17 Differential geometry of membranes

17.1 Differential geometry of curves

Consider a continuous curve $\boldsymbol{r}(t) \in \mathbb{R}^3$, where $t \in [0, T]$. The length of the curve is given by

$$L = \int_0^T dt\, ||\dot{\boldsymbol{r}}(t)|| \tag{1}$$

where $\dot{\boldsymbol{r}}(t) = d\boldsymbol{r}/dt$ and $||\cdot||$ denotes the Euclidean norm. The local unit tangent vector is defined by

$$\boldsymbol{t} = \frac{\dot{\boldsymbol{r}}}{||\dot{\boldsymbol{r}}||}. \tag{2}$$

The unit normal vector, or unit curvature vector, is

$$\boldsymbol{n} = \frac{(\boldsymbol{I} - \boldsymbol{tt}) \cdot \ddot{\boldsymbol{r}}}{||(\boldsymbol{I} - \boldsymbol{tt}) \cdot \ddot{\boldsymbol{r}}||}. \tag{3}$$

Unit tangent vector $\hat{\boldsymbol{t}}(t)$ and unit normal vector $\hat{\boldsymbol{n}}(t)$ span the *osculating* ('kissing') plane at point t. The unit binormal vector is defined by

$$\boldsymbol{b} = \frac{(\boldsymbol{I} - \boldsymbol{tt}) \cdot (\boldsymbol{I} - \boldsymbol{nn}) \cdot \dddot{\boldsymbol{r}}}{||(\boldsymbol{I} - \boldsymbol{tt}) \cdot (\boldsymbol{I} - \boldsymbol{nn}) \cdot \dddot{\boldsymbol{r}}||}\,. \tag{4}$$

The orthonormal basis $\{\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t)\}$ spans the local Frenet frame.

The local curvature $\kappa(t)$ and the associated radius of curvature $\rho(t) = 1/\kappa$ are defined by

$$\kappa(t) = \frac{\dot{\boldsymbol{t}} \cdot \boldsymbol{n}}{||\dot{\boldsymbol{r}}||}, \tag{5}$$

and the local torsion $\tau(t)$ by

$$\tau(t) = \frac{\dot{\boldsymbol{n}} \cdot \boldsymbol{b}}{||\dot{\boldsymbol{r}}||}. \tag{6}$$

Plane curves satisfy, by definition, $\boldsymbol{b} = const.$ or, equivalently, $\tau = 0$.

Given $||\dot{\boldsymbol{r}}||$, $\kappa(t)$, $\tau(t)$ and the initial values $\{\boldsymbol{t}(0), \boldsymbol{n}(0), \boldsymbol{b}(0)\}$, the Frenet frames along the curve can be obtained by solving the Frenet-Serret system

$$\frac{1}{||\dot{\boldsymbol{r}}||} \begin{pmatrix} \dot{\boldsymbol{t}} \\ \dot{\boldsymbol{n}} \\ \dot{\boldsymbol{b}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}. \tag{7a}$$

The above formulas simplify if t is the arc length, for in this case $||\dot{\boldsymbol{r}}|| = 1$.

As a simple example (which is equivalent to our shortest path problem) consider a polymer confined in a plane. Assume the polymer's end-points are fixed at $(x, y) = (0, 0)$ and $(x, y) = (0, L)$, respectively, and that the ground-state configuration corresponds to a straight line connecting these two points. Denoting the tension[1] by γ, adopting the parameterization $y = h(x)$ for the polymer and assuming that the bending energy is negligible, the energy relative to the ground-state is given by

$$E = \gamma \left[\int_0^L dx \sqrt{1 + h_x^2} - L \right], \tag{8}$$

where $h_x = h'(x)$. Restricting ourselves to small deformations, $|h_x| \ll 1$, we may approximate

$$E \simeq \frac{\gamma}{2} \int_0^L dx\, h_x^2. \tag{9}$$

Minimizing this expression with respect to the polymer shape h yields the Euler-Lagrange equation

$$h_{xx} = 0. \tag{10}$$

17.2 2D differential geometry

We now consider an orientable surface in $\mathbb{R}^3$. Possible local parameterizations are

$$\boldsymbol{F}(s_1, s_2) \in \mathbb{R}^3 \tag{11}$$

where $(s_1, s_2) \in U \subseteq \mathbb{R}^2$. Alternatively, if one chooses Cartesian coordinates $(s_1, s_2) = (x, y)$, then it suffices to specify

$$z = f(x, y) \tag{12a}$$

or, equivalently, the implicit representation

$$\Phi(x, y, z) = z - f(x, y). \tag{12b}$$

The vector representation (11) can be related to the 'height' representation (12a) by

$$\boldsymbol{F}(x, y) = \begin{pmatrix} x \\ y \\ f(x, y) \end{pmatrix} \tag{13}$$

Denoting derivatives by $\boldsymbol{F}_i = \partial_{s_1} \boldsymbol{F}$, we introduce the surface metric tensor $g = (g_{ij})$ by

$$g_{ij} = \boldsymbol{F}_i \cdot \boldsymbol{F}_j, \tag{14a}$$

abbreviate its determinant by

$$|g| := \det g, \tag{14b}$$

[1] γ carries units of energy/length.

and define the associated Laplace-Beltrami operator ∇^2 by

$$\nabla^2 h = \frac{1}{\sqrt{|g|}} \partial_i (g_{ij}^{-1} \sqrt{|g|} \partial_j h), \tag{14c}$$

for some function $h(s_1, s_2)$. For the Cartesian parameterization (13), one finds explicitly

$$\boldsymbol{F}_x(x,y) = \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix}, \qquad \boldsymbol{F}_y(x,y) = \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix} \tag{15}$$

and, hence, the metric tensor

$$g = (g_{ij}) = \begin{pmatrix} \boldsymbol{F}_x \cdot \boldsymbol{F}_x & \boldsymbol{F}_x \cdot \boldsymbol{F}_y \\ \boldsymbol{F}_y \cdot \boldsymbol{F}_x & \boldsymbol{F}_y \cdot \boldsymbol{F}_y \end{pmatrix} = \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_y f_x & 1 + f_y^2 \end{pmatrix} \tag{16a}$$

and its determinant

$$|g| = 1 + f_x^2 + f_y^2, \tag{16b}$$

where $f_x = \partial_x f$ and $f_y = \partial_y f$. For later use, we still note that the inverse of the metric tensor is given by

$$g^{-1} = (g_{ij}^{-1}) = \frac{1}{1 + f_x^2 + f_y^2} \begin{pmatrix} 1 + f_y^2 & -f_x f_y \\ -f_y f_x & 1 + f_x^2 \end{pmatrix}. \tag{16c}$$

Assuming the surface is regular at (s_1, s_2), which just means that the tangent vectors $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are linearly independent, the local unit normal vector is defined by

$$\boldsymbol{N} = \frac{\boldsymbol{F}_1 \wedge \boldsymbol{F}_2}{||\boldsymbol{F}_1 \wedge \boldsymbol{F}_2||}. \tag{17}$$

In terms of the Cartesian parameterization, this can also be rewritten as

$$\boldsymbol{N} = \frac{\nabla \Phi}{||\nabla \Phi||} = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}. \tag{18}$$

Here, we have adopted the convention that $\{\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{N}\}$ form a right-handed system.

To formulate 'geometric' energy functionals for membranes, we still require the concept of curvature, which quantifies the local bending of the membrane. We define a 2×2-curvature tensor $R = (R_{ij})$ by

$$R_{ij} = \boldsymbol{N} \cdot (\boldsymbol{F}_{ij}) \tag{19}$$

and local *mean curvature* H and local *Gauss curvature* K by

$$H = \frac{1}{2} \mathrm{tr}\,(g^{-1} \cdot R)\,, \qquad K = \det(g^{-1} \cdot R). \tag{20}$$

Adopting the Cartesian representation (12a), we have

$$\boldsymbol{F}_{xx} = \begin{pmatrix} 0 \\ 0 \\ f_{xx} \end{pmatrix}, \qquad \boldsymbol{F}_{xy} = \boldsymbol{F}_{yx} = \begin{pmatrix} 0 \\ 0 \\ f_{xy} \end{pmatrix}, \qquad \boldsymbol{F}_{yy} = \begin{pmatrix} 0 \\ 0 \\ f_{yy} \end{pmatrix} \tag{21a}$$

yielding the curvature tensor

$$(R_{ij}) = \begin{pmatrix} \boldsymbol{N}\cdot\boldsymbol{F}_{xx} & \boldsymbol{N}\cdot\boldsymbol{F}_{xy} \\ \boldsymbol{N}\cdot\boldsymbol{F}_{yx} & \boldsymbol{N}\cdot\boldsymbol{F}_{yy} \end{pmatrix} = \frac{1}{\sqrt{1+f_x^2+f_y^2}} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \tag{21b}$$

Denoting the eigenvalues of the matrix $g^{-1}\cdot R$ by κ_1 and κ_2, we obtain for the mean curvature

$$H = \frac{1}{2}(\kappa_1+\kappa_2) = \frac{(1+f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1+f_x^2)f_{yy}}{2(1+f_x^2+f_y^2)^{3/2}} \tag{22}$$

and for the Gauss curvature

$$K = \kappa_1\cdot\kappa_2 = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1+f_x^2+f_y^2)^2}. \tag{23}$$

An important result that relates curvature and topology is the Gauss-Bonnet theorem, which states that any compact two-dimensional Riemannian manifold M with smooth boundary ∂M, Gauss curvature K and geodesic curvature k_g of ∂M satisfies the integral equation

$$\int_M K\, dA + \oint_{\partial M} k_g\, ds = 2\pi\,\chi(M). \tag{24}$$

Here, dA is the area element on M, ds the line element along ∂M, and $\chi(M)$ the Euler characteristic of M. The latter is given by $\chi(M) = 2-2g$, where g is the *genus* (number of handles) of M. For example, the 2-sphere $M = \mathbb{S}^2$ has $g=0$ handles and hence $\chi(\mathbb{S}^2) = 2$, whereas a two-dimensional torus $M = \mathbb{T}^2$ has $g=1$ handle and therefore $\chi(\mathbb{T}^2) = 0$.

Equation (24) implies that, for any closed surface, the integral over K is always a constant. That is, for closed membranes, the first integral in Eq. (24) represents just a trivial (constant) energetic contribution.

17.3 Minimal surfaces

Minimal surfaces are surfaces that minimize the area within a given contour ∂M,

$$A(M|\partial M) = \int_M dA = \text{min!} \tag{25}$$

Assuming a Cartesian parameterization $z = f(x,y)$ and abbreviating $f_i = \partial_i f$ as before, we have

$$dA = \sqrt{|g|}\, dxdy = \sqrt{1+f_x^2+f_y^2}\, dxdy =: \mathcal{L}\, dxdy, \tag{26}$$

and the minimum condition (25) can be expressed in terms of the Euler-Lagrange equations

$$0 = \frac{\delta A}{\delta f} = -\partial_i \frac{\partial \mathcal{L}}{\partial f_i}. \tag{27}$$

Inserting the Lagrangian $\mathcal{L} = \sqrt{|g|}$, one finds

$$0 = -\left[\partial_x \left(\frac{f_x}{\sqrt{1+f_x^2+f_y^2}}\right) + \partial_y \left(\frac{f_y}{\sqrt{1+f_x^2+f_y^2}}\right)\right] \tag{28}$$

which may be recast in the form

$$0 = \frac{(1+f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1+f_x^2)f_{yy}}{(1+f_x^2+f_y^2)^{3/2}} = -2H. \tag{29}$$

Thus, minimal surfaces satisfy

$$H = 0 \qquad \Leftrightarrow \qquad \kappa_1 = -\kappa_2, \tag{30}$$

implying that each point of a minimal surface is a saddle point.

17.4 Helfrich's model

Assuming that lipid bilayer membranes can be viewed as two-dimensional surfaces, Helfrich proposed in 1973 the following geometric curvature energy per unit area for a closed membrane

$$\epsilon = \frac{k_c}{2}(2H - c_0)^2 + k_G\, K, \tag{31}$$

where constants k_c, k_G are bending rigidities and c_0 is the spontaneous curvature of the membrane. The full free energy for a closed membrane can then be written as

$$E_c = \int dA\, \epsilon + \sigma \int dA + \Delta p \int dV, \tag{32}$$

where σ is the surface tension and Δp the osmotic pressure (outer pressure minus inner pressure). Minimizing F with respect to the surface shape, one finds after some heroic manipulations the shape equation[2]

$$\Delta p - 2\sigma H + k_c(2H - c_0)(2H^2 + c_0 H - 2K) + k_c \nabla^2 (2H - c_0) = 0, \tag{33}$$

where ∇^2 is the Laplace-Beltrami operator on the surface. The derivation of Eq. (33) uses our earlier result

$$\frac{\delta A}{\delta f} = -2H, \tag{34}$$

[2]The full derivation can be found in Chapter 3 of Z.-C. Ou-Yang, *Geometric Methods in the Elastic Theory of Membranes in Liquid Crystal Phases*(World Scientific,Singapore, 1999).

and the fact that the volume integral may be rewritten as[3]

$$V = \int dV = \int dA \, \frac{1}{3} \boldsymbol{F} \cdot \boldsymbol{N} \, , \tag{35}$$

which gives

$$\frac{\delta V}{\delta f} = 1, \tag{36}$$

corresponding to the first term on the rhs. of Eq. (33).

For open membranes with boundary ∂M, there is no volume constraint and a plausible energy functional reads

$$E_o = \int dA \, \epsilon + \sigma \int dA + \gamma \oint_{\partial M} ds, \tag{37}$$

where γ is the line tension of the boundary. In this case, variation yields not only the corresponding shape equation but also a non-trivial set of boundary conditions.

[3]Here, we made use of the volume formula $dV = \frac{1}{3} h \, dA$ for a cone or pyramid of height $h = \boldsymbol{F} \cdot \boldsymbol{N}$.