
18.354 – Nonlinear Dynamics II: Continuum systems

Problem Set 1: RANDOM WALKS

Due: Monday, February 24 (by 3pm in class)

Problem 1: BIASED RANDOM WALKS

The goal of this problem is to derive the continuum approximation for a particle undergoing a biased random walk. Consider a random walker on a one dimensional lattice with:

- (a) probability of 1/3 of staying still and probability of 1/3 of moving to the left or to the right.
- (b) probability of 3/4 of moving to the left and probability of 1/4 of moving to the right.

Following the procedure from class, write down the continuum equation for a cloud of random walkers obeying these rules. Use both particle fluxes and probabilities (see class notes, Lecture 2) to derive the continuum equation, and check they agree. If not, why not? For each case, sketch what happens to an initial distribution of particles at the origin. What is the diffusion constant? Case (b) is an example of an advection-diffusion equation, in which you will find that the cloud of particles drifts to one side as well as diffusing. What is the drift velocity? How does the maximum density of the distribution decrease with time in each case?

Problem 2: GREEN'S FUNCTIONS

- (a) Use the Fourier method to solve for the Green's function of the diffusion equation.
- (b) Find the Greens functions for the continuum equations you derived in Problem 1. *Hint: to account for the drift in the second case move into the frame of reference of the translating cloud by considering a solution $f(x - ct, t) = g(x, t)$.*

Problem 3: RANDOM WALK SIMULATION

In class we discussed the random walk. The point of this problem is to use MATLAB to verify our assertions that a cloud of random walkers is well described by the diffusion equation.

- (a) Write a simple MATLAB program to simulate the motion of random walkers. Start with all the walkers at a fixed point and show that if they are allowed to take random steps to either the left or the right, they disperse according to the \sqrt{t} law. (*Hint: If you prefer, an alternative to letting the random walkers take discrete steps to the left or right of size $\delta = 1$ is to use the function `randn` to generate random data with zero mean and unit variance*). Provide a printout of your MATLAB code.

- (b) Show that the distribution of walkers after many time steps of the cloud is well approximated by the analytic solution to the diffusion equation.

Problem 4: RANDOM WALKERS AND THE DIFFUSION EQUATION

The aim of this problem is to recover a famous result: a random walker starting at the origin will (with probability 1) return to the origin in 1 and 2 dimensions, but not in 3! (ie. there is only some probability less than 1 that the walker will return in 3D). First, we wish to understand one random walker. To do this, let $p(x, t)$ denote the probability distribution for a walker (Note: the concentration $n(x, t)$ is related to $p(x, t)$ by the total number of particles N in the system: $n(x, t) = Np(x, t)$). Therefore, the probability of finding the walker at time t , between x and $x + dx$ is $\int_x^{x+dx} p(x, t) dx \approx p(x, t) dx$. In addition, since $n(x, t)$ satisfies the diffusion equation, so does $p(x, t)$.

Part A Consider a random walker meandering around in an infinite domain. Take the walkers starting location at the origin, so that $p(x, 0) = \delta(x)$. Since, $p(x, t)$ solves the diffusion equation, $\frac{\partial}{\partial t} p(x, t) = \nabla^2 p(x, t)$, the solution for $p(x, t)$ is exactly the Greens function. An extension of the calculation in class to any dimension, modifies the Greens function to:

$$p(x, t) = \frac{1}{(4\pi t)^{d/2}} e^{-r^2/4t} \quad (1)$$

Here d is the dimension of the domain, while r is the distance from the origin (the radial polar coordinate).

- (a) For each dimension, $d = 1, 2, 3$: if you observed the walker from time $t = 1$ to time $t = T$, how much time would you expect to see the walker near the origin? (*Hint: If V is the volume of a small ball around the origin, then $Vp(0, t)$ is roughly the probability that the walker is contained inside V at time t . If τ is the time spent inside V , then the expected value of τ is $\langle \tau \rangle = V \int_1^T p(0, t) dt$, ie. think of the particle having probability $p(0, t)$ of spending time dt near the origin.*)
- (b) After observing for time T , for which dimension would you expect to see the walker near the origin the most (and which dimension the least)?
- (c) What happens to $\langle \tau \rangle$ for $d = 1, 2, 3$ when $T \rightarrow \infty$? Convince yourself that this implies the particle should return to the origin infinitely often in 1 and 2 dimensions, while in 3D, the random walker will only spend a finite amount of time near the origin (ie. if you get lucky may return once or twice, but after a time will eventually be lost).

Part B The calculation in part A explicitly requires an infinite domain. The result no longer holds if we restrict the walker to live inside a 3D container (regardless of the shape!). Consider a 3D domain with volume Ω . Again let $p(x, 0) = \delta(x)$. After long times, solutions to the diffusion equation approach a constant. The only appropriate constant is $p(x, t) \rightarrow 1/\Omega$ (ie. a uniform probability distribution). Argue that the walker spends an expected time $\langle \tau \rangle \rightarrow \infty$ at the origin, as $T \rightarrow \infty$. After watching a random walker for a very long time, which would you expect to see spend more time at the origin: a walker on a 1D infinite line OR a walker contained inside a 3D box?