. However, given that the aspect ratio of a wing is roughly 10:1 it is acceptable to consider 2D flow, and it is unlikely that changing this will generate the lift and drag we are missing. Indeed, we hope that this is not the case, otherwise theoretical aerodynamics would become even more complex. Lastly, we have neglected viscosity, which we considered to be small enough to be neglected. This last point turns out to be the source of our troubles, but let's first convince ourselves that altering the shape of the wing doesn't change things. To do so will require *conformal mapping.*

18.354/12.207
Spring 2014

21 Classical aerofoil theory

There is a useful device for thinking about two dimensional flows, called the *stream function* of the flow. The stream function $\psi(x, y)$ is defined as follows

$$(u, v) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right). \tag{1}$$

The velocity field described by ψ automatically satisfies the incompressibility condition, and it should be noted that

$$\boldsymbol{u}\cdot\nabla\psi = u\frac{\partial\psi}{\partial x} + v\frac{\partial\psi}{\partial y} = 0. \tag{2}$$

Thus ψ is constant along streamlines of the flow. Besides it's physical convenience, another great thing about the stream function is the following. By definition

$$u \;=\; \frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x}, \tag{3a}$$

$$v \;=\; -\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y}, \tag{3b}$$

where ϕ is the velocity potential for an irrotational flow. Thus, both ϕ and ψ obey the well known *Cauchy-Riemann* equations of complex analysis.

21.1 The Cauchy-Riemann equations

In complex analysis you work with the complex variable $z = x + iy$. Thus, if you have some complex function $f(z)$ what is df/dz? Well, $f(z)$ can be separated into a real part $u(x, y)$ and an imaginary part $v(x, y)$, where u and v are real functions, i.e.:

$$f(z) = f(x + iy) = u(x, y) + iv(x, y). \tag{4}$$

For example, if $f(z) = z^2$ then $u = x^2 + y^2$ and $v = 2xy$. What then is df/dz? Since we are now in two-dimensions we can approach a particular point z from the x-direction or the y-direction (or any other direction, for that matter). On one hand we could define

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{5a}$$

Or, alternatively

$$\frac{df}{dz} = \frac{\partial f}{\partial(iy)} = -i\frac{\partial f}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{5b}$$

For the definition of the derivative to make sense requires $\partial u/\partial x = \partial v/\partial y$ and $-\partial u/\partial y = \partial v/\partial x$, the Cauchy-Riemann equations. If this is true then $f(z)$ is said to be analytic and we can simply differentiate with respect to z in the usual manner. For our simple example $f(z) = z^2$ we have that $df/dz = 2z$ (confirm for yourself that z^2 is analytic as there are many functions that are not, e.g., $|z|$ is not an analytic function.)

21.2 Conformal mapping

We can now use the power of complex analysis to think about two dimensional potential flow problems. Since ϕ and ψ obey the Cauchy-Riemann equations, this implies that $w = \phi + i\psi$ is an analytic function of the complex variable $z = x + iy$. We call w the *complex potential.* Another important property of 2D incompressible flow is that both ϕ and ψ satisfy Laplace's equation. For example, using the Cauchy-Riemann equations we see that

$$\frac{\partial\psi}{\partial x^2} + \frac{\partial\psi}{\partial y^2} = -\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial y\partial x} = 0. \tag{6}$$

The same proof can be used for ϕ. We can therefore consider any analytic function (e.g., $\sin z$,z^4,...), calculate the real and imaginary parts and both of them satisfy Laplace's equation.

The velocity components u and v are directly related to dw/dz, which is conveniently calculated as follows:

$$\frac{dw}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = u - iv. \tag{7}$$

As a simple example consider uniform flow at an angle α to the x-axis. The corresponding complex potential is $w = u_0 z e^{-i\alpha}$. In this case $dw/dz = u_0 e^{-i\alpha}$. Using the above relation, this tells us that $u = u_0 \cos\alpha$ and $v = u_0 \sin\alpha$.

We can also determine the complex potential for flow past a cylinder since we know that

$$\phi = u_0\left(r + \frac{R^2}{r}\right)\cos\theta, \tag{8}$$

and this is just the real part of the complex potential

$$w = u_0\left(z + \frac{R^2}{z}\right). \tag{9}$$

Check this by substituting in $z = re^{i\theta}$. What is the corresponding stream function? Also $w(z) = -i\ln z$ is the complex potential for a point vortex since

$$Re\,(w(z)) = Re\left(-i\ln(re^{i\theta})\right) = \theta, \tag{10}$$

and we know that $\phi = \theta$ is the real potential for a point vortex. Thus

$$w(z) = u_0\left(z + \frac{R^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln z \tag{11}$$

is the complex potential for flow past a cylinder with circulation Γ.

So let's assume that the only problem we know how to solve is flow past a cylinder, when really we want to know how to solve for flow past an aerofoil. The idea is to now consider two complex planes (x, y) and (X, Y). In the first plane we have the complex variable $z = x + iy$ and in the latter we have $Z = X + iY$. If we construct a mapping $Z = F(z)$ which is analytic, with an inverse $z = F^{-1}(Z)$, then $W(Z) = w(F^{-1}(Z))$ is also analytic, and may be considered a complex potential in the new co-ordinate system. Because $W(Z)$

and $w(z)$ take the same value at corresponding points of the two planes it follows that Ψ and ψ are the same at corresponding points. Thus streamlines are mapped into streamlines. In particular a solid boundary in the z-plane, which is necessarily a streamline, gets mapped into a streamline in the Z-plane, which could accordingly be viewed as a rigid boundary. Thus all we have done is distort the streamlines and the boundary leaving us with the key question: Given flow past a circular cylinder in the z-plane can we choose a mapping so as to obtain in the Z-plane uniform flow past a more wing-like shape? (Note that we have brushed passed some technical details here, such as the requirement that $dF/dz \neq 0$ at any point, as this would cause a blow-up of the velocity).

21.3 Simple conformal maps

The simplest map is

$$Z = F(z) = z + b, \tag{12}$$

which corresponds to a translation. Then there is

$$Z = F(z) = ze^{i\alpha}, \tag{13}$$

which corresponds to a rotation through angle α. In this case, the complex potential for uniform flow past a cylinder making angle α with the stream is

$$W(Z) = u_0 \left(Ze^{-i\alpha} + \frac{R^2}{Z} e^{i\alpha} \right) - \frac{i\Gamma}{2\pi} \ln Z. \tag{14}$$

Note, that this expression could also include the term $\ln e^{i\alpha} = i\alpha$ which I have neglected. This is just a constant however and doesn't change the velocity.

Finally there is the non-trivial *Joukowski transformation*,

$$Z = F(z) = z + \frac{c^2}{z}. \tag{15}$$

What does this do to the circle? Well, $z = ae^{i\theta}$ becomes

$$Z = ae^{i\theta} + \frac{c^2}{a} e^{-i\theta} = (a + \frac{c^2}{a}) \cos\theta + i(a - \frac{c^2}{a}) \sin\theta. \tag{16}$$

Defining $X = Re(Z)$, $Y = Im(z)$, it is easily shown that

$$\left(\frac{X}{a + \frac{c^2}{a}} \right)^2 + \left(\frac{Y}{a - \frac{c^2}{a}} \right)^2 = 1, \tag{17}$$

which is the equation of an ellipse, provided $c < a$.