

## 10 Dimensional Analysis

Before moving on to more ‘sophisticated things’, we pause to think a little about dimensional analysis and scaling. On the one hand these are trivial, and on the other they give a simple method for getting answers to problems that might otherwise be intractable. The idea behind dimensional analysis is very simple: *any physical law must be expressible in any system of units that you use*. There are two consequences of this:

- One can often guess the answer just by thinking about what the dimensions of the answer should be, and then expressing the answer in terms of quantities that are known to have those dimensions<sup>1</sup>.
- The scientifically interesting results are *always* expressible in terms of quantities that are *dimensionless*, not depending on the system of units that you are using.

One example of a dimensionless number relevant for fluid dynamics that we have already encountered is the *Reynolds number*. Another example of dimensional analysis that we have seen already is the solution to the diffusion equation for the spreading of a point source. The only relevant physical parameter is the diffusion constant  $D$ , which has dimensions of  $L^2T^{-1}$ . We denote this by writing  $[D] = L^2T^{-1}$ . Therefore the characteristic scale over which the solution varies after time  $t$  must be  $\sqrt{Dt}$ . This might seem like a rather simple result, but it expresses the essence of solutions to the diffusion equation. Of course, we were able to solve the diffusion equation exactly, so this argument wasn’t really necessary. In practice, however, we will rarely find useful exact solutions to the Navier-Stokes equations, and so dimensional analysis will often give us insight before diving into the mathematics or numerical simulations. Before formalising our approach, let us consider a few examples where simple dimensional arguments intuitively lead to interesting results.

### 10.1 The pendulum

This is a trivial problem that you know quite well. Consider a pendulum with length  $L$  and mass  $m$ , hanging in a gravitational field of strength  $g$ . What is the period of the pendulum? We need a way to construct a quantity with units of time involving these numbers. The only possible way to do this is with the combination  $\sqrt{L/g}$ . Therefore, we know immediately that

$$\tau = c\sqrt{L/g}. \tag{1}$$

This result might seem trivial to you. However, remember that you have only previously solved the problem for small amplitude oscillations, with  $c = 2\pi$ . The above formula works even for large amplitude oscillations.

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<sup>1</sup>Be careful to distinguish between *dimensions* and *units*. For example mass ( $M$ ), length ( $L$ ) and time ( $T$ ) are dimensions, and they can have different units of measurement (e. g. length may be in feet or meters)

## 10.2 Pythagorean Theorem

Now we try to prove the Pythagorean theorem by dimensional analysis. Suppose you are given a right triangle, with hypotenuse length  $L$  and smallest acute angle  $\phi$ . The area of the triangle is clearly

$$A = A(L, \phi). \quad (2)$$

Since  $\phi$  is dimensionless, it must be that

$$A = L^2 f(\phi), \quad (3)$$

where  $f$  is some function we don't know.

Now the triangle can be divided into two little right triangles by dropping a line from the right angle which is perpendicular to the hypotenuse. The two right triangles have hypotenuses that happen to be the other two sides of our original right triangle, let's call them  $a$  and  $b$ . So we know that the areas of the two smaller triangles are  $a^2 f(\phi)$  and  $b^2 f(\phi)$  (where elementary geometry shows that the acute angle  $\phi$  is the same for the two little triangles as the big triangle). Moreover, since these are all right triangles, the function  $f$  is the same for each. Therefore, since the area of the big triangle is just the sum of the areas of the little ones, we have

$$L^2 f = a^2 f + b^2 f, \quad (4)$$

or

$$L^2 = a^2 + b^2. \quad (5)$$

## 10.3 The oscillation of a star

It is known that the sun, and many other stars undergo some mode of oscillation. The question we might ask is how does the frequency of oscillation  $\omega$  depend on the properties of that star? The first step is to identify the physically relevant variables. These are the density  $\rho$ , the radius  $R$  and the gravitational constant  $G$  (as the oscillations are due to gravitational forces). So we have

$$\omega = \omega(\rho, R, G). \quad (6)$$

The dimensions of the variables are  $[\omega] = T^{-1}$ ,  $[\rho] = ML^{-3}$ ,  $[R] = L$  and  $[G] = M^{-1}L^3T^{-2}$ . The only way we can combine these to give as a quantity with the dimensions of time, is through the relation

$$\omega = c\sqrt{G\rho}. \quad (7)$$

Thus, we see that the frequency of oscillation is proportional to the square root of the density and independent of the radius. The determination of  $c$  requires a real stellar observation, but we have already determined a lot of interesting details from dimensional analysis alone. For the sun,  $\rho = 1400\text{kgm}^{-3}$ , giving  $\omega = 3 \times 10^{-4}\text{s}^{-1}$ . The period of oscillation is approximately 1 hour, which is reasonable. However, for a neutron star ( $\rho = 7 \times 10^{11}\text{kgm}^{-3}$ ) we predict  $\omega = 7000\text{s}^{-1}$ .

## 10.4 The oscillation of a droplet

What happens if instead of considering a large body of fluid, such as a star, we consider a smaller body of fluid, such as a raindrop. Well, in this case we argue that surface tension  $\gamma$  provides the relevant restoring force and we can neglect gravity.  $\gamma$  has dimensions of energy/area, so that  $[\gamma] = MT^{-2}$ . The only quantity we can now make with the dimensions of  $T^{-1}$  using our physical variables is

$$\omega = c\sqrt{\frac{\gamma}{\rho R^3}}. \quad (8)$$

which is not independent of the radius. For water  $\gamma = 0.07\text{Nm}^{-1}$  giving us a characteristic frequency of 3Hz for a raindrop.

One final question we might ask ourselves before moving on is how big does the droplet have to be for gravity to have an effect? We reason that the crossover will occur when the two models give the same frequency of oscillation. Thus, when

$$\sqrt{\rho G} = \sqrt{\frac{\gamma}{\rho R^3}} \quad (9)$$

we find that

$$R_c \sim \left(\frac{\gamma}{\rho^2 G}\right)^{\frac{1}{3}} \quad (10)$$

This gives a crossover radius of about 10m for water.

## 10.5 Water waves

This is a subject we will deal with in greater detail towards the end of the course, but for now we look to obtain a basic understanding of the motion of waves on the surface of water. For example, how does the frequency of the wave depend on the wavelength  $\lambda$ ? This is called the *dispersion relation*.

If the wavelength is long, we expect gravity to provide the restoring force, and the relevant physical variables in determining the frequency would appear to be the mass density  $\rho$ , the gravitational acceleration  $g$  and the wave number  $k = 2\pi/\lambda$ . The dimensions of these quantities are  $[\rho] = ML^{-3}$ ,  $[g] = LT^{-2}$  and  $[k] = L^{-1}$ . We can construct a quantity with the dimensions of  $T^{-1}$  through the relation

$$\omega = c\sqrt{gk}. \quad (11)$$

We see that the frequency of water waves is proportional to the square root of the wavenumber, in contrast to light waves for which the frequency is proportional to the wavenumber.

As with a droplet, we might worry about the effects of surface tension when the wavelength gets small. In this case we replace  $g$  with  $\gamma$  in our list of physically relevant variables. Given that  $[\gamma] = MT^{-2}$ , the dispersion relation must be of the form

$$\omega = c\sqrt{\gamma k^3/\rho}, \quad (12)$$

which is very different to that for gravity waves. If we look for a crossover, we find that the frequencies of *gravity waves* and *capillary waves* are equal when

$$k \sim \sqrt{\rho g / \gamma}. \quad (13)$$

This gives a wavelength of 1cm for water waves.

## 11 Dimensionless Groups

A formal justification of the dimensional analysis approach in the previous section comes from Buckingham's Pi Theorem. Consider a physical problem in which the dependent parameter is a function of  $n - 1$  independent parameters, so that we may express the relationship among the variables in functional form as

$$q_1 = g(q_2, q_3, \dots, q_n), \quad (1a)$$

where  $q_1$  is the dependent parameter, and  $q_2, \dots, q_n$  are the  $n - 1$  independent parameters. Mathematically, we can rewrite the functional relationship in the equivalent form

$$0 = f(q_1, q_2, \dots, q_n). \quad (1b)$$

where  $f = q_1 - g(q_2, q_3, \dots, q_n)$ . For example, for the period of a pendulum we wrote  $\tau = \tau(l, g, m)$ , but we could just as well have written  $f(\tau, l, g, m) = 0$ . The Buckingham Pi theorem states that given a relation of the form (1), the  $n$  parameters may be grouped into  $n - d$  independent dimensionless ratios, or dimensionless groups  $\Pi_i$ , expressible in functional form by

$$\Pi_1 = G(\Pi_2, \Pi_3, \dots, \Pi_{n-d}), \quad (2a)$$

or, equivalently,

$$0 = F(\Pi_1, \Pi_2, \dots, \Pi_{n-d}), \quad (2b)$$

where  $d$  is the number of independent dimensions (mass, length, time...). The formal proof can be found in the book *Scaling, Self Similarity and Intermediate Asymptotics* by Barenblatt. The Pi theorem does not predict the functional form of  $F$  or  $G$ , and this must be determined experimentally. The  $n - d$  dimensionless groups  $\Pi_i$  are independent. A dimensionless group  $\Pi_i$  is not independent if it can be formed from a product or quotient of other dimensionless groups in the problem.

### 11.1 The pendulum

To develop an understanding of how to use Buckingham's Pi theorem, let's first apply it to the problem of a swinging pendulum, which we considered in the previous lecture. We argued that the period of the pendulum  $\tau$  depends on the length  $l$  and gravity  $g$ . It cannot depend on the mass  $m$  since we cannot form a dimensionless parameter including  $m$  in our list of physical variables. Thus

$$\tau = \tau(l, g), \quad (3a)$$

or alternatively

$$0 = f(\tau, l, g). \quad (3b)$$

We have  $n = 3$  and  $d = 2$ , so the problem has one dimensionless group

$$\Pi_1 = \tau l^\alpha g^\beta. \quad (4)$$

The relevant dimensions are  $[\tau] = T$ ,  $[l] = L$ ,  $[g] = LT^{-2}$ , so for  $\Pi_1$  to be dimensionless equate the exponents of the dimension to find

$$\begin{aligned} 1 - 2\beta &= 0, \\ \alpha + \beta &= 0, \end{aligned}$$

which are satisfied if  $\alpha = -\frac{1}{2}$  and  $\beta = \frac{1}{2}$ . Thus

$$\Pi_1 = \tau\sqrt{g/l}. \quad (5)$$

We thus see  $\Pi_1$  is just the constant of proportionality  $c$  from above. Thus we have

$$c = \tau\sqrt{g/l} \quad (6)$$

where  $c$  is a constant to be determined from an experiment.

## 11.2 Taylor's blast

This is a famous example, of some historical and fluid mechanical importance. The story goes something like this. In the early 1940's there appeared a picture of an atomic blast on the cover of Life magazine. GI Taylor, a fluid mechanician at Cambridge, wondered what the energy of the blast was. When he called his colleagues at Los Alamos and asked, they informed him that it was classified information, so he resorted to dimensional analysis. In a nuclear explosion there is an essentially instantaneous release of energy  $E$  in a small region of space. This produces a spherical shock wave, with the pressure inside the shock wave several thousands of times greater than the initial air pressure, that can be neglected. How does the radius  $R$  of this shock wave grow with time  $t$ ? The relevant parameters are  $E$ , the density of air  $\rho$  and time  $t$ . Thus

$$R = R(E, \rho, t) \quad (7a)$$

or

$$0 = f(R, E, \rho, t). \quad (7b)$$

The dimensions of the physical variables are  $[E] = ML^2T^{-2}$ ,  $[t] = T$ ,  $[R] = L$  and  $[\rho] = ML^{-3}$ . We have  $n = 4$  physical variables and  $d = 3$  dimensions, so the Pi theorem tells us there is one dimensionless group,  $\Pi_1$ . To form a dimensionless combination of parameters we assume

$$\Pi_1 = Et^\alpha \rho^\beta R^\gamma \quad (8)$$

and equating the exponents of dimensions in the problem requires that

$$\begin{aligned} 1 + \beta &= 0, \\ \alpha - 2 &= 0, \\ 2 - 3\beta + \gamma &= 0. \end{aligned}$$

It follows that  $\alpha = 2$ ,  $\beta = -1$  and  $\gamma = -5$ , giving

$$\Pi_1 = \frac{Et^2}{\rho R^5}. \quad (9)$$

Assuming that  $\Pi_1$  is constant gives

$$R = c \left( \frac{E}{\rho} \right)^{\frac{1}{5}} t^{\frac{2}{5}}. \quad (10)$$

The relation shows that if one measures the radius of the shock wave at various instants in time, the slope of the line on a log-log plot should be  $2/5$ . The intercept of the graph would provide information about the energy  $E$  released in the explosion, if the constant  $c$  could be determined. Since information about the development of blast with time was provided by the sequence of photos on the cover of Life magazine, Taylor was able to determine the energy of the blast to be  $10^{14}$  Joules (he estimated  $c$  to be about 1 by solving a model shock-wave problem), causing much embarrassment.

### 11.3 The drag on a sphere

Now what happens if you have two dimensionless groups in a problem? Let's consider the problem of the drag on a sphere. We reason that the drag on a sphere  $D$  will depend on the relative velocity,  $U$ , the sphere radius,  $R$ , the fluid density  $\rho$  and the fluid viscosity  $\mu$ . Thus

$$D = D(U, R, \rho, \mu) \quad (11a)$$

or

$$0 = F(D, U, R, \rho, \mu). \quad (11b)$$

Since the physical variables are all expressible in terms of dimensions  $M, L$  and  $T$ , we have  $n = 5$  and  $d = 3$ , so there are two dimensionless groups. There is now a certain amount of arbitrariness in determining these, however we look for combinations that make some physical sense. For our first dimensionless group, we choose the Reynolds number

$$\Pi_1 = \frac{\rho U R}{\mu}, \quad (12)$$

as we know that it arises naturally when you nondimensionalise the Navier-Stokes equations. For the second we choose the combination

$$\Pi_2 = D \rho^\alpha U^\beta R^\gamma, \quad (13)$$

which, if we replaced  $D$  with  $\mu$ , would just give the Reynolds number. Equating the exponents of mass length and time gives,  $\alpha = -1$ ,  $\beta = -2$  and  $\gamma = -2$ . Thus

$$\Pi_2 = \frac{D}{\rho U^2 R^2}, \quad (14)$$

and this is called the dimensionless drag force. Buckingham's Pi theorem tells us that we must have the functional relationship

$$\Pi_2 = G(\Pi_1) \quad (15)$$

or alternatively

$$\frac{D}{\rho U^2 R^2} = G(\text{Re}). \quad (16)$$

The functional dependence is determined by experiments. It is found that at high Reynolds numbers  $G(\text{Re}) = 1$ , so that

$$D = \rho U^2 R^2. \quad (17)$$

This is known as form drag, in which resistance to motion is created by inertial forces on the sphere. At low Reynolds numbers  $G(\text{Re}) \propto 1/\text{Re}$  so that

$$D \propto \mu U R. \quad (18)$$

This is Stokes drag, caused by the viscosity of the fluid.

The power of taking this approach can now be seen. Without dimensional analysis, to determine the functional dependence of the drag on the relevant physical variables would have required four sets of experiments to determine the functional dependence of  $D$  on velocity, radius, viscosity and density. Now we need only perform one set of experiments using our dimensionless parameters and we have all the information we need.



## 12 Scaling

Scaling can be defined as the determination of the interdependency of variables in a physical system. It involves a combination of dimensional analysis and physical reasoning, and typically requires a good deal of physical intuition. It can, however, be used to great advantage in solving otherwise intractable problems. The best way to introduce scaling is through examples, so we shall do a few here.

### 12.1 Hydrodynamic drag on a sphere

In the last section we used dimensional analysis to show that the drag  $D$  on a sphere of radius  $R$  moving with velocity  $U$  through a fluid is

$$D = \rho U^2 R^2 F(\text{Re}). \quad (1)$$

From experiments one can obtain  $F(\text{Re})$  and determine the behaviour of the drag at high and low Reynolds numbers, and we identified the existence of *form drag* and *Stokes drag*, respectively, in these two limits.

An alternative to this formal approach is to use scaling arguments. At low  $\text{Re}$ , it is plausible to assume that viscosity dominates, so the motion will be resisted by viscous forces. A characteristic viscous stress,  $\sigma$ , in the problem is

$$\sigma \sim \frac{\mu U}{R}. \quad (2)$$

This is a force per unit area. The surface area of the sphere is proportional to  $R^2$ , so scaling implies that the total viscous drag acting on the surface area of the sphere is

$$D \sim \frac{\mu U}{R} R^2 \sim \mu U R, \quad (3)$$

which is Stokes drag.

Alternatively, at high Reynolds numbers inertial forces dominate. In this case, the characteristic surface force is due to pressure. From balancing the inertial and pressure terms in the Navier Stokes equations we know that this scales like

$$p \sim \rho U^2. \quad (4)$$

Pressure is also a force per unit area, so the retarding pressure force acting over the surface of the sphere is

$$D \sim \rho U^2 R^2, \quad (5)$$

which is the form drag.

## 12.2 The scaling of flight

Another interesting scaling argument arises if we consider the flight of birds. From dimensional analysis we argue that the drag force  $F_D$  and the lift force  $F_L$  are given by

$$F_D = F_D(L, b, U, \mu, \rho, \theta) \quad (6)$$

and

$$F_L = F_L(L, b, U, \mu, \rho, \theta), \quad (7)$$

where  $L$  is the wing length,  $b$  is width of the wing,  $U$  is the speed of flight,  $\mu$  is the viscosity,  $\rho$  is the density of air and  $\theta$  is the angle of attack. We have  $n = 7$  physical variables and  $d = 3$  dimensions, giving us 4 dimensionless groups. In addition to the Reynolds number and the drag/lift coefficient, we choose  $L/b$  and  $\theta$ . Thus, by Buckingham's Pi Theorem,

$$\frac{F_L}{\rho L b U^2} = G\left(\text{Re}, \frac{L}{b}, \theta\right). \quad (8)$$

The dimensions of birds are found to be geometrically similar, or *isometric*. In this case  $L \propto m^{1/3}$ , so that the projected area of the wings is proportional to  $m^{2/3}$  and the wing loading (loaded weight per wing area) increases as  $m/m^{2/3} = m^{1/3}$ . In Fig. 1, the wing loading is shown as a function of the body mass for a large number of flying objects; the solid line corresponds to a slope 1/3.

As birds are isometric, the ratio  $b/L$  is a constant. The wing angle of attack is also pretty much constant, so we can reduce the above relation to

$$\frac{F_L}{\rho L^2 U^2} = G(\text{Re}), \quad (9)$$

where we have used the fact that  $b$  is proportional to  $L$ . Over the range of Reynolds numbers for bird flight ( $10^3$  to  $10^5$ ) it is found that the lift coefficient does change with the Reynolds number, but not by very much, so we have that

$$\frac{F_L}{\rho L^2 U^2} = k_1. \quad (10)$$

Now we argue that for a bird to be able to fly the lift force must balance the weight of the bird ( $F_L = mg$ ). Thus

$$\frac{mg}{\rho L^2 U^2} = k_1, \quad (11)$$

and we have the prediction that

$$U = k_2 \left(\frac{m}{L^2}\right)^{1/2}. \quad (12)$$

The ratio  $m/L^2$  is called the span loading, and the above scaling argument tells us how it is related to the cruising speed of a bird. There is very good agreement with the calculated values for many different birds, as seen in the attached figure. Note that from self similarity,  $m \propto L^3$ , we come up with the prediction that

$$U \sim L^{1/2} \sim m^{1/6}, \quad (13)$$

implying that larger birds fly faster (do you believe this?).

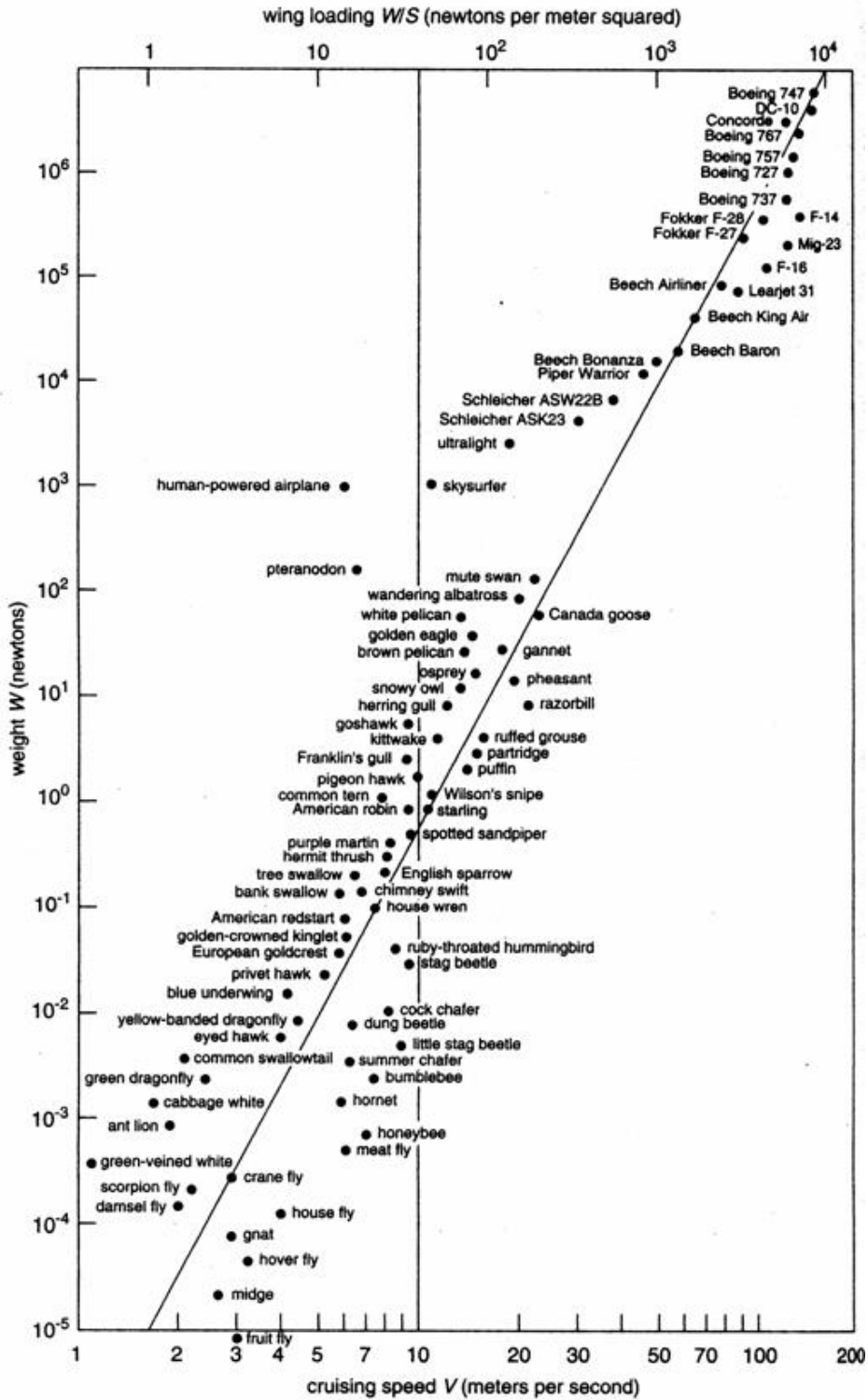


Figure 1: Taken from *The Simple Science of Flight: From Insects to Jumbo Jets* (Tenneke 1996).

### 12.3 McMahon's rowers

This example, and the preceding ones can all be found in the excellent book *On size and life*, by Thomas McMahon. He was a professor at Harvard and asked the following simple question in 1970. Consider the crew boats which row on the Charles. How does the speed of the boats scale with the number of rowers?

The boats are rowed by one, two, four or eight rowers, and if one examines the proportions, they are very close to being geometrically similar. This is estimated by measuring the length of the boat,  $L$ , and the width,  $b$ . The ratio of these two values remains fairly constant, although the length almost triples. Even the boatmass per oarsman is similar. The way that the boat works is that power being imparted to the water from the rowers is balanced by the drag on the boat. If there are  $N$  oarsmen, the power that is provided is proportional to the number of oarsmen:  $P = k_1 N$ .

What is the drag? The drag could arise from the viscosity or inertia of the water, and is also a function of the shape of the boat. Following our methods of dimensional analysis we propose

$$D = D(\rho, \mu, L, b, U). \quad (14)$$

In this case there are  $n = 6$  variables and  $m = 3$  dimensions, so we know there are three dimensionless groups in the problem (in the case of drag on a sphere we only had two dimensionless groups, because there was only one lengthscale). We choose our third dimensionless group to be  $L/b$ , the aspect ratio of the boat. Thus

$$\frac{D}{\rho U^2 L^2} = f\left(\text{Re}, \frac{L}{b}\right). \quad (15)$$

But  $L/b$  is constant for the rowing boats, so we only need to know

$$\frac{D}{\rho U^2 L^2} = f(\text{Re}). \quad (16)$$

It turns out that  $f(\text{Re})$  is a constant for streamlined rowing boats, which tells us that

$$D = k_2 \rho U^2 L^2. \quad (17)$$

(note that as  $L \propto b$  for our boat, the drag is proportional to the surface area of the boat in contact with the water). The power required to maintain a constant speed  $U$  is  $DU$ , and this must be balanced by the input of the rowers, giving

$$U^3 L^2 = k_3 N, \quad (18)$$

where  $k_3 = k_1/\rho k_2$  is a constant.

Now we need a relation between  $L$  and  $N$  to complete our scaling. From the data we have that the boat mass increases linearly with the number of oarsmen (which makes sense if you consider Archimedes) so from isometry we have that  $L \propto N^{1/3}$ , so that

$$L^2 \sim N^{2/3}. \quad (19)$$

Putting this back into relation (18), we get the scaling

$$U \sim N^{1/9}. \quad (20)$$

Fig. 2 shows data from the 2010 Lucerne finals. The theoretical lines  $\propto N^{1/9}$  fit the data quite well.

Discipline	Winner	Crew weight (ave)	average speed	Average Stroke Rate	average power	Power/kg
M8+	GER	87.4	6.05	38.94	610.97	6.99
M4x	CRO	92.0	5.76	37.84	644.83	7.01
M4-	GBR	96.0	5.63	39.40	643.92	6.71
M2x	FRA	90.5	5.28	36.25	617.86	6.83
M2-	NZL	91.0	5.19	38.48	613.78	6.74
M1x	CZE	98.0	4.83	35.45	644.05	6.57
W8+	USA	82.0	5.37	36.67	408.93	4.99
W4x	GBR	78.8	5.22	35.93	424.69	5.39
W2x	GBR	79.5	4.87	35.52	447.10	5.62
W2-	NZL	72.0	4.66	38.54	379.99	5.28
W1x	BLR	82.0	4.43	34.20	437.92	5.34
LM4-	GBR	70.0	5.54	37.86	492.07	7.03
LM2x	NZL	70.0	5.24	38.16	511.34	7.30
W2x	AUS	57.0	4.67	34.66	315.63	5.54

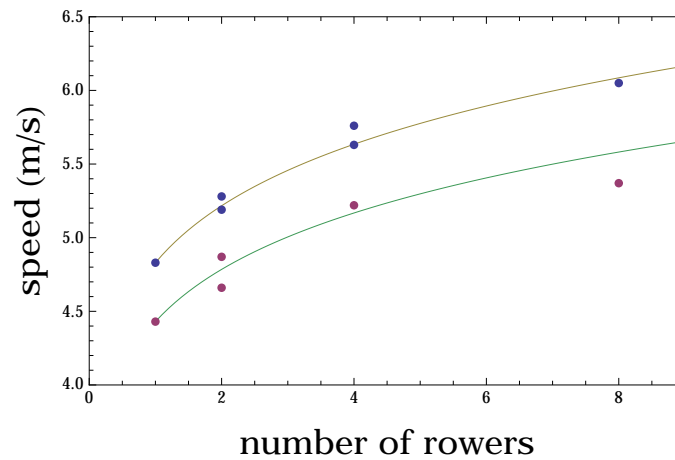


Figure 2: Rowing data from Lucerne 2010 A-finals (Sunday 11 July 2010; from <http://sanderroosendaal.wordpress.com/>). The theoretical lines with  $\propto N^{1/9}$  fit the data quite well.